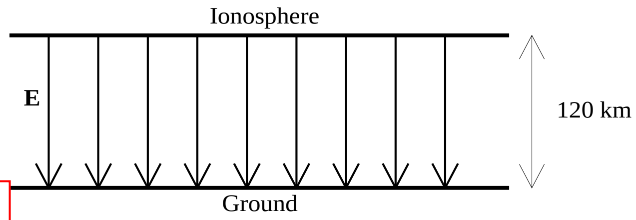
ELECTRIC AND MAGNETIC FIELDS
ANSWERS TO ASSIGNMENT 5

Q1: 30



(a) Energy density of the electric field is

$$u = \frac{1}{2}\varepsilon_0 E^2.$$

10 for formula

In this case, E is constant: $E = 100\ \mathrm{V\ m^{-1}} \Rightarrow$ $\mathbf{u = 4.43 \times 10^{-8}\ J\ m^{-3}}$.

5 for result

(b) The total electric energy of the atmosphere is U_{tot} = (Energy density)(Volume of the atmosphere)

$\Rightarrow U_{tot}$ = (Energy density)(Area of Earth's surface)(Height of ionosphere)

$\Rightarrow U_{tot} = (4.43 \times 10^{-8})[4\pi(6.4 \times 10^{6})^2](1.2 \times 10^{5}) = \mathbf{2.73 \times 10^{12}\ J.}$

15 = 10 for method and 5 for result

Slightly more accurate calculation:

Vol. of atmosphere = $\frac{4}{3}\pi\left[(R_E + 120\mathrm{km})^3 - (R_E)^3\right] = 6.29 \times 10^{19}\ \mathrm{m}^3$ $\Rightarrow \mathbf{U_{tot} = 2.79 \times 10^{12}\ J.}$

Q2: The energy of a system of point charges is

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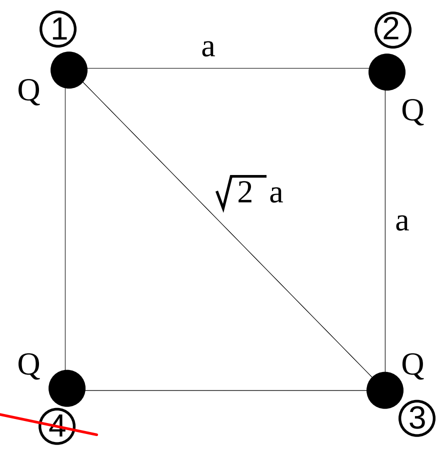
$$U_{tot} = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i.$$



8 for good diagram

By symmetry,

- All four charges are equal to Q, and they all have the same energy. Therefore,

$$U_{tot} = (1/2)(4)QV_1 = 2QV_1$$

where V_1 is the potential at charge 1 (for instance) due to all of the other charges.

10 for getting to here or equivalent

- V_1 has three contributions, from charges 2, 3, and 4: $V_1 = V_{12} + V_{13} + V_{14}$

  where V_{12} = potential at position 1 due to the charge at position 2, etc.

10 for knowing what V_I means and expressing it as three contributions

- $V_{12} = V_{14} = Q/(4\pi\varepsilon_0 a)$ $\qquad V_{13} = Q/(4\pi\varepsilon_0 a\sqrt{2})$

So $U_{tot} = 2Q\left[\frac{Q}{4\pi\varepsilon_0 a} + \frac{Q}{4\pi\varepsilon_0 a\sqrt{2}} + \frac{Q}{4\pi\varepsilon_0 a}\right] = \frac{Q^2}{4\pi\varepsilon_0 a}\left[2 + 2 + \frac{2}{\sqrt{2}}\right] = \frac{Q^2}{4\pi\varepsilon_0 a}\left[4 + \sqrt{2}\right]$

7 for final answer

Q3: (i) By spherical symmetry, the field lines are radial.

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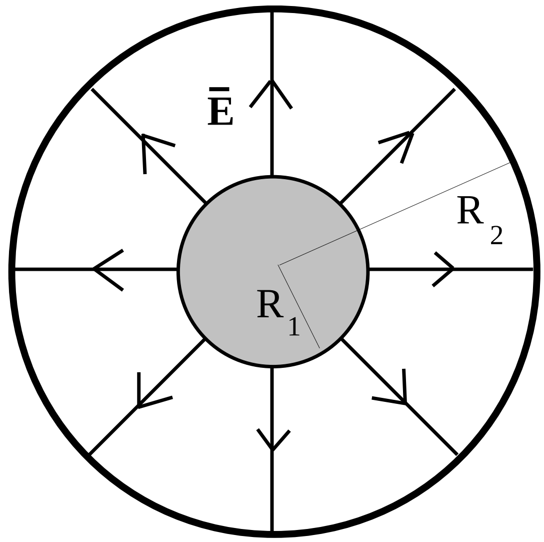


E = 0 inside the solid conducting sphere.

Field lines originate on positive charges on the surface of the inner conductor, and terminate on negative charges on the inner surface of the shell.

(ii) The energy density of the electric field is given by

$u = \frac{1}{2}\varepsilon_0 E^2$ J m^{-3} 3

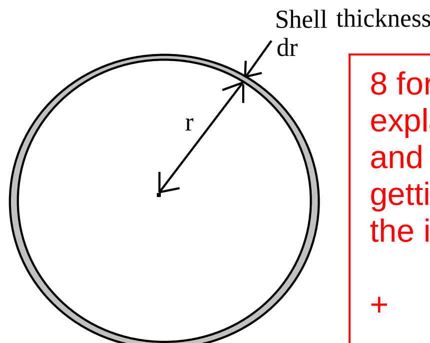
The field in the region between the conductors is the same as for a point charge, because the charge distribution is spherically symmetric:

6 for good diagram (note – question doesn't ask for explanation

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$ 3

The energy density is therefore $u = \frac{1}{2}\varepsilon_0 E^2 = \frac{Q^2}{32\pi^2\varepsilon_0^2 r^4}$ 3

(iii) The total energy is obtained by integrating this expression over the entire volume between a and b. As the incremental volume element, choose a spherical shell of radius r, thickness dr.



8 for good explanation and getting to the integral

+

5 for integrating to get to the answer

Volume of shell is $d(\text{Volume}) = 4\pi r^2 dr$

The energy contained in the shell is

$$dU = u(r)d(\text{Volume}) = \frac{Q^2}{32\pi^2\varepsilon_0 r^4}4\pi r^2 dr$$

The total energy is obtained by integrating this over the region in which E exists - i.e., between R_1 and R_2.

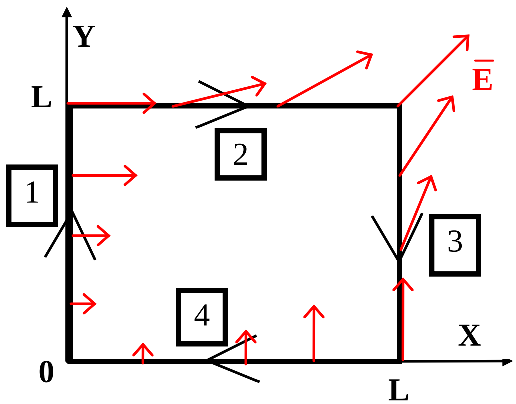
Therefore

$$U_{tot} = \int_a^b \frac{Q^2}{8\pi\varepsilon_0} r^{-2} dr = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] \Rightarrow U_{tot} = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{b-a}{ab}\right]$$

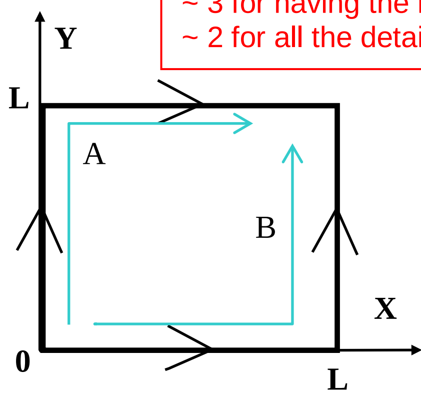
(iv) If b >> a then $U_{tot} = \dfrac{Q^2}{8\pi\varepsilon_0 a}$ ≡ formula derived in the lectures for total electric energy of an isolated conducting sphere of radius a. $\boxed{2}$

Q4: $\boxed{5}$

It is very useful to draw the electric field vector at various positions, as shown



~ 3 for having the right idea
~ 2 for all the details



.If $\overline{\mathbf{E}}$ is conservative, then $\oint \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} = 0$ (or $\int_0 \overline{\mathbf{E}}.d\overline{\mathbf{L}}$ is the same for paths A and B).

We can integrate $\overline{\mathbf{E}}\cdot d\overline{\mathbf{L}}$ around the square loop. Splitting the integral into four sections, we can see from the filed pattern that:

Sections 1 and 4 : $\overline{\mathbf{E}}$ and $d\overline{\mathbf{L}}$ are perpendicular $\Rightarrow$ $\overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} = 0$

Section 2 : The x component of $\overline{\mathbf{E}}$ is parallel to $d\overline{\mathbf{L}}$ $\Rightarrow$ $\overline{\mathbf{E}}\cdot d\overline{\mathbf{L}}$ is positive

Section 3 : The y component of $\overline{\mathbf{E}}$ is anti-parallel to $d\overline{\mathbf{L}}$ $\Rightarrow$ $\overline{\mathbf{E}}\cdot d\overline{\mathbf{L}}$ is negative

$$\oint \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} = \int_{\mathbf{1}} \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} + \int_{\mathbf{2}} \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} + \int_{\mathbf{3}} \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} + \int_{\mathbf{4}} \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}}$$

$$\int_{\mathbf{1}} \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} = \int (ay\,\hat{\mathbf{i}} + bx\,\hat{\mathbf{j}})\cdot d\overline{\mathbf{y}} = \int (bx\,\hat{\mathbf{j}})\cdot d\overline{\mathbf{y}} = 0 \qquad \text{as } x = 0$$

$$\int_{\mathbf{2}} \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} = \int (ay\,\hat{\mathbf{i}} + bx\,\hat{\mathbf{j}})\cdot d\overline{\mathbf{x}} = \int (ay\hat{\mathbf{i}})\cdot d\overline{\mathbf{x}} = \int (aL\,\hat{\mathbf{i}})\cdot d\overline{\mathbf{x}} = \int aLdx = aL\int dx \qquad \text{as } y = L \text{ and } \hat{\mathbf{i}}\cdot d\overline{\mathbf{x}} = dx$$

The length of the path is L, so $\int_2 \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} = aL^2$

$$\int_3 \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} = \int (ay\,\hat{\mathbf{i}} + bx\,\hat{\mathbf{j}})\cdot -d\overline{\mathbf{y}} = -\int (bx\,\hat{\mathbf{j}}).d\overline{\mathbf{y}} = -\int (bL\,\hat{\mathbf{j}}).d\overline{\mathbf{y}} = -\int bLdy = -bL\int dy \qquad \text{as } x = L \text{ and } \hat{\mathbf{j}}\cdot d\overline{\mathbf{y}} = dy$$

The length of this path is L, so $\int_2 \overline{\mathbf{E}}\cdot d\overline{\mathbf{L}} = -bL^2$

Difficult – some may get this as positive. Don't penalise harshly

$$\int_{4} \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = \int (ay\,\hat{\mathbf{i}} + bx\;\hat{\mathbf{j}}) \cdot -d\overline{\mathbf{x}} = -\int (ay\,\hat{\mathbf{i}}) \cdot d\overline{\mathbf{x}} = 0 \qquad \text{as } y = 0$$

Therefore