SOLUTIONS TO HOMEWORK 2

1) Before the step, V= 0 so the TISE takes the form:

For $x < 0$: $-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} = Eu$

Or $\frac{d^2u}{dx^2} + k^2 u = 0$ with $k^2 = \frac{2mE}{\hbar^2}$

This has solutions e^{+ikx}, e^{-ikx}.

The e^{+ikx} part represents particles moving to the right (incident from the left).

The Re^{-ikx} represents particles reflected by the step. **[5]**

(a) For $x \geq 0$, the TISE is :

$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + (V_0 - E)u = 0$ or

$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} - p^2 u = 0$ where $p^2 = \frac{2m}{\hbar^2}(V_0 - E)$

This has solutions $e^{\pm px}$. For $x \to +\infty$ the e^{+px} solution diverges (we take $p>0$) so we must set $Q=0$ and hence $u(x) = Te^{-px}$.

NB for $E > V_0$, the equivalent transmitted wave is $u(x) = Te^{+ipx}$. **[5]**

(b) Match $u(x)$ at $x=0$.

$e^{+ikx} + Re^{-ikx} = Te^{-px}$ for $x=0$

$\Rightarrow 1 + R = T$ ①

Match $u'(x)$ at $x=0$,

$\Rightarrow ik - ikR = -pT$ ②

Solving these simultaneous equations gives:

$T = 2ik/(ik-p) = \frac{2k}{k+ip}$ ③

and $R = (k-ip)/(k+ip)$ ④ **[5]**

(c) The current (flux) density is:

$j(x) = \hbar/2im\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]$

Substituting $u(x) = e^{ikx} + Re^{-ikx}$ in the expression above, we get:

$j(x) = \frac{\hbar k}{m}\left[1 - |R|^2\right]$

From ④ we see that $|R|^2 = RR^* = 1$ so $j = 0$. We have total reflection.

For $x > 0$, if we substitute $u = Te^{-px}$ in the expression for the current, we easily see that the answer is zero. In fact, this must be the case since u is real so $\psi^*\frac{\partial\psi}{\partial x} = \psi\frac{\partial\psi^*}{\partial x}$. **[5]**

2) (a) Before the barrier, $u(x) = e^{ikx} + Re^{-ikx}$

after the barrier, $u(x) = Te^{ikx}$ **[5]**

(b) Before the barrier, $j(x) = \frac{\hbar k}{m}[1 - |R|^2]$

while after the barrier, $j(x) = \frac{\hbar k}{m}|T|^2$

hence, conservation of flux implies that $1 - |R|^2 = |T|^2$.

We can write $R = [1 + \alpha/\beta]^{-1/2}$ and $T = [1 + \beta/\alpha]^{-1/2}$

So, $|R|^2 + |T|^2 = R^2 + T^2 = \beta/\alpha+\beta + \frac{\alpha}{\beta+\alpha} = 1$

So flux is conserved. **[5]**

(d) There is perfect transmission if $T = 1$ ie if $\sin^2 ka = 0$.

Hence $ka = 2\pi a/\lambda = 0, \pi, 2\pi \ldots n\pi$.

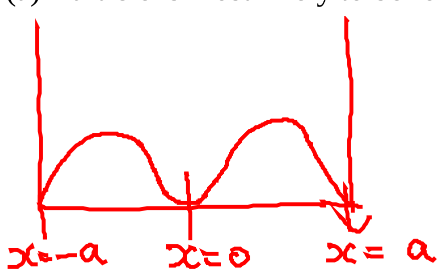
So $a = n\lambda/2$. **[5]**

3) For the second lowest energy state (first excited state) we have

$u(x) = \frac{1}{\sqrt{a}} \sin\frac{\pi x}{a}$. The probability distribution is

$P(x) = |U|^2 = \frac{1}{a}\sin^2\frac{\pi x}{a}$

(a) Particle is most likely to be found at maxima of $P(x)$.



$dP(x)/dx = 0$ for $x = 0, \pm a/2$.

$x = \pm a/2$ are maxima. **[5]**

(b) We want:

$$\int_0^{a/2} P(x)\,dx = \frac{1}{a}\int_0^{a/2} \sin^2\frac{\pi x}{a}\,dx$$

$$= \frac{1}{a}\int \frac{1}{2}\left(1 - \cos\frac{2\pi}{a}x\right)dx$$

$$= \frac{1}{2a}\left[x - \frac{a}{2\pi}\sin\frac{2\pi x}{a}\right]_0^{a/2} = 1/4$$ **[5]**

(c) $\bar{x} = \langle x \rangle = \frac{1}{a}\int_{-a}^{a} \sin\frac{\pi x}{a}\; x \sin\frac{\pi x}{a}\,dx$ (an odd function so…)

$\langle x \rangle = 0$. **[5]**