SOLUTIONS TO PROBLEMS 8A

1) a)

$$\frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial\psi}{\partial r}\right)}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]+V(r)\psi$$

$$=E\psi$$

LET $\psi = R(r)\,Y(\theta,\varphi) = RY$ ✓

$$\frac{-\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial\left(r^2\frac{\partial RY}{\partial r}\right)}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial RY}{\partial\theta}\right)}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 RY}{\partial\varphi^2}\right]+VRY=ERY$$

✓ SUBSTITUTION

$$\frac{-\hbar^2}{2m}\left[\frac{Y}{r}\frac{d\left(r^2\frac{dR}{dr}\right)}{dr}+\frac{R}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)}{\partial\theta}+\frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right]+VRY=ERY$$

✓ TAKE CONSTANT OUT

$\left(\times\frac{-2m}{\hbar^2}\right)$

$$\frac{Y}{r}\frac{d\left(r^2\frac{dR}{dr}\right)}{dr}+\frac{R}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)}{\partial\theta}+\frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}-\frac{2m}{\hbar^2}VRY=-\frac{2m}{\hbar^2}ERY$$

✓ MANIPULATION

$\left(\times\frac{1}{RY}\right)$

$$\frac{1}{rR}\frac{d\left(r^2\frac{dR}{dr}\right)}{dr}+\frac{1}{r^2\sin\theta\,Y}\frac{\partial\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)}{\partial\theta}+\frac{1}{r^2\sin^2\theta\,Y}\frac{\partial^2 Y}{\partial\varphi^2}-\frac{2mV}{\hbar^2}=\frac{-2mE}{\hbar^2}$$

$\times r^2$ ✓

$$\frac{r}{R}\frac{d\left(r^2\frac{dR}{dr}\right)}{dr}+\frac{1}{Y\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)}{\partial\theta}+\frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}-\frac{2mVr^2}{\hbar^2}=-\frac{2mEr^2}{\hbar^2}$$

$$\frac{r}{R}\frac{d\left(r^2\frac{dR}{dr}\right)}{dr} + \frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right\} - (V-E)\frac{2m}{\hbar^2} = 0$$

$$\underbrace{\frac{r}{R}\frac{d\left(r^2\frac{dR}{dr}\right)}{dr} - \frac{2m}{\hbar^2}(V-E)}_{\text{FUNCTIONS OF } r} = \underbrace{-\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right\}}_{\text{FUNCTIONS OF } \theta, \varphi} \checkmark$$

$= \text{CONSTANT}$, LET CONSTANT $= l(l+1)$ ✓

$$-\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right\} = l(l+1)$$

AS REQUIRED

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b) LET $Y(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi) = \Theta\Phi$ ✓

$$-\frac{1}{\Theta\Phi}\left\{\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial\Theta\Phi}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\Theta\Phi}{\partial\varphi^2}\right\} = l(l+1)$$

✓ SUBSTITUTION

$$-\frac{1}{\Theta\Phi}\left\{\frac{\Phi}{\sin\theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} + \frac{\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}\right\} = l(l+1)$$

✓ TAKE CONSTANT OUT

$$-\frac{1}{\Theta\sin\theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} - \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = l(l+1)$$

$\times \sin^2\theta$

✓ MANIPULATION

$$-\frac{\sin\theta}{\Theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} - \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = l(l+1)\sin^2\theta$$

$$\frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{\sin\theta}{\Theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} + l(l+1)\sin^2\theta$$

$$\underbrace{\frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\text{FUNCTIONS OF } \phi} \qquad \underbrace{\frac{\sin\theta}{\Theta}\frac{d\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{d\theta} + l(l+1)\sin^2\theta}_{\text{FUNCTIONS OF } \theta}$$

MUST BE EQUAL TO A CONSTANT, let CONSTANT $= M_l^2$ ✓

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = M_l^2 \qquad \text{AS REQUIRED}$$

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c) REARRANGE TO GET

$$\frac{d^2\Phi}{d\phi^2} = -M_l^2\Phi$$ ✓

SOLUTION $\Phi(\phi) = Ae^{iM_l\phi} + Be^{-iM_l\phi}$

ALLOW M_l TO BE BOTH POSITIVE AND NEGATIVE

$$\Phi(\phi) = Ae^{iM_l\phi}$$ ✓

NOW $\Phi(\theta) = \Phi(\theta + 2\pi)$ ✓ SINCE ϕ IS ANGLE FROM x-AXIS IT REPEATS EVERY 2π (PERIODIC BOUNDARY CONDITION)

$$Ae^{iM_l\phi} = Ae^{iM_l(\phi+2\pi)}$$ ✓

$$e^{iM_l\phi} = e^{iM_l\phi}\cdot e^{iM_l 2\pi} \quad \therefore \quad e^{iM_l 2\pi} = 1$$ ✓

$\therefore$ M_l IS AN INTEGER

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2)

a) $$E_n = -\left\{ \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \right\} \frac{1}{n^2} \approx -13.6 \times \frac{1}{n^2} \text{ eV}$$

$n=1$ $E_1 = -13.6$ eV ✓

$n=2$ $E_2 = -3.4$ eV ✓

$n=3$ $E_3 = -1.51$ eV ✓

$n=4$ $E_4 = -0.85$ eV ✓ /4

b) $n=1$ $l=0$ $m_l=0$ ✓

$n=2$ $l=0$ $m_l=0$
$l=1$ $m_l=-1,0,+1$ ✓

$n=3$ $l=0$ $m_l=0$
$l=1$ $m_l=-1,0,+1$
$l=2$ $m_l=-2,-1,0,1,2$ ✓

$n=4$ $l=0$ $m_l=0$
$l=1$ $m_l=-1,0,1$
$l=2$ $m_l=-2,-1,0,1,2$
$l=3$ $m_l=-3,-2,-1,0,1,2,3$ ✓

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c) DEGENERACY (IGNORING SPIN)

$n=1$ 1 FOLD ✓

$n=2$ 4 FOLD ✓

$n=3$ 9 FOLD ✓

$n=4$ 16 FOLD ✓ /4

(it's n^2)

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d) DEGENERACY INCLUDING SPIN = 2 X DEGENERACY IGNORING SPIN

SO $2n^2$ SINCE TWO SPIN STATES ↑ ↓

$n=1$ 2 FOLD

$n=2$ 8 FOLD

$n=3$ 18 FOLD

$n=4$ 32 FOLD.

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