PHAS2222 QUANTUM PHYSICS
Problem Sheet 3 (P2222.3)
Answers to be handed in by 5pm on Friday 23 November 2007

1. From the definition of a Hermitian operator $\hat{O}$, show that the quantity $\int_{-\infty}^{\infty} \psi^*[\hat{O}\psi]\mathrm{d}x$ is real for *any* function $\psi(x)$ vanishing at infinity (not necessarily an eigenfunction). What is the physical interpretation of such an integral in quantum mechanics? **[3]**

2. Suppose that two functions ϕ_1 and ϕ_2 are both normalized eigenfunctions of the same linear, Hermitian operator $\hat{O}$ with the *same* eigenvalue λ, but are not orthogonal to one another: $\int \phi_1^*\phi_2 \mathrm{d}x \neq 0$. Show that a pair of orthonormal eigenfunctions can be constructed from them as follows.
(a) Using the definition of a linear operator, show that a new function $\tilde{\phi}_2 = \phi_2 + c\phi_1$ (where *c* is an arbitrary constant) is also an eigenfunction of $\hat{O}$ with eigenvalue λ. **[2]**
(b) Show that $\tilde{\phi}_2$ can be made orthogonal to ϕ_1, i.e. $\int \phi_1^*\tilde{\phi}_2 \mathrm{d}x = 0$, provided we choose the coefficient *c* as $c = -\int \phi_1^*\phi_2 \mathrm{d}x$. **[4]**
(c) Show that the functions ϕ_1 and $\tilde{\phi}_2$ can now be made orthonormal if $\tilde{\phi}_2$ is multiplied by the constant $\left[1-|c|^2\right]^{-1/2}$, where the *c* takes the same value as found in part (b). **[4]**
[All the integrals in this question go over all of space.]

3. A particle of mass *m* moving in a simple-harmonic oscillator potential with angular frequency ω_0 has the following normalized wavefunction at time $t = 0$:

$$\Psi(x, t=0) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{m\omega_0}{\hbar}}x\right)\exp\left(-\frac{m\omega_0 x^2}{2\hbar}\right).$$

(a) Use the expansion postulate to write this wavefunction in terms of the energy eigenfunctions $\psi_n(x)$. In other words, find the coefficients a_n in the expansion $\Psi(x, t=0) = \sum_n a_n\psi_n(x)$. **[3]**

*[**HINT** – it is much easier to do this by inspection using the results below than by using the integral formula derived in the lectures. You should find that only two of the a_n are non-zero.]*

(b) Hence write down the wavefunction $\Psi(x,t)$ at subsequent times *t*. **[4]**

(c) Find and sketch the probability distributions for the particle at (i) $t = 0$; (ii) $t = \pi/\omega_0$; (iii) $t = 2\pi/\omega_0$. **[5]**

[The two lowest normalized energy eigenfunctions are:

$$\psi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega_0 x^2}{2\hbar}\right);\ \psi_1(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\sqrt{2}\sqrt{\frac{m\omega_0}{\hbar}}x\exp\left(-\frac{m\omega_0 x^2}{2\hbar}\right).$$

The corresponding energy eigenvalues are $E_0 = \hbar\omega_0/2$ *and* $E_1 = 3\hbar\omega_0/2$ *.]*

4. ***[For tutorial discussion – no marks.]*** What happens if you apply the rule for the time-variation of the expectation value of a physical quantity *Q*, $d\langle\hat{Q}\rangle/dt = \left\langle \partial\hat{Q}/\partial t\right\rangle + \frac{1}{i\hbar}\left\langle[\hat{Q},\hat{H}]\right\rangle$, to the position operator $\hat{x}$? Assume a one-dimensional Hamiltonian of the form $\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})$.