ELECTRIC AND MAGNETIC FIELDS

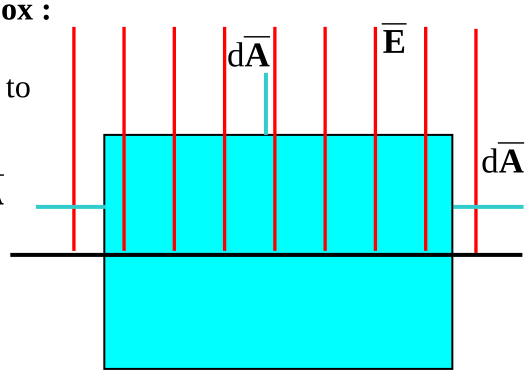
ANSWERS TO ASSIGNMENT 3

Q1: E = 80 N C^{-1} and points downwards. Cylindrical box has radius 4 m.

35

Ground ≡ a perfect conductor ⇒ Electric field is zero inside the ground
⇒ Electric field lines terminate at negative charges on the surface of the ground.

(i) Total electric flux through the box :



~ 20 for this bit. The answer here is a bit more thourough than is necessary but an understanding of the main steps must be evident.

Deduct 5 marks if the sign is wrong at the end.

- For the top side, $\overline{\mathbf{E}}$ is anti-parallel to the normal vector, so

  $\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = -EdA$

  The magnitude of the field, E, is also the same everywhere over the top side.

- $\overline{\mathbf{E}}$ is zero over the bottom surface ⇒ no contribution to Φ

- $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ are perpendicular for the sides ⇒ $\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = 0$ ⇒ no contribution to Φ

Therefore

$$\oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \int_{top} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} + \int_{bottom} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} + \int_{sides} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \int_{top} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} + 0 + 0$$

$$\text{So} \quad \Phi = \int_{top} \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \int_{top} EdA = -E\int_{top} dA = -(E)(\text{Area of top}) = -(80)\pi(4^2) = \mathbf{-\,4021.76\,N\,C^{-1}\,m^2}$$

The negative sign signifies electric flux going **into** the box.

(ii) The total enclosed charge is $Q_{enc} = \varepsilon_0\Phi = \mathbf{-\ 3.56 \times 10^{-8}\ C}$ **.** 8

This charge is located at the surface of the ground, where the electric field lines terminate.

(iii) Spherical box of same radius:

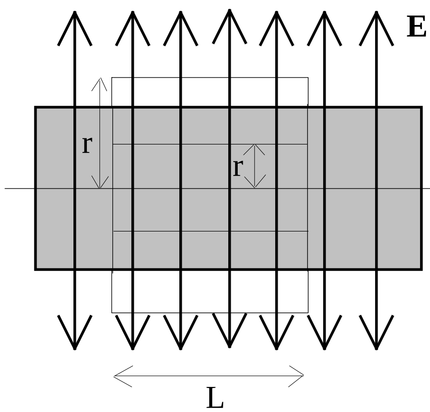
The enclosed charge is the same as before since the area of ground inside the box is the same. 7
Since Q_{enc} is the same, the total flux, Q_{enc}/ε_0 is also the same.

Q2: Long plastic (i.e., insulating) rod. Radius a. Uniform charge density inside.

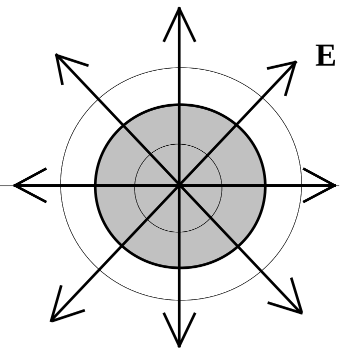
30

Charge per unit length = λ.

(a) By symmetry, the field lines will radiate outwards, pointing perpendicular to the axis of the cylinder.



SIDE VIEW



AXIAL VIEW

6 for a good diagram

(b) To use Gauss's Law to find E:

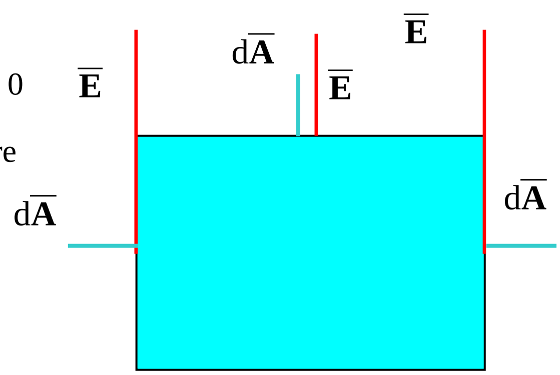
1. Diagram with field pattern: as above

2. Best Gaussian surface: Choose a cylinder, radius r (two cases: r < a and r > a).

For either of the two Gaussian cylinders,

- Flat ends: $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ are perpendicular $\Rightarrow \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = 0$
- Curved side: $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ are parallel

$\Rightarrow \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = EdA$



and: because r is the same over the whole of the curved side, so is E.

Step-by-step procedure has been taught explicitly so will probably be followed.But as long as the main ideas are clear it's OK

3. Work out Φ:

$$\Phi = \oint \overline{\mathbf{E}}.d\overline{\mathbf{A}} = \int_{\text{top}} \overline{\mathbf{E}}.d\overline{\mathbf{A}} + \int_{\text{bottom}} \overline{\mathbf{E}}.d\overline{\mathbf{A}} + \int_{\text{side}} \overline{\mathbf{E}}.d\overline{\mathbf{A}} = 0 + 0 + \int_{\text{side}} \overline{\mathbf{E}}.d\overline{\mathbf{A}} = \int_{\text{side}} EdA = E\int_{\text{side}} dA$$

So, $\Phi = (E)(\text{Area of curved surface}) = (2\pi rL)E$

8 for correct Φ and correct derivation

4. Find $Q_{enclosed}$:

(i) $r < a$: $\quad Q_{enc} = (\text{Charge on length L of rod}) \times \dfrac{\text{Volume of Gaussian cylinder}}{\text{Volume of rod}}$

$$\Rightarrow \quad Q_{enc} = (\lambda L)\,\frac{\pi r^2 L}{\pi a^2 L} = \frac{r^2}{a^2}\lambda L$$

3 + 3 for correct Q_{enc} values

(i) $r > a$: $\quad Q_{enc} = (\text{Charge on length L of rod}) = \lambda L$

5. Equate Φ and Q_{enc}/ε_0 to find E:

(i) $r < a$: $\quad (2\pi rL)E = (r^2/a^2)\lambda L/\varepsilon_0 \quad \Rightarrow \quad \boxed{E(r) = \dfrac{\lambda r}{2\pi a^2 \varepsilon_0}}$

(ii) $r > a$: $\quad (2\pi rL)E = \lambda L/\varepsilon_0 \quad \Rightarrow \quad \boxed{E(r) = \dfrac{\lambda}{2\pi \varepsilon_0 r}}$

3 + 3 for correct final answers

(c) The flux through a cylinder of radius a/2 is

$$\Phi = \oint \mathrm{E.ds} = E\oint ds = (E)(Area\ of\ curved\ surface) = \frac{\lambda(a/2)}{2\pi a^2 \varepsilon_0}2\pi(a/2)(5)$$

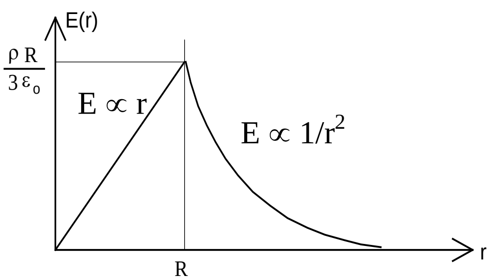
So $\quad \boxed{\Phi = \dfrac{5\lambda}{4\varepsilon_0}}$

4

30

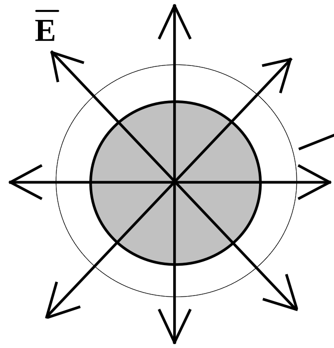
Q3: Given $\quad E = \dfrac{\rho r}{3\varepsilon_0}$ for $r < R \qquad E = \dfrac{\rho R^3}{3\varepsilon_0 r^2}$ for $r > R$

(i) Sketch:



6 for a good sketch - doesn't need to be labeled iin as much detail as this.

(ii) By symmetry, the electric field pattern of a spherically symmetric charge distribution must be be radially symmetric.



Therefore, if we choose a sphere of radius r (> R) as the Gaussian surface, the field will be perpendicular to the surface at all points.

Therefore, if E is the magnitude of the field at radius r, the total flux through the Gaussian sphere is

$$\Phi = (4\pi r^2)E$$

The charge enclosed is Q, so $E = Q/(4\pi\varepsilon_0 r^2)$,

6 for any convincing answer based on Gauss's Law

as for a point charge at the centre.

(iii) $r_1 = 10$ mm, $E_1 = 3.77 \times 10^5$ N C^{-1} $r_2 = 40$ mm, $E_2 = 1.88 \times 10^5$ N C^{-1}

If we determine whether the points are inside or outside, we can use the above formulas to find ρ, R and Q.

If both points are inside, $\dfrac{E_1}{E_2} = \dfrac{r_1}{r_2} \Rightarrow \dfrac{3.77}{1.88} = \dfrac{10}{40} \Rightarrow 2 = 0.25$ (NOT TRUE)

If both are outside, $\dfrac{E_1}{E_2} = \dfrac{r_2{}^2}{r_1{}^2} \Rightarrow \dfrac{3.77}{1.88} = \dfrac{40^2}{10^2} \Rightarrow 2 = 8$ (NOT TRUE)

Therefore the only possibility is that r_1 is inside and r_2 is outside.

3 for actually proving this

(a) r_1 inside $\Rightarrow \rho(0.010)/(3\varepsilon_0) = 3.77 \times 10^5 \Rightarrow$ $\boldsymbol{\rho = 1.0 \times 10^{-3}}$ **C m^{-3}** $\boxed{5}$

(b) Therefore, r_2 outside $\Rightarrow \dfrac{(1.0\times10^{-3})R^3}{3\varepsilon_0(0.040)^2} = 1.88\times10^5 \Rightarrow$ **R = 20 mm** $\boxed{5}$

(c) $Q = \dfrac{4}{3}\pi R^3\rho \Rightarrow$ $\mathbf{Q = 3.35 \times 10^{-8}}$ **C** $\boxed{5}$

5

Q4: $\rho = \dfrac{e}{8\pi b^3}\exp\left[-\dfrac{r}{b}\right]$ $\quad$ C m^{3} $\qquad$ where $b = 2.3 \times 10^{-16}$ m

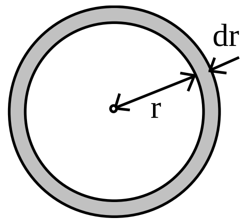
Follow the procedure for using Gauss's law using a spherical Gaussian surface of radius r.

2 for applying Gauss's Law and using correct volume element etc.

Steps 1 - 3 give $\Phi = (4\pi r^2)E$ where E is the magnitude of the field at r.

Step 4: $\qquad$ Find $Q_{enclosed}$:

Define a thin spherical shell, radius r, thickness dr:



The charge contained in the shell is

dq = ρ(r)(volume of shell)

$$\text{So,}\quad dq = \frac{e}{8\pi b^3}\exp\left[-\frac{r}{b}\right]\left[4\pi r^2 dr\right] = \frac{er^2}{2b^3}\exp\left[-\frac{r}{b}\right]dr$$

The total charge inside a Gaussian sphere of radius r is therefore

$$q(r) = \frac{e}{2b^3}\int_0^r r^2 \exp\left[-\frac{r}{b}\right]dr$$

2 for coping with the integration

The standard integral, with x=r/b gives

$$Q_{enc} = \frac{e}{2}\int_0^{r/b} x^2 \exp[-x]dx = \frac{e}{2}\left[-x^2e^{-x} - 2e^{-x}(x+1)\right]_0^{r/b}$$

So

$$Q_{enc} = \frac{e}{2}\left[2 - \left(\frac{r}{b}\right)^2 e^{-r/b} - 2e^{-r/b}\left(\frac{r}{b}+1\right)\right]$$

Putting $\Phi = Q_{enc}/\varepsilon_0$ gives

$$E_r = \frac{e}{8\pi\varepsilon_0 r^2}\left[2 - \frac{r^2}{b^2}\exp\left(-\frac{r}{b}\right) - 2\left(\frac{r}{b}+1\right)\exp\left(-\frac{r}{b}\right)\right]$$

Putting in the numerical values gives $E = 2.13 \times 10^{21}$ N C^{-1} at $r = 5 \times 10^{-16}$ m $\quad$ 0.5

This is 2.7 times smaller than the value we would get if we regarded the proton as a point charge, ($E = 5.67 \times 10^{21}$ N C^{-1}). $\quad$ 0.5

Round up any half marks at the end