SUMMARY OF SECTIONS 4-5 2B22 QUANTUM PHYSICS PROBLEMS:

See one / Do one problem:

1a) An atom is in a superposition of normalised eigenstates of hydrogen $\psi_{nlm}(\underline{r}) = R_{nl}(r)\, Y_{lm}(\theta,\phi)$ given by:

$$\psi(\underline{r}) = \sqrt{3/5}\,\psi_{311} + \sqrt{2/5}\,\psi_{310} + \sqrt{5/5}\,\psi_{31-1} + \sqrt{\tfrac{4}{5}}\,\psi_{411} + \psi_{421}$$

a) is $\psi(\underline{r})$ normalised? If not give it in normalised form.

b) What is the probability that , in a single measurement, we will measure L^2 to be equal to $6\hbar^2$?

c) What is the probability that we will measure the value of $L_z = \hbar$?

d) What value for L_z will we obtain by averaging many measurements on identically prepared atoms?

e) What is the expectation value of r?

ANSWER

The 3 eigenvalues of $\hat{H}$, $\hat{L}^2$, $\hat{L}_z$ are

$E_n = -\frac{1}{2n^2}$, $l(l+1)\hbar^2$, $m\hbar$ respectively.

a) NO. The squares of the coefficients add up to 4 rather than 1. To normalise, multiply all the coefficients by $1/\sqrt{4} = 1/2$.

b) This means that $l(l+1)\hbar^2 = 6\hbar^2$ so $l = 2$. The only component with $l=2$ is ψ_{421}, with coefficient ½ so the probability is = 1/4

c) This means the components with $m=1$ (ie the ψ_{311}, ψ_{411}, ψ_{421}). The total probability is the sums of those coefficients so = $3/20 + 1/5 + 1/4 = 3/5$.

d) $\langle \hat{a} \rangle = \sum_n |c_n|^2 \lambda_n$ ie the expectation value of the operator is a sum of eigenvalues weighted by the respective probabilities. Here $= \hbar \cdot [3/20 + 0 - 5/20 + \frac{4}{20} + 5/20] = 3/10\,\hbar$.

e) In this case it is the weighted sum $\langle r \rangle = \sum_{nlm} |c_{nlm}|^2 \langle r \rangle_{nlm}$ of the expectation values obtained from each term, where $\langle r \rangle_{nlm} = \int |\psi_{nlm}|^2\, r \cdot r^2 dr \sin\theta\, d\theta\, d\phi$.

1b) A quantum particle is in a superposition of normalised eigenstates, $\psi_n(x)$ of the QHO (for which the eigen-energy, $E_n = (n + 1/2)\hbar\omega$):

$$\Psi(x) = \sqrt{1/6}\,\psi_0(x) + \sqrt{\tfrac{1}{3}}\,\psi_1(x) + \sqrt{5/6}\,\psi_2(x) + \sqrt{2/3}\,\psi_3(x)$$

a) Check the normalisation and give Ψ in normalised form.

b) Here $\omega = 4$. What is the probability that , in a single measurement, we will measure the energy to be = $10\hbar$?

d) What value for the energy will we obtain by averaging many measurements?

e) Give the expectation value of *x* (at least explain the maths you need , even if you can't solve it) . But in fact you can write the answer if you consider the symmetry of the $\psi_n(x)$.

Q.4c of Homework 3 :

We have a wavefunction

$$\psi(\theta,\phi) = \sqrt{\tfrac{15}{14\pi}}\cos 2\theta \sin\phi$$

Work out the probability that a measurement of L^2 will yield the value $2\hbar^2$.

ANSWER: We know that

$$\psi(\theta,\phi) = \sqrt{\tfrac{15}{14\pi}}\cos 2\theta \sin\phi = \sum_{\ell m} C_{\ell m} Y_{\ell m}(\theta,\phi)$$

What we need to calculate are the $C_{\ell m}$.

Now if we measure the eigenvalue $\ell(\ell+1)\hbar^2 = 2\hbar^2$ then we know that $\ell = 1$.

fr all terms in the expansion $\sum_{\ell m} C_{\ell m} Y_{\ell m}$.

Hence $m = -1, 0, 1$.

So we can have $Y_{10}(\theta,\phi)$, $Y_{11}(\theta,\phi)$, $Y_{1-1}(\theta,\phi)$ in our expansion. We know:

$$C_{\ell m} = \int Y^{*}_{\ell m}(\theta,\phi)\,\psi(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

and we want C_{11}, C_{10}, C_{1-1}.

$$C_{10} = \int Y^{*}_{10}\,\psi(\theta,\phi)\sin\theta\, d\theta\, d\phi =$$

$$= \int_0^{\pi} \sqrt{\tfrac{3}{4\pi}}\cos\theta \sqrt{\tfrac{15}{14\pi}}\cos 2\theta \sin\theta\, d\theta \int_0^{2\pi}\sin\phi\, d\phi = 0$$

since $\int_0^{2\pi}\sin\phi\, d\phi = 0$

$$C_{11} = \int Y^{*}_{11}\,\psi(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$C_{11} = -\left(\tfrac{45}{112\pi^2}\right)^{1/2} \underbrace{\int_0^{\pi}\cos 2\theta \sin^2\theta\, d\theta}_{\pi/4} \underbrace{\int_0^{2\pi}\sin\phi\, e^{i\phi}\, d\phi}_{\pi/i} = C_{1-1}$$

Probability $= |C_{11}|^2 + |C_{1-1}|^2 = 45/896\,\pi^2$

ANOTHER example

An electron is described by a wavefunction $u(\theta,\phi) = \sqrt{\tfrac{15}{16\pi}}\sin^2\theta\cos 2\phi$. Work out the probability that a measurement of L_z 1) that will give the value $2\hbar$. 2) that it will give the value 0 . and 3) It will give the value $-2\hbar$.

You can use:

$$Y_{20}(\theta,\phi) = \sqrt{\tfrac{5}{16}}(3\cos^2\theta - 1)$$

$$Y_{2\pm1}(\theta,\phi) = \pm\sqrt{\tfrac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{2\pm2}(\theta,\phi) = \sqrt{\tfrac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

Useful integral : $\int_0^{\pi}\sin^5\theta\, d\theta = 16/15$

(HINT: do the ϕ part of the overlap integral first. The answers should be

1) probability = ½ 2) probability = 0 and 3) probability = ½ .)