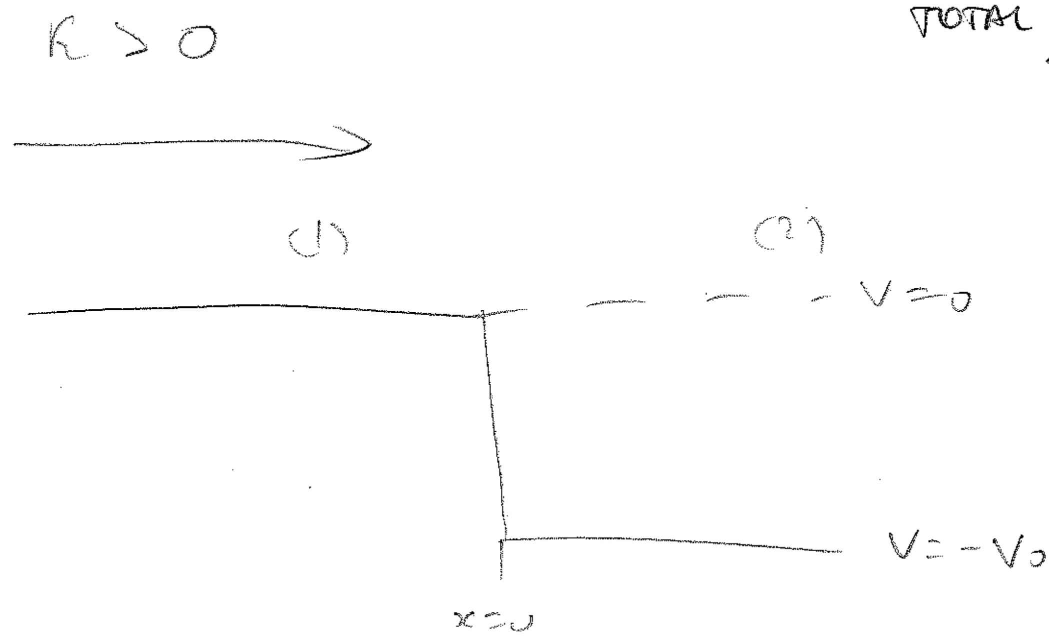
QM A SOLUTIONS 5A

TOTAL $\frac{\ }{38}$ + 8 BONUS



a) Region 1 $\quad V = 0 \quad \therefore$ TISE $\Rightarrow$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad ✓$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

SOLUTIONS

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad ✓$$

RIGHT TRAVELLING INCIDENT ✓ $\qquad$ LEFT TRAVELLING, REFLECTED

Region 2 $\quad V = -V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \quad ✓$$

$$\frac{d^2\psi}{dx^2} = \frac{-2m(E+V_0)}{\hbar^2}\psi = -q^2\psi \qquad q = \sqrt{\frac{2m(V_0+E)}{\hbar^2}}$$

2

solution $\psi(x) = Ce^{iqx} + De^{-iqx}$

RIGHT TRAVELLING, TRANSMITTED

LEFT TRAVELLING, NO SOURCE AT $+\infty$

$\therefore D = 0$

6

b) @ $x = 0$ ψ CONTINUOUS AND $\frac{d\psi}{dx}$ CONTINUOUS

$$Ae^{ikx} + Be^{-ikx} = Ce^{iqx}$$

$$\Rightarrow A + B = C \Rightarrow \frac{A}{C} + \frac{B}{C} = 1 \text{ or } 1 + \frac{B}{A} = \frac{C}{A}$$

$$ikAe^{ikx} - ikBe^{-ikx} = iqCe^{iqx}$$

$$\Rightarrow kA - kB = qC \Rightarrow \frac{A}{C} - \frac{B}{C} = \frac{q}{k}$$

NOW...

$$J_{INC} = |A|^2 \frac{\hbar k}{m} \quad J_{TRANS} = |C|^2 \frac{\hbar q}{m} \quad |J_{REF}| = |B|^2 \frac{\hbar k}{m}$$

$$T = \left|\frac{J_{TRANS}}{J_{INC}}\right| = \left|\frac{C}{A}\right|^2 \frac{q}{k} \qquad R = \left|\frac{J_{REF}}{J_{INC}}\right| = \left|\frac{B}{A}\right|^2$$

$$\frac{A}{C}+\frac{B}{C}=1 \qquad \text{AND} \qquad \frac{A}{C}-\frac{B}{C}=\frac{q}{k}$$

$$\left.\begin{aligned} 2\frac{A}{C} &= 1+\frac{q}{k} \quad ✓ \\ \frac{A}{C} &= \frac{k+q}{2k} \qquad \frac{C}{A}=\frac{2k}{k+q} \quad ✓ \end{aligned}\right\} \text{setting up}$$

$$T=\left|\frac{C}{A}\right|^2\frac{q}{k}=\frac{4k^2}{(k+q)^2}\cdot\frac{q}{k}$$

$$T=\frac{4kq}{(k+q)^2} \quad ✓$$

AND ALSO

$$\left.\begin{aligned} 1+\frac{B}{A} &= \frac{C}{A} \qquad \text{WITH} \quad \frac{C}{A}=\frac{2k}{k+q} \\ \frac{B}{A} &= \frac{C}{A}-1=\frac{2k}{k+q}-1 \quad ✓ \; ✓ \\ \frac{B}{A} &= \frac{2k-(k+q)}{(k+q)}=\frac{k-q}{k+q} \end{aligned}\right\} \text{setting up}$$

$$R=\left|\frac{B}{A}\right|^2=\left(\frac{k-q}{k+q}\right)^2 \quad ✓$$

$\overline{\;12}$

NOW $k = \sqrt{\frac{2mE}{\hbar^2}}$ $\quad q = \sqrt{\frac{2m(V_0+E)}{\hbar^2}}$

$k = \sqrt{\frac{2m}{\hbar^2}}\sqrt{E}$ $\quad q = \sqrt{\frac{2m}{\hbar^2}}\sqrt{V_0+E}$

$E = 32\,eV$

$V_0 = 16\,eV$

SUBSTITUTION INTO T, R

$$T = \frac{4kq}{(k+q)^2} = \frac{4\cdot\frac{2m}{\hbar^2}\cdot\sqrt{E}\cdot\sqrt{V_0+E}}{\left(\sqrt{\frac{2m}{\hbar^2}}\left(\sqrt{E}+\sqrt{V_0+E}\right)\right)^2} = \frac{4\cdot\cancel{\frac{2m}{\hbar^2}}\cdot\sqrt{E}\cdot\sqrt{V_0+E}}{\cancel{\frac{2m}{\hbar^2}}\left(\sqrt{E}+\sqrt{V_0+E}\right)^2}$$

$$T = \frac{4\sqrt{32}\sqrt{48}}{(\sqrt{32}+\sqrt{48})^2} \approx 0.98979$$

$$R = \left(\frac{k-q}{k+q}\right)^2 = \left(\frac{\sqrt{E}-\sqrt{V_0+E}}{\sqrt{E}+\sqrt{V_0+E}}\right)^2 = \left(\frac{\sqrt{32}-\sqrt{48}}{\sqrt{32}+\sqrt{48}}\right)^2$$

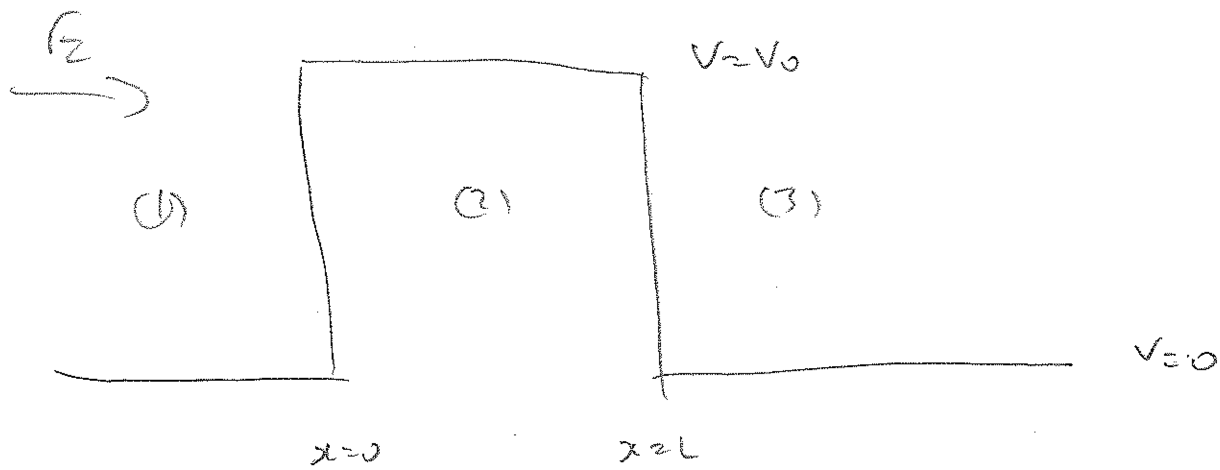
$$R \simeq 0.0102$$

$T + R \simeq 1$ OK.

NB: WE DON'T NEED THE MASS!

$\overline{5}$

TOTAL Q1 $\overline{23}$



a) $E < V_0$

(1) $\psi = Ae^{ikx} + Be^{-ikx}$ FROM Q.1 $k = \sqrt{\frac{2mE}{\hbar^2}}$

(3) $\psi = Fe^{ikx}$ AS IN (1) BUT RIGHT TRAVELLING (TRANSMITTED) ONLY

(2) $E < V_0$ TISE $\Rightarrow$ $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$

$$\therefore \frac{d^2\psi}{dx^2} = \frac{2m(V_0-E)}{\hbar^2}\psi$$

$$\frac{d^2\psi}{dx^2} = \alpha^2\psi \qquad \alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

IS REAL

SOLUTIONS $\psi = Ce^{\alpha x} + De^{-\alpha x}$ BOTH SOLUTIONS FOR REGION $0 \le x \le L$

AT $x = 0$ AND $x = L$ NEED ψ CONTINUOUS
AND $\frac{d\psi}{dx}$ CONTINUOUS

$\boxed{6}$

$x = 0$ $x = L$

BONUS

$A + B = C + D$ ✓ $Ce^{\alpha L} + De^{-\alpha L} = Fe^{ikL}$ ✓

AND $ik(A - B) = \alpha(C - D)$ ✓ AND $\alpha(Ce^{\alpha L} - De^{-\alpha L}) = ikFe^{ikL}$ ✓

/9 + 4 BONUS

b) (1) & (3) SAME AS a) ✓

(2) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$ $E > V_0$

$\frac{d^2\psi}{dx^2} = -\frac{2m(E - V_0)\psi}{\hbar^2} = -q^2\psi$ ✓ where $q = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

SOLUTIONS

$\psi = Ce^{iqx} + De^{-iqx}$ ✓

TO RIGHT ✓ TO LEFT, REFLECTED AT L

BOTH GOOD SOLUTIONS

ψ continuous & $\frac{d\psi}{dx}$ continuous ✓ @ $x = 0$ AND $x = L$ ✓

$x = 0$ $x = L$

$A + B = C + D$ ✓ ✓ $Ce^{iqL} + De^{-iqL} = Fe^{ikL}$

AND $ik(A - B) = iq(C - D)$ ✓ ✓ $iq(Ce^{iqL} - De^{-iqL}) = ikFe^{ikL}$

BONUS

/6 + 4 BONUS

Q2 /15