PHAS2222 QUANTUM PHYSICS
Problem sheet 4 (P2222.4)
To be handed in by 5pm on Friday 7 December 2007

1. The normalized wavefunction of an electron in an atom can be written in spherical polar coordinates as

   $\psi(r,\theta,\phi) = R(r)\left[\sqrt{\frac{1}{6}}Y_2^2(\theta,\phi) + \sqrt{\frac{2}{3}}Y_2^0(\theta,\phi) + \sqrt{\frac{1}{6}}Y_2^{-2}(\theta,\phi)\right]$, where R is some normalized function of r only and Y_l^m is a spherical harmonic as defined in the lectures.

   (a) Find the action of the operators $\hat{L}^2$ and $\hat{L}_z$ on ψ. Is it an eigenfunction of either of these operators? **[4]**

   (b) What are the possible results that could be obtained when the quantities (i) $\hat{L}^2$ and (ii) $\hat{L}_z$ are measured in this system, and what are the associated probabilities? **[5]**

2. Consider a particle of mass m moving in three dimensions with Hamiltonian $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x,y,z)$. Provided that the potential V can be written as $V(x,y,z) = V_x(x) + V_y(y) + V_z(z)$ (where V_x is an arbitrary function of x alone, and so on), show that separable solutions of the time-independent Schrödinger equation exist in the form $\psi(x,y,z) = X(x)Y(y)Z(z)$, and find the equations satisfied by X, Y and Z. **[6]**

3. An atom of tritium (the unstable heavy isotope of hydrogen, ^3H) has the electron in its ground state. As a result of β-decay, the atom's nucleus suddenly changes from hydrogen (^3H, Z=1) to helium (^3He, Z=2), forming a He^+ ion.

   What is the ground-state energy (in atomic units) of He^+? Using atomic units and the results given in the lectures, write down correctly normalized wavefunctions for (a) the electron in the hydrogen atom before the decay, and (b) the lowest energy eigenfunction for the electron in the He^+ ion after the decay. **[3]**

   Using the expansion postulate, it is possible to write the old (tritium) ground state in terms of the new (helium) energy eigenfunctions. Find the expansion coefficient of the helium ground state in this expansion. Hence show that if a measurement is made of the electron's energy *immediately* after the nuclear decay (i.e. before there can be any change in the electronic wave function), the probability that the He^+ ion is found in its ground state is approximately 0.702. **[7]**

   [The l=0. m=0 spherical harmonic is $Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$. You may assume this spherical harmonic is correctly normalized, and also the result that $\int_0^\infty r^p \exp(-\alpha r)\,\mathrm{d}r = \frac{p!}{\alpha^{p+1}}$.]

4. **For tutorial discussion (no marks).** In atomic units, the effective potential for radial motion of an electron in a hydrogenic atom of nuclear charge Z with angular momentum quantum number l becomes $V_{\text{eff}}(r) = \frac{-Z}{r} + \frac{l(l+1)}{2r^2}$. Unless l=0, this potential has a well-defined minimum value; what is this minimum value of V_{eff}, and at what distance r from the origin does it occur? Compare the minimum value of V_{eff} with the energy of the lowest energy eigenfunction having angular momentum l.