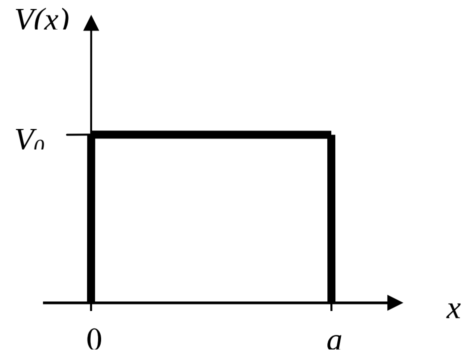
PHAS2222 QUANTUM PHYSICS
Problem Sheet 2 (P2222.2)
Answers to be handed in by 5pm on Friday 2 November 2007



1. Consider a beam of particles in one dimension, of mass m and energy E, incident from the left on a rectangular potential barrier of height V_0 located between x=0 and x=a, as shown in the figure. Suppose that the incoming beam is normalized to one particle per unit length and that the energy is greater than the barrier height (i.e. is such that $E > V_0$, in contrast to the case considered in the lectures).
(a) Explain why the solutions for the time-independent Schrődinger equation to the left and right of the barrier can be taken as
$$\psi(x) = \exp(ikx) + B\exp(-ikx) \quad \text{(left of barrier, } x \le 0)$$
$$\psi(x) = F\exp(ikx) \quad \text{(right of barrier, } x \ge a)$$
(where B and F are arbitrary constants), and state the value of k. Write down an appropriate solution in the barrier region $0 < x < a$, in terms of the variable $k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}$. **[5]**
[HINT – your solution in the barrier region should contain two arbitrary constants.]
(b) Write down four conditions on the four unknown constants (B, F and your two new constants for the solution in region II) coming from the continuity of ψ and its derivative at x=-a and x=+a. **[4]**
(c) These conditions can be solved to find $F = \frac{4kk'\exp(-ika)}{(k+k')^2\exp(-ik'a)-(k-k')^2\exp(ik'a)}$. Find an expression for the transmission probability T for particles through the well into region III. **[4]**
(d) Using a computer package of your choice (e.g. Mathematica, or Excel) plot the transmission probability against energy E for the case $V_0 = 1.0$ and $a = 1.0$, using units in which $m = \hbar = 1$, over the range E=1.0 to E=20.0. Attach a printout of the graph to your solutions. **[2]**
(e) What happens to the transmission probability when $\sin(k'a) = 0$? What is the physical significance of this condition? **[5]**

2. We showed in the lectures that the lowest-energy stationary state of the time-independent Schrődinger equation for a simple harmonic oscillator of frequency ω_0 can be written as $\psi(y) = A\exp(-y^2/2)$, where $y = \left(\frac{m\omega_0}{\hbar}\right)^{1/2} x$, x is the displacement from equilibrium, m is the mass of the particle, and A is a normalization constant.
(a) Find a value of the constant A which correctly normalizes the wavefunction (i.e. ensures that $\int_{-\infty}^{\infty} |\psi|^2 dx = 1$), and hence find an expression for the probability density for the particle as a function of x. **[4]**
(b) What is the mean-squared displacement of the particle from the origin (i.e. the average value of the quantity x^2) in this state? Hence deduce also the mean of the potential energy $V(x) = \frac{1}{2} m\omega_0^2 x^2$, and compare your answer with the ground-state energy $\frac{1}{2}\hbar\omega_0$. **[6]**

[You will find the values of the following integrals useful in your answer:
$$\int_{-\infty}^{\infty} \exp(-x^2/a^2)dx = \sqrt{\pi}a; \quad \int_{-\infty}^{\infty} x^2\exp(-x^2/a^2)dx = \frac{\sqrt{\pi}a^3}{2}$$
]

3. **[For tutorial discussion – no marks.]** The *correspondence principle* states that the predictions of quantum mechanics should approach those of classical mechanics in the appropriate limit (e.g. for large energies, or very massive objects). What would the classical probability distribution look like for the position of a particle bouncing backwards and forwards between two hard walls with a fixed energy E? What do the quantum-mechanical probability distributions look like for the stationary states of an infinite square well in the limit of large n? Compare the quantum and classical results. What would the quantum distributions look like if they were averaged over a range of states with nearby energies?