SOLUTIONS TO HOMEWORK 3

1) (a) $u(x) = \frac{1}{\sqrt{a}} \cos \frac{\pi x}{2a}$ **[2]**

(b) $\langle \hat{x} \rangle = \int_{-a}^{a} \frac{1}{a}\, x \cos^2 \frac{\pi x}{2a}\, dx = 0$ **[4]**

(c) $\langle \hat{P}_x \rangle = \frac{\hbar}{i} \int_{-a}^{a} \frac{\pi}{2a^2} \cdot \cos\frac{\pi x}{2a} \cdot -\sin\frac{\pi x}{2a}\, dx = 0$

(note that both integrands in (b) and (c) are odd so must $= 0$ **[4]**

2) a) Hamiltonian is just the KE operator

$\hat{H} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$ **[2]**

(b) $\hat{P}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$ so

$[\hat{H}, P_x]\psi = (\hat{H}\hat{P}_x - \hat{P}_x\hat{H})\psi = \frac{i\hbar^3}{2m}\frac{\partial^3\psi}{\partial x^3} - \frac{i\hbar^3}{2m}\frac{\partial^3\psi}{\partial x^3} = 0$ **[4]**

(c) We can have simultaneous knowledge of the momentum and the energy. Further, the momentum is conserved since an observable whose operator commutes wth the Hamiltonian is conserved. **[2]**

(d) The eigenfunctions of H are obtained from:

$\hat{H} u = E u$

$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} = E u$

$\Rightarrow \frac{d^2u}{dx^2} + k^2 u = 0 \qquad k^2 = 2mE/\hbar^2$

$\Rightarrow u = e^{\pm ikx}$

In fact these are also momentum eigenfunctions since:

$\hat{P}_x e^{ikx} = \frac{\hbar}{i}\frac{\partial}{\partial x} e^{ikx} = \hbar k e^{ikx}$

so $e^{\pm ikx}$ are eigenfunctions of momentum, with eigenvalues $\pm\hbar k$ **[4]**

e)

$[\hat{H},\hat{P}]\psi = (\hat{H}\hat{P} - \hat{P}\hat{H})\psi$

$= \left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\right]\psi$

$= \frac{\hbar}{i}\left[V(x)\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}(V(x)\psi)\right]$

$= \frac{\hbar}{i}\psi\frac{\partial V}{\partial x}$

Hence, $[\hat{H},\hat{P}] \neq 0$ unless V is a constant **[2]**

(f) If V is a constant, then no forces act on the particle $(F = -\partial V/\partial x)$ so the momentum is conserved, as there is no acceleration. **[2]**

3) $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

$\Rightarrow [\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2, \hat{L}_z]$ since $[\hat{L}_z^2, \hat{L}_z] = 0$

Now, $[\hat{L}_x^2, \hat{L}_z] = \hat{L}_x[\hat{L}_x, \hat{L}_z] + [\hat{L}_x, \hat{L}_z]\hat{L}_x$ (see notes)

Ie $[\hat{L}_x^2, L_z] = -i\hbar(\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x)$

Similarly we can show that

$$[\hat{L}_y^2, L_z] = i\hbar(\hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y) = -[\hat{L}_x^2, \hat{L}_z]$$

hence, $[\hat{L}^2, \hat{L}_z] = 0$ **[6]**

(b) Consider $\hat{L}_-\hat{L}_+ = (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y)$

$= \hat{L}_x^2 + \hat{L}_y^2 + i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)$

$= \hat{L}^2 - \hat{L}_z^2 + i[\hat{L}_x, \hat{L}_y]$

$\Rightarrow \hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 + i(i\hbar\hat{L}_z)$

$\Rightarrow \hat{L}^2 = \hat{L}_-\hat{L}_+ + \hbar\hat{L}_z + \hat{L}_z^2$ **[6]**

3) (a) $\int \phi_n^* \psi \, dx = \int \phi_n^* \sum_m c_m \phi_m$

$$= \sum_m c_m \int \phi_n^* \phi_m \, dx$$

since $\int \phi_n^* \phi_m \, dx = 0$ if $m \neq n$ (and =1 if $m = n$)

$$\sum_m c_m \int \phi_n^* \phi_m \, dx = c_n$$ **[4]**

(b) It is $c_m^* c_m = |c_m|^2$ **[2]**

(c) The idea is similar to (a) and (b) except we have angles instead of ‘x’ and we have two quantum numbers ℓm instead of n.

$$\psi(\theta,\phi) = \sqrt{\frac{15}{14\pi}} \cos 2\theta \sin\phi = \sum_{\ell m} c_{\ell m} Y_{\ell m}(\theta,\phi)$$

What we need to calculate are the $c_{\ell m}$.

Now if we measure the eigenvalue $\ell(\ell+1)\hbar^2 = 2\hbar^2$ then we know that $\ell = 1$.

fr all terms in the expansion $\sum_{\ell m} c_{\ell m} Y_{\ell m}$.

Hence $m = -1, 0, 1$.

So we can have $Y_{10}(\theta,\phi)$, $Y_{11}(\theta,\phi)$, $Y_{1-1}(\theta\phi)$ in our expansion. We know:

$c_{\ell m} = \int Y_{\ell m}^*(\theta,\phi)\,\psi(\theta,\phi) \sin\theta \, d\theta \, d\phi$

and we want c_{11}, c_{10}, c_{1-1}.

$$c_{10} = \int Y_{10}^* \psi(\theta,\phi) \sin\theta \, d\theta \, d\phi =$$

$$= \int_0^\pi \sqrt{\frac{3}{4\pi}} \cos\theta \sqrt{\frac{15}{14\pi}} \cos 2\theta \sin\theta \, d\theta \int_0^{2\pi} \sin\phi \, d\phi = 0$$

since $\int_0^{2\pi} \sin\phi \, d\phi = 0$

$c_{11} = \int Y_{11}^* \psi(\theta,\phi) \sin\theta \, d\theta \, d\phi$

$$c_{11} = -\left(\frac{45}{112\pi^2}\right)^{1/2} \underbrace{\int_0^\pi \cos 2\theta \sin^2\theta \, d\theta}_{\pi/4} \underbrace{\int_0^{2\pi} \sin\phi \, e^{i\phi} \, d\phi}_{\pi/i} = c_{1-1}$$

Probability $= |c_{11}|^2 + |c_{1-1}|^2 = 45/896\,\pi^2$ **[6]**