Quantum Mechanics PHYS2B22
Problem Sheet 3
Work in by Monday February 28th

1) A particle is confined in a 1D box with walls of infinite height at $x = -a$ and $x = +a$. At $t = 0$ it is found to have the wavefunction

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2a}} \left[\cos\left(\frac{\pi x}{2a}\right) + \sin\left(\frac{\pi x}{a}\right) \right],$$

for $-a \le x \le a$ and $\Psi = 0$ otherwise.

a) Sketch the probability density for $t = 0$. [**3**]
b) Since the energy eigenfunctions $\psi_n(x)$ for the infinite well form a complete set, we can expand the initial wavefunction in terms of them, i.e. $\Psi(x, t = 0) = \sum_{n=0}^{\infty} a_n \psi_n(x)$. Find (by inspection or otherwise) the coefficients a_n in this expansion. Hence confirm that the wavefunction is correctly normalized. [**4**]
c) What are the possible results of a measurement of the particle's energy at $t = 0$? What are the corresponding probabilities? [**4**]
d) What would be the average value of the energy if many measurements were made on identically prepared particles? [**2**]
e) Use your expansion of $\Psi(x, t = 0)$ in terms of the energy eigenfunctions to write down the wavefunction $\Psi(x, t)$ at an arbitrary time t. [Hint: use your knowledge of the time-evolution of the energy eigenfunctions and the superposition principle for the time-dependent Schrödinger equation.] [**4**]
f) Hence sketch the probability density at $t = 8ma^2/\pi\hbar$. [**3**]

[You may use:
Normalized energy eigenfunctions are:

$$\psi_n(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2a}\right), \quad \textit{(n odd: n = 1,3,5 ...)},$$
$$= \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right), \quad \textit{(n even: n = 2,4,6 ...)}.$$

Corresponding energy eigenvalues:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{8ma^2}.$$

PTO

2) Suppose the functions $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$ are normalized eigenfunctions of a Hermitian operator $\hat{O}$ with eigenvalues $\{\lambda_1 = 1, \lambda_2 = 5, \lambda_3 = 9\}$ respectively. Suppose also that a system has the (un-normalized) wavefunction $\psi(x) = \phi_1(x) + 2\phi_2(x) + 3\phi_3(x)$.

(a) Normalize $\psi(x)$. [**4**]
(b) What are the possible results that can be obtained by measuring the physical quantity corresponding to $\hat{O}$ in the state $\psi(x)$, and what is the probability of each outcome? [**3**]
(c) What is $\langle\hat{O}\rangle$, i.e. the expectation value of the physical quantity corresponding to $\hat{O}$ in the state $\psi(x)$? [**3**]
(d) If $\hat{O}$ is measured and the result $\lambda_2 = 5$ is obtained, what is the wavefunction immediately after the measurement? [**2**]

3) Consider the following superposition of right and left moving waves: $\psi(x) = Ae^{ikx} + Be^{-ikx}$, where A and B are constants. Show that the particle flux in this state is $j = \frac{\hbar k}{m}(|A|^2 - |B|^2)$. [**4**] Apply this result to evaluate the particle flux in the first excited state of an infinite square well of width 2a for which the wavefunction is $\psi(x) = \frac{1}{\sqrt{a}}\sin(\frac{2\pi x}{2a})$ [**2**]. Does your answer make sense physically? [**2**]

[HINT: remember $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$*].*