MTH4101 Calculus II

Lecture Notes for Week 1
Series I

Thomas' Calculus, Section 10.1

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Sequences

A **sequence** is a list of numbers in a given *order*:

$$a_1, a_2, a_3, \ldots, a_n, \ldots .$$

Each of the a_1, a_2, etc. represents a number; these are the *terms* of the sequence. For example

$$2, 4, 6, 8, \ldots, 2n, \ldots$$

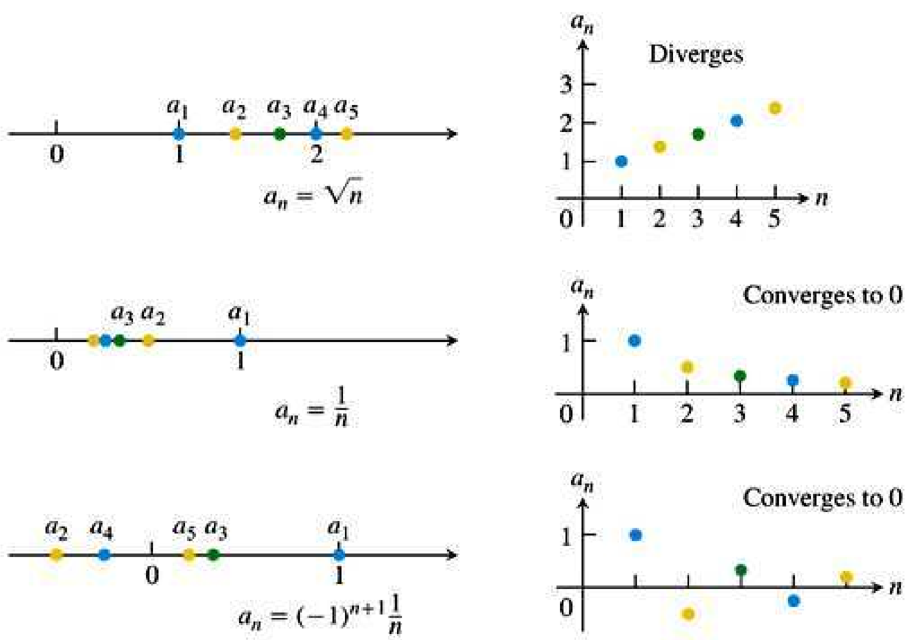
has first term $a_1 = 2$, second term $a_2 = 4$ and nth term $a_n = 2n$. The integer n is called the *index* of a_n and denotes where a_n occurs in the list.
We can consider the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ as a function that sends 1 to a_1, 2 to a_2, etc. and in general sends the positive integer n to the nth term a_n.

DEFINITION Infinite Sequence
An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

Sequences can be described by *rules* or by *listing terms*. For example,

$$a_n = \sqrt{n} \qquad \{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$
$$b_n = (-1)^{n+1}(1/n) \qquad \{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$
$$c_n = (n-1)/n \qquad \{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots\right\}$$
$$d_n = (-1)^{n+1} \qquad \{d_n\} = \left\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\right\}$$

Sequences can be illustrated graphically either as points on a real axis or as the graph of a function defining the sequence:



Consider the following sequences:

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\} \qquad \text{terms approach 0 as } n \text{ gets large}$$

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1 - \frac{1}{n}, \ldots\right\} \qquad \text{terms approach 1 as } n \text{ gets large}$$

$$\left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots\right\} \qquad \text{terms get larger than any number as } n \text{ increases}$$

$$\left\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\right\} \qquad \text{terms oscillate between 1 and } -1\text{, never converging to a single value}$$

This leads to the definition of **convergence**, **divergence** and a **limit**:

DEFINITIONS Converges, Diverges, Limit

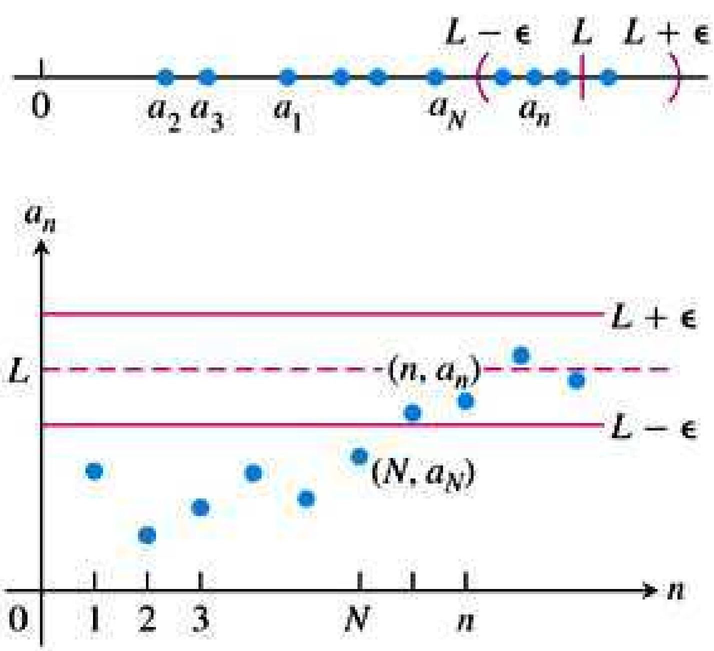
The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence

The concept of a limit is illustrated in the following figure:



Here $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$.

We will now consider two examples of the application of the definitions.

Example:
We want to prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Let $\epsilon > 0$ be given. We need to find an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon\,.$$

This condition will be satisfied provided $1/n < \epsilon$, which means $n > 1/\epsilon$. Therefore if N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. Hence $\lim_{n\to\infty}(1/n) = 0$. For example, suppose we take $\epsilon = 0.01$ then the condition is just $n > 100$.

Example:
We want to prove that the sequence

$$\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\} \quad \text{diverges.}$$

proof by contradiction: Assume that the sequence converges to some number L. Choose $\epsilon = \frac{1}{2}$ in the definition of the limit and so all terms a_n of the sequence with n larger than some N must lie within $\epsilon = \frac{1}{2}$ of L:

$$n > N \quad \Rightarrow \quad |a_n - L| < \frac{1}{2}\,.$$

Since 1 is in every other term of the sequence, 1 must lie within ϵ of L. Hence

$$|1 - L| = |L - 1| < \frac{1}{2} \quad \text{or} \quad \frac{1}{2} < L < \frac{3}{2}\,.$$

Then -1 is also in every other term and so we must have

$$|L - (-1)| < \frac{1}{2} \quad \text{or} \quad -\frac{3}{2} < L < -\frac{1}{2}\,.$$

However, this is a *contradiction*: Both conditions cannot be satisfied simultaneously. Therefore no such limit exists and so the sequence diverges.
There is a second type of divergence:

DEFINITION Diverges to Infinity

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty\,.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty\,.$$

Example:

$$\lim_{n\to\infty} \sqrt{n} = \infty \quad \text{(proof?)}$$

note: The sequence $\{1, -2, 3, -4, 5, \ldots\}$ *also* diverges, but not to ∞ or $-\infty$.

Sequences are functions with domain restricted to $n \in \mathbb{N}$, hence:

THEOREM 1
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

We can use these rules to help us to calculate limits of sequences.

Example:
Find $\lim\limits_{n\to\infty} \dfrac{n-1}{n}$.

$$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1\,.$$

Example:
Find $\lim\limits_{n\to\infty} \dfrac{5}{n^2}$.

$$\lim_{n\to\infty} \frac{5}{n^2} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0\,.$$

The **Sandwich Theorem for Sequences** provides another method for finding the limits of sequences:

THEOREM 2 The Sandwich Theorem for Sequences
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Note that if $|b_n| \le c_n$ and $c_n \to 0$ as $n \to \infty$, then $b_n \to 0$ also, because $-c_n \le b_n \le c_n$.

Example:
Find $\lim\limits_{n\to\infty} \dfrac{\sin n}{n}$.
By the properties of the sine function we have $-1 \le \sin n \le 1$ for all n. Therefore

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

because of $\lim_{n\to\infty}(-1/n) = \lim_{n\to\infty}(1/n) = 0$ and the use of the Sandwich Theorem.

Example:

Find $\lim_{n\to\infty} \dfrac{1}{2^n}$.

$1/2^n$ must always lie between 0 and $1/n$ (e.g. $\frac{1}{2} < 1, \frac{1}{4} < \frac{1}{2}, \frac{1}{8} < \frac{1}{3}, \frac{1}{16} < \frac{1}{4}, \ldots$). Therefore

$$0 \le \frac{1}{2^n} \le \frac{1}{n} \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{1}{2^n} = 0\,.$$

The limits of sequences can also be determined by using the following theorem:

THEOREM 3 The Continuous Function Theorem for Sequences
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Example:

Determine the limit of the sequence $\left\{2^{1/n}\right\}$ as $n \to \infty$.

We already know that the sequence $\left\{\frac{1}{n}\right\}$ converges to 0 as $n \to \infty$. Let $a_n = 1/n$, $f(x) = 2^x$ and $L = 0$ in the continuous function theorem for sequences. This gives

$$2^{1/n} = f(1/n) \to f(L) = 2^0 = 1 \quad \text{as} \quad n \to \infty\,.$$

Hence the sequence $\left\{2^{1/n}\right\}$ converges to 1.

We can also make use of l'Hôpital's Rule to find the limits of sequences. To do so we need to make use of the following theorem:

THEOREM 4
Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n = L\,.$$

Example:

Show that $\lim_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = 0$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} &= \lim_{n\to\infty} \frac{1/n}{(1/2)n^{-1/2}} \\
&\quad \text{(using l'Hôpital's Rule \textit{by treating n as a continuous real variable})} \\
&= \lim_{n\to\infty} 2 \cdot \frac{n^{1/2}}{n} = 2 \lim_{n\to\infty} \frac{1}{n^{1/2}} = 0\,.
\end{aligned}$$

Example:

Does the sequence whose nth term is $a_n = ((n+1)/(n-1))^n$ converge? If so, find $\lim_{n\to\infty} a_n$.

If we just take the straightforward limit we get the indeterminate form 1^∞. Typically with questions of this type we take the logarithm. This gives:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\,\ln\left(\frac{n+1}{n-1}\right).$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n\ln\left(\frac{n+1}{n-1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \\
&= \lim_{n\to\infty} \frac{\ln(n+1)-\ln(n-1)}{1/n} \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \quad \text{(using l'Hôpital's Rule)} \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2\,.
\end{aligned}$$

Let $b_n = \ln a_n$ Then $\lim_{n\to\infty} b_n = 2$ and since $f(x) = e^x$ is continuous we have by the continuous function theorem for sequences

$$a_n = e^{\ln a_n} = e^{b_n} \to e^2 \quad \text{as} \quad n\to\infty\,.$$

Therefore the sequence $\{a_n\}$ converges to e^2.

MTH4101 Calculus II

Lecture notes for Week 2
Series I and II

Thomas' Calculus, Sections 10.1 to 10.3

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The following Theorem summarizes some common results for the limits of sequences:

THEOREM 5
The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

The first result can be proved using l'Hôpital's rule. The second and third results can be proved using logarithms and applying the previous theorems. Proofs of the remaining results are given in Appendix 5 of Thomas' Calculus.

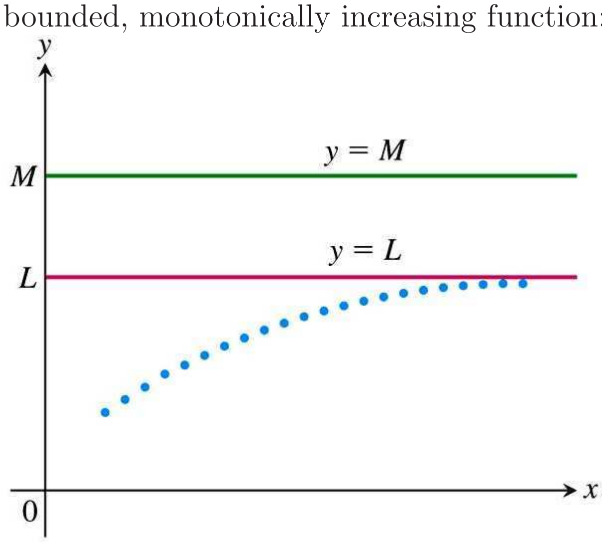
Example:
Show that $\lim_{n\to\infty} \sqrt[n]{n^2} = 1$.

$$\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} \left(n^{1/n}\right)^2 = (1)^2 = 1\,.$$

For *bounded, monotonic* sequences there is the following theorem:

THEOREM 6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

For example, look at a bounded, monotonically increasing function:



Example:

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1\,.$$

Infinite series

An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots .$$

DEFINITIONS **Infinite Series, *n*th Term, Partial Sum, Converges, Sum**

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

A *geometric series* has the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

where a and r are fixed real numbers and $a \neq 0$. The quantity r is called the *ratio* of the geometric series and can be positive or negative.
In the special case where $r = 1$ the nth partial sum is

$$s_n = a + a \cdot 1 + a \cdot 1^2 + \cdots + a \cdot 1^{n-1} = na$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$ depending on the sign of a. If $r = -1$ the series diverges because either $s_n = a$ or $s_n = 0$ depending on the value of n.

Now consider the case of a geometric series with $|r| \neq 1$. We have

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ s_n - rs_n &= a - ar^n \quad \text{or} \quad s_n(1-r) = a(1-r^n) \\ \Rightarrow s_n &= \frac{a(1-r^n)}{1-r} \quad (r \neq 1)\,. \end{aligned}$$

Therefore, if $|r| < 1$ then $r^n \to 0$ as $n \to \infty$ and hence $s_n \to a/(1-r)$. If $|r| > 1$ then $|r^n| \to \infty$ and the series diverges. So we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for} \quad |r| < 1$$

and the geometric series converges.
For example,

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{(1/9)}{1-(1/3)} = \frac{1}{6} \qquad (a = 1/9,\ r = 1/3)$$

and

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \frac{5}{1+(1/4)} = 4 \qquad (a = 5,\ r = -1/4)\,.$$

Example:
Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\,.$$

Note that we can use partial fractions to write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\,.$$

Hence the sum of the first k terms is

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and so the kth partial sum is

$$\begin{aligned} s_k &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{k} + \frac{1}{k}\right) - \frac{1}{k+1} \end{aligned}$$

Hence $s_k \to 1$ as $k \to \infty$ and so the series converges giving

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1\,.$$

Suppose the series $\sum_{n=1}^{\infty} a_n$ converges to a sum S and the nth partial sum of the series is $s_n = a_1 + a_2 + \cdots + a_n$. When n is large, both s_n and s_{n-1} are close to S and therefore their difference a_n is close to zero. Using the Difference Rule for sequences we have

$$a_n = s_n - s_{n-1} \quad \longrightarrow \quad S - S = 0 \quad \text{as} \quad n \to \infty\,.$$

Hence:

THEOREM 7

If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

This, in turn, leads to

The nth-Term Test for Divergence

$\displaystyle\sum_{n=1}^{\infty} a_n$ diverges if $\displaystyle\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Example:

$$\sum_{n=1}^{\infty} n^2 \quad \text{diverges because} \quad n^2 \to \infty$$

$$\sum_{n=1}^{\infty} \frac{n+1}{n} \quad \text{diverges because} \quad \frac{n+1}{n} \to 1$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \quad \text{diverges because} \quad \lim_{n\to\infty} (-1)^{n+1} \quad \text{does not exist}$$

$$\sum_{n=1}^{\infty} \frac{-n}{2n+5} \quad \text{diverges because} \quad \lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0\,.$$

Note that the *converse of the above theorem is false*: If $a_n \to 0$ this does **not** imply that the series $\sum_{n=1}^{\infty} a_n$ converges.

Example:
Consider the unusual case of a series where $a_n \to 0$ but the series itself diverges:

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n} + \cdots$$

where there are two terms of $1/2$, four terms of $1/4$, ..., 2^n terms of $1/2^n$, etc. In this case each grouping of terms adds up to 1 so the partial sums must increase without bound and so the series diverges, even though the terms of the series form a sequence that converges to 0.

If we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series:

THEOREM 8
If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (Any number k).

Example:
Find $\sum_{n=1}^{\infty}(3^{n-1} - 1)/6^{n-1}$.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} - \sum_{n=1}^{\infty}\frac{1}{6^{n-1}} \\ &= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} \quad \text{(two geometric series)} \\ &= 2 - \frac{6}{5} = \frac{4}{5}.\end{aligned}$$

We can add a finite number of terms or delete a finite number of terms without altering the convergence or divergence of a series but if the series is convergent this will usually alter the sum. Consider the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n .$$

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$. Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
Note that re-indexing a series (e.g. changing the starting value of the index) does not alter its convergence, provided the order of the terms is preserved.
For example, raise the starting value of the index h units:

$$n = k - h : \quad \sum_{n=1}^{\infty} a_n = \sum_{k=1+h}^{\infty} a_{k-h} = a_1 + a_2 + a_3 + \cdots .$$

Lower the starting value of the index h units:

$$n = k + h : \quad \sum_{n=1}^{\infty} a_n = \sum_{k=1-h}^{\infty} a_{k+h} = a_1 + a_2 + a_3 + \cdots .$$

The Integral Test

For a given series $\sum a_n$ we want to know: (1) Does it converge? (2) If it converges, what is its sum?
A corollary of the Monotonic Sequence Theorem is that the series $\sum_{n=1}^{\infty} a_n$ of *non-negative* terms converges *if and only if* (why?) its partial sums are bounded above.

Example:
Consider the **harmonic series**:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots .$$

This series is actually divergent even though the nth term $1/n \to 0$ as $n \to \infty$, cf. the *n-th term test* seen before. However, the series has no upper bound for its partial sums. We can see this by writing the series as

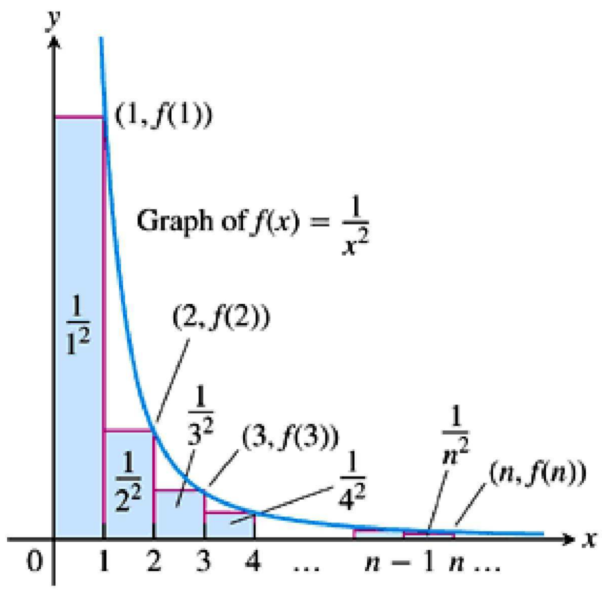
$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots .$$

Now $\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$, $\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{8}{16} = \frac{1}{2}$ and so on. Therefore the sum of the 2^n terms ending with $1/2^{n+1}$ is $> 2^n/2^{n+1} = 1/2$. Therefore the sequence of partial sums is not bounded from above, and so the harmonic series diverges.

Now consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Does it converge or diverge? To answer this question we will consider a new approach involving the use of integration. What we need to do is to compare the series $\sum_{n=1}^{\infty} 1/n^2$ with the integral $\int_1^{\infty} 1/x^2 \, \mathrm{d}x$.



$$\begin{aligned}
s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\
&= f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 \cdots + f(n) \cdot 1 \\
&< f(1) + \int_1^n \frac{1}{x^2} \, \mathrm{d}x \quad \text{lower sum} \\
&< 1 + \int_1^{\infty} \frac{1}{x^2} \, \mathrm{d}x
\end{aligned}$$

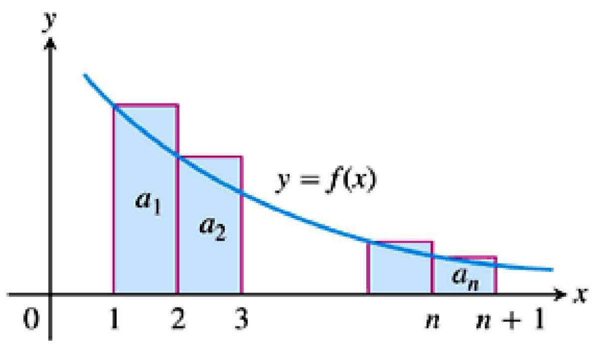
Therefore

$$s_n < 1 + \int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x = 1 + \left[-\frac{1}{x}\right]_1^{\infty} = 2\,.$$

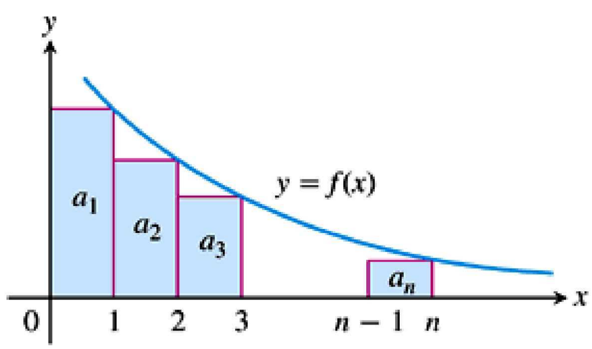
Thus $s_n < 2$ for all n, the partial sums are bounded from above (by 2) and therefore (why?) the series converges. Note that the series and the integral need not have the same value in the convergent case.
The approach we have just taken leads us to

THEOREM 9 The Integral Test
Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

We will consider the proof for the case $N = 1$ and we start with the asumption that f is a decreasing function with $f(n) = a_n$ for every n.



(a)



(b)

In part (a) of the above figure, the areas of the rectangles $a_1, a_2, \ldots, a_n$ enclose more area than that under the curve $y = f(x)$ between $x = 1$ and $x = n+1$. Therefore we can write

$$\int_1^{n+1} f(x)\,\mathrm{d}x \leq a_1 + a_2 + \cdots + a_n\,.$$

Now consider the rectangles as shown in part (b) above. If we ignore the first rectangle we can write

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,\mathrm{d}x\,.$$

Adding the area a_1 to each side gives

$$a_1 + a_2 + a_3 + \cdots + a_n \le a_1 + \int_1^n f(x)\,\mathrm{d}x$$

Combining the two inequalities gives

$$\int_1^{n+1} f(x)\,\mathrm{d}x \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,\mathrm{d}x\,.$$

These inequalities will hold as $n \to \infty$.
Therefore, if $\int_1^n f(x)\,\mathrm{d}x$ is finite, the right-hand part of the inequality shows that $\sum a_n$ is also finite. Similarly, if $\int_1^{n+1} f(x)\,\mathrm{d}x$ is infinite, then $\sum a_n$ is infinite by the left-hand part of the inequality.

The Integral Test can be used to show that the *p-series* $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \le 1$.[1]

Example:
Show that the series $\sum_{n=1}^{\infty} 1/(n^2+1)$ converges by the integral test.
The function $f(x) = 1/(x^2+1)$ is positive, continuous and decreasing for $x \ge 1$. Also

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x &= \lim_{b\to\infty} [\arctan x]_1^b = \lim_{b\to\infty} [\arctan b - \arctan 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

and so the series converges (but we do not know its sum).

[1] See the Thomas' Calculus Section 10.3, p.555 for a proof.

Queen Mary
University of London

MTH4101 Calculus II

Lecture notes for Week 3
Series II and III

Thomas' Calculus, Sections 10.5 to 10.8

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The Ratio Test

THEOREM 12—The Ratio Test Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

A proof of the above results is given in the textbook.
The two series we looked at in the last section are good examples of cases where $\rho = 1$ and the test is inconclusive:

$$\sum \frac{1}{n} \quad : \qquad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1 \quad (n \to \infty)$$

$$\sum \frac{1}{n^2} \quad : \qquad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1^2 = 1 \quad (n \to \infty)\,.$$

In each case $\rho = 1$ (i.e. the test is inconclusive) and yet we know that $\sum 1/n$ diverges whereas $\sum 1/n^2$ converges.

Example:
Use the Ratio Test to investigate the convergence of the following series:

$$\text{(a)} \ \sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}\,, \quad \text{(b)} \ \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}\,, \quad \text{(c)} \ \sum_{n=1}^{\infty} \frac{n!}{n^n}\,.$$

(a)

$$a_n = \frac{2^n + 5}{3^n}; \qquad a_{n+1} = \frac{2^{n+1} + 5}{3^{n+1}};$$

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3}\left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right)$$

$$\to \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3} < 1 \text{ as } n \to \infty \text{ and the series converges.}$$

(b)

$$a_n = \frac{(2n)!}{(n!)^2}; \qquad a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^2};$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!\,n!}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$= \frac{4n+2}{n+1} = \frac{4 + 2/n}{1 + 1/n} \to 4 > 1 \text{ and the series diverges.}$$

(c)

$$a_n = \frac{n!}{n^n}; \qquad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}};$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\,n^n}{(n+1)^{n+1}n!} = \frac{(n+1)n^n}{(n+1)^n(n+1)}$$

$$= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n \to \frac{1}{e} < 1$$

and the series converges.

As we can see, the Ratio Test is often useful when the terms of a series contain factorials involving n or expressions raised to the power involving n.

Power Series

A **power series** is like an "infinite polynomial", i.e., it is an infinite series in powers of some variable, usually x:

DEFINITIONS Power Series, Center, Coefficients

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \qquad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

If they converge, such series can be *added, subtracted, multiplied, differentiated and integrated* to give new power series.

Example:
Consider the case where the coefficients in (1) in the definition above are all unity.:

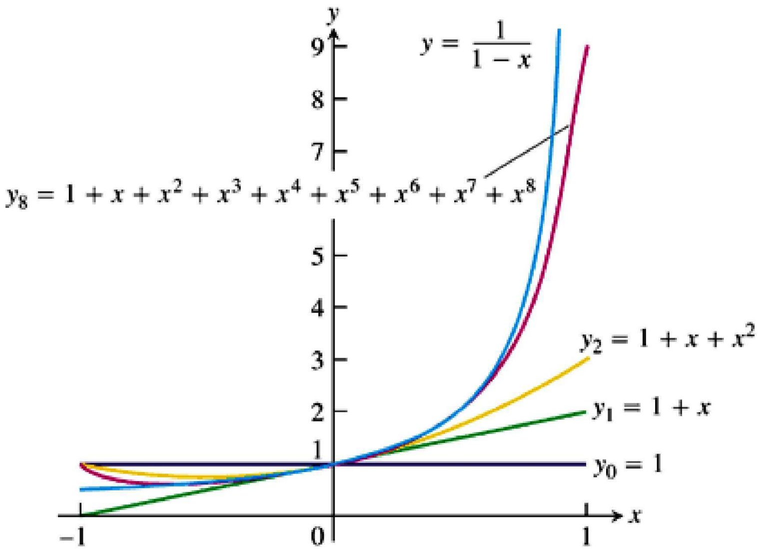
$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots .$$

This is just a geometric series with first term 1 and ratio x $(a = 1, r = x)$. We know from the properties of geometric series that it converges to $1/(1-x)$ for $|x| < 1$. Hence

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots , \qquad -1 < x < 1 .$$

We can think of the right-hand side of this equation as a sequence of partial sums which are polynomials $P_n(x)$ that *approximate* the function on the left:

$$
\begin{aligned}
f(x) = \frac{1}{1-x}; \quad P_0(x) &= 1 = y_0 && \text{(horizontal line)} \\
P_1(x) &= 1 + x = y_1 && \text{(straight line, slope 1)} \\
P_2(x) &= 1 + x + x^2 = y_2 && \text{(quadratic curve [parabola])} \\
&\vdots \\
&\text{etc.}
\end{aligned}
$$



Example:

Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots .$$

This matches the form of (2) in the former definition with $a = 2$, $c_n = (-1/2)^n$. It is a geometric series with the first term 1 and ratio $r = -(x-2)/2$. The series converges for $|(x-2)/2| < 1$ or $0 < x < 4$. The sum is

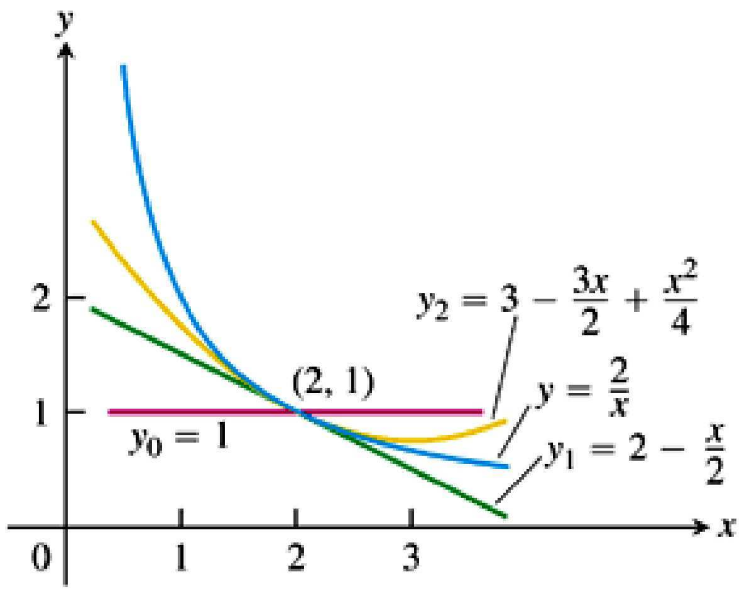
$$\frac{1}{1-r} = \frac{1}{1+(x-2)/2} = \frac{2}{x}.$$

Hence

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^2 (x-2)^n + \cdots, \quad 0 < x < 4.$$

Again we can consider the series as a sequence of partial sums which are polynomials $P_n(x)$ that approximate $2/x$:

$$f(x) = \frac{2}{x}; \quad \begin{aligned} P_0(x) &= 1 = y_0 \\ P_1(x) &= 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{2} = y_1 \\ P_2(x) &= 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4} = y_2 \\ &\vdots \\ &\text{etc.} \end{aligned}$$



A series $\sum a_n$ **converges absolutely** if the corresponding series of absolute values, $\sum |a_n|$, converges. Most importantly, it can be shown that if a series converges *absolutely*, then it *converges*.[1] This enables us to apply the ratio test and the integral test, which only test the convergence of series of positive terms.

A series that converges but does not converge absolutely **converges conditionally**.

THEOREM 18 The Convergence Theorem for Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

[1] See Section 10.6 for a short, clever proof.

COROLLARY TO THEOREM 18

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

Here R is called the **radius of convergence** and the interval of radius R centred at $x = a$ is called the **interval of convergence**.

Example:
Find the values of x for which the series

$$\sum_{n=0}^{\infty}(2x)^n$$

converges absolutely, specifying both the radius and interval of convergence.

This is a geometric series with first term $a = 1$ and ratio $r = 2x$. It **converges absolutely** for $|r| < 1$, that is, $|2x| < 1$ or $-1/2 < x < 1/2$, and diverges elsewhere. Hence, the **radius of convergence** is $R = 1/2$ and the **interval of convergences** $-1/2 < x < 1/2$.

When studying the convergence of power series such as these, alternating series frequently arise. Here we can make use of an additional test. The **Alternating Series Test** (or Leibniz's Test) states that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + u_5 - \cdots$$

converges if all three of the following conditions hold:

1. The u_n are all positive,
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N and
3. $u_n \to 0$ as $n \to \infty$.

Example:
The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

satisfies all of the above three requirements with $N = 1$ and hence converges. But it does *not* converge absolutely, as we have shown before, hence it converges **conditionally**.
We can test a power series for convergence using several methods:

1. Use a test such as the *ratio test* to find the interval where the series converges absolutely.
2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint using a test such as the integral test or the alternating sequences test.
3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$.

Example:
Use the ratio test to determine the convergence of

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

We have

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{2n+1}}{2n+1} \frac{2n-1}{x^{2n-1}} \right| = \frac{2n-1}{2n+1} x^2 \to x^2 .$$

Therefore the series converges absolutely for $x^2 < 1$ and diverges for $x^2 > 1$. At $x = 1$ the series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ which converges by the alternating sequences test. The series also converges at $x = -1$, as can be shown by the alternating sequences test.

Taylor and Maclaurin Series

Assume that the function $f(x)$ can be represented as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - a)^n = a_0 + a_1(x - a) + \cdots + a_n (x - a)^n + \cdots$$

which converges for $a - R < x < a + R$ with $R > 0$. Can we calculate the coefficients a_n in terms of $f(x)$?
It can be shown[2] that $f(x)$ has *derivatives of all orders* inside this interval by *differentiating the power series term by term*:

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x - a) + \cdots + na_n(x - a)^{n-1} + \cdots \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x - a) + \cdots + n(n-1)a_n(x - a)^{n-2} + \cdots \\ &\vdots \\ f^{(n)}(x) &= n!\, a_n + \text{ a sum of terms with } (x - a) \text{ as a factor.} \end{aligned}$$

Therefore

$$f'(a) = a_1 ,\ f''(a) = 1 \cdot 2a_2 ,\ f'''(a) = 1 \cdot 2 \cdot 3a_3 ,\ \ldots ,\ f^{(n)}(a) = n!\, a_n .$$

This gives us a formula for the coefficients in the power series:

$$a_n = \frac{f^{(n)}(a)}{n!} .$$

[2] This is a theorem, which can be proved. Likewise, it can be proved that $f(x)$ can be *integrated term by term*; see Thomas' Calculus, end of Section 10.7. for details.

It also suggests that *if* f has a power series representation *then* it must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

leading us to the following definition:

DEFINITIONS **Taylor Series, Maclaurin Series**

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots ,$$

the Taylor series generated by f at $x = 0$.

Example:
Find the Taylor series generated by $f(x) = 1/x$ at $x = 2$. Where, if anywhere, does the series converge to $1/x$?

$$\begin{aligned}
f(x) &= x^{-1}; \quad f(2) = 2^{-1} = \frac{1}{2} \\
f'(x) &= -x^{-2}; \quad f'(2) = -\frac{1}{2^2} \\
f''(x) &= 2!\, x^{-3}; \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3} \\
&\vdots \\
f^{(n)}(x) &= (-1)^n n!\, x^{-(n+1)}; \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}} .
\end{aligned}$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots .$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$, or $0 < x < 4$ with sum

$$S = \frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x} .$$

Related to the Taylor *series* is the Taylor *polynomial* of order n:

DEFINITION Taylor Polynomial of Order n

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

There is a similar definition for Maclaurin polynomials.

MTH4101 Calculus II

Lecture notes for Week 4
Series III and Derivatives IV

Thomas' Calculus, Sections 10.9 to 10.10 and 14.1

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Spring 2015

Example:
Find the Taylor polynomials of order 0, 2 and 4 for the function $f(x) = \cos x$ at $x = 0$.
We have

$$f(x) = \cos x\,, \quad f'(x) = -\sin x\,, \quad f''(x) = -\cos x\,, \quad f'''(x) = \sin x\,, \quad f^{(4)}(x) = \cos x$$

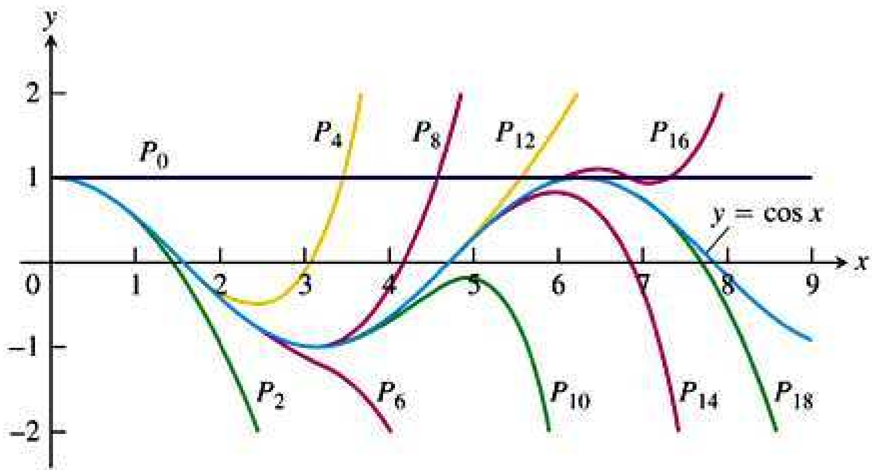
and

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1\,.$$

By using the previous definition, the first three Taylor polynomials of $f(x) = \cos x$ about $x = 0$ are

$$\begin{aligned} P_0(x) &= 1 \\ P_2(x) &= 1 - \frac{x^2}{2!} \\ P_4(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\,. \end{aligned}$$

The following figure shows how successive Taylor polynomials provide better and better approximations to the function as $n \to \infty$:



Below we give the Taylor series expansions for a variety of functions about $x = 0$ and $x = 1$. These can all be derived using the methods in this section.
Taylor series about $x = 0$:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\,. \end{aligned}$$

Taylor series about $x = 1$:

$$\begin{aligned} \ln x &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \\ \sqrt{x} &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \cdots\,. \end{aligned}$$

Convergence of Taylor Series and Error Estimates

There are still two unanswered questions about Taylor series:

1. **When** does a Taylor series **converge** to the function that generated it?
2. **How accurately** do a function's Taylor polynomials **approximate** the function on a given interval?

To answer these questions we need to make use of **Taylor's Formula**:

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \qquad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \qquad (2)$$

This formula is a special case of *Taylor's Theorem*, which in addition requires differentiability at the end points I. This theorem can in turn be understood as a generalization of the Mean Value Theorem (set $n = 0$ in the above formula).
The quantity $R_n(x)$ in Taylor's Formula is called the **remainder of order** n or the **error term** for the approximation of f by $P_n(x)$ over I. If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series *converges* to f on I and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Finally we can use the **Remainder Estimation Theorem** to provide an estimate of the error:

THEOREM 23 The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)}(t)| \le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

The usefulness of this theorem is demonstrated by the following example:

Example:

Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

We have

$$f(x) = \sin x\,, \quad f'(x) = \cos x\,, \quad f''(x) = -\sin x\,, \ldots$$

and, in general,

$$f^{(2k)}(x) = (-1)^k \sin x\,, \quad f^{(2k+1)}(x) = (-1)^k \cos x$$

Therefore, evaluating at $x = 0$ gives $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$. Hence the Taylor series for $\sin x$ at $x = 0$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x)\,.$$

Applying the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k+1}(x)| \leq 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!} \to 0 \text{ as } k \to \infty \text{ for all } x.$$

(cf. the list of sequences and their limits discussed in Week 1) Therefore $R_{2k+1}(x) \to 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x.

Applications of Power Series

Binomial series

The Taylor series generated by $f(x) = (1+x)^m$ (around $x = 0$) where m is a constant is

$$\begin{aligned} f(x) &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \\ &\quad + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k + \cdots\,. \end{aligned}$$

This is called the **binomial series**.

If $m \geq 0$ is an integer, the series stops after $(m+1)$ terms because coefficients from $k = m+1$ onwards are zero.

If m is not a positive integer the series is *infinite*. From the Ratio Test for absolute convergence it follows that this series converges absolutely for $|x| < 1$. It can also be shown that the series converges to $(1+x)^m$.[1]

We can define this series conveniently as follows:

[1] see Thomas' Calculus Section 10.10 for details

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

Note that $m \in \mathbb{R}$. In the case of $m \in \mathbb{N}$ we recover the familiar *binomial coefficients*. Note also the relation between the binomial series and the binomial formula.
In the case where $m = -1$,

$$\binom{-1}{1} = -1\,, \quad \binom{-1}{2} = 1 \quad \text{and} \quad \binom{-1}{k} = (-1)^k\,.$$

For example,

$$\begin{aligned}
\frac{x}{1+x^2} &= x(1+x^2)^{-1} \\
&= x\left(1 - x^2 + \frac{(-1)(-2)}{2!}x^4 + \frac{(-1)(-2)(-3)}{3!}x^6 + \cdots\right) \\
&= x(1 - x^2 + x^4 - x^6 + \cdots) \\
&= x - x^3 + x^5 - x^7 + \cdots
\end{aligned}$$

which is a geometric series.

Reading assignment: Work yourself through the following two examples. (cf. Examples 3 and 7 in Thomas' Calculus, Section 10.10)

Evaluation of non-elementary integrals

We can use the term-by-term integration property of power series to allow us to do non-elementary integrals.

Example:
Express $\int \sin x^2 \, \mathrm{d}x$ as a power series.
Recall that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Hence

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

and so

$$\int \sin x^2 \,\mathrm{d}x = C + \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \cdots .$$

where C is a constant of integration.

Evaluating indeterminate forms

Power series also provide an alternative to L'Hôpital's rule for evaluating indeterminate forms.

Example:
Find

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) .$$

We can write

$$\begin{aligned}\frac{1}{\sin x} - \frac{1}{x} &= \frac{x - \sin x}{x \sin x} = \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} \\ &= \frac{x^3\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right)}{x^2\left(1 - \frac{x^2}{3!} + \cdots\right)} = x\frac{\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right)}{\left(1 - \frac{x^2}{3!} + \cdots\right)} .\end{aligned}$$

Hence

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\left(x\frac{\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right)}{\left(1 - \frac{x^2}{3!} + \cdots\right)}\right) = 0 .$$

Functions of Several Variables

Reminder: What is a function?
In *Calculus 1* and in *Mathematical Structures* you have learned the following:

Definition
A **function** from a set D (domain) to a set Y (codomain) is a rule that assigns a *unique* (single) element $y \in Y$ to each element $x \in D$.

So far you have dealt with functions of a *single* variable, such as

$$f : \mathbb{R} \to \mathbb{R} \quad , \quad x \mapsto y = f(x)$$

with, for example, $f(x) = x^2$.
Functions of *several variables* are defined in complete analogy to functions of one variable in terms of uniqueness, domain, codomain, range, etc. (without involving complex numbers):

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

In the following we will focus on functions of two variables.

Examples:

$$\begin{aligned} V &= V(r,h) = \pi r^2 h \quad \text{(volume of cylinder, radius } r\text{, height } h) \\ M &= M(r,\rho) = \frac{4}{3}\pi r^3 \rho \quad \text{(mass of sphere, radius } r\text{, density } \rho) \end{aligned}$$

In the case of V the quantities r and h are the input (*independent*) variables and V is the *unique* output (*dependent*) variable.

If f is a function of two independent variables, x and y, the domain of f is a region in the x-y plane.

Example:

(Natural) domains and ranges for function of two variables

Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Entire plane	$[-1, 1]$

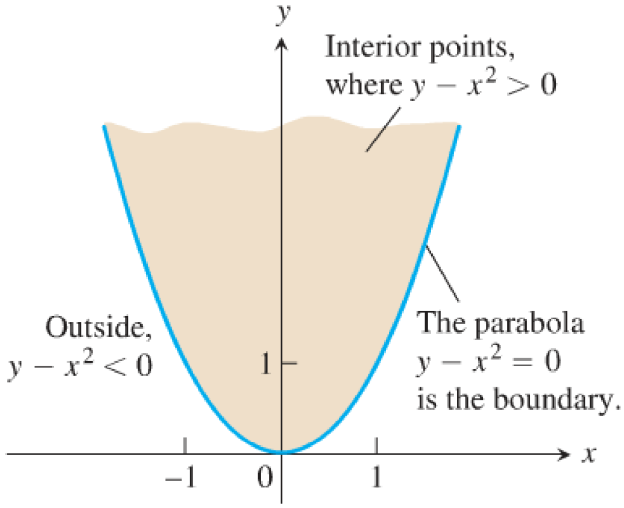
Interior points, boundary points, open and closed sets are defined in higher dimensions in analogy to dealing with intervals on the real line.[2]

Example:

Describe the domain of the function $f(x,y) = \sqrt{y - x^2}$.

Since f is defined only where $y - x^2 \geq 0$, the domain is the *closed* (the set contains all boundary points), *unbounded* (why?) region shown below (shaded). The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior.

[2] If you are not satisfied with this statement, please check out Thomas' Calculus p.749 for details.



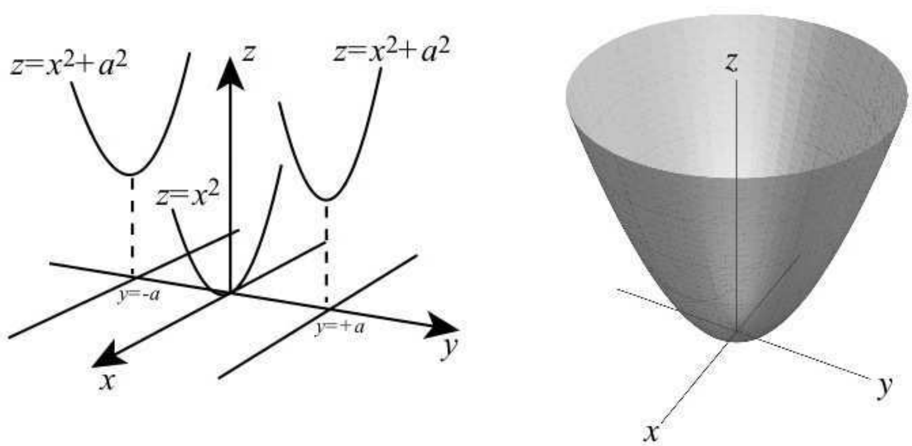
There are two ways to visualise a function $f(x, y)$:

1. Sketch the **graph**, or **surface** $z = f(x, y)$ in space.
2. Draw and label **level curves** in the domain on which f has a constant value.

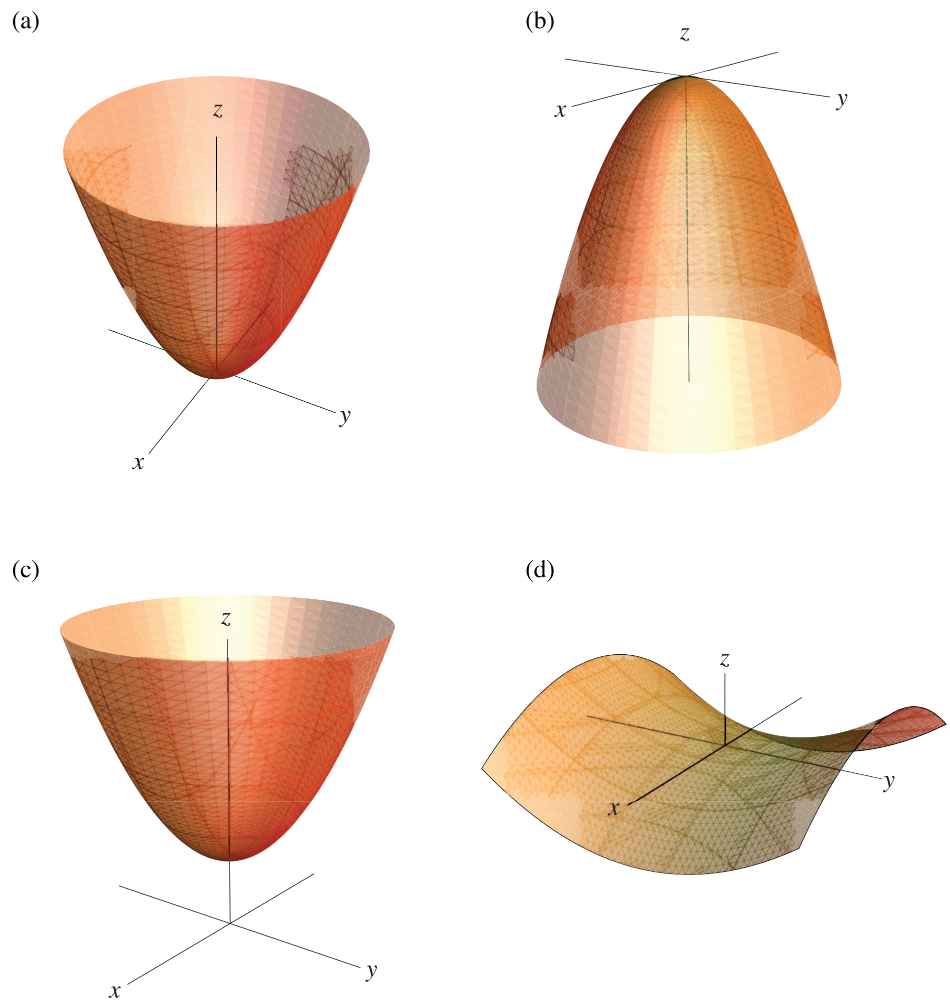
As an example for 1., we will consider the function

$$f(x, y) = x^2 + y^2 .$$

To visualise the surface, consider the nature of f for a fixed value of y, say $y = a$. In this case $z = x^2 + a^2$ and $z = z(x)$. The equation $z = x^2 + a^2$ defines a parabola in the plane $y = a$, perpendicular to the y-axis. Each different value of a gives a different parabola. For example, for $y = a = 0$ we have $z = x^2$. Therefore the required surface is made up of parabolas and forms a *paraboloid* as shown below.



Examples of other surfaces are shown in the following figure. It displays the three dimensional surfaces defined by the functions (a) $f(x, y) = x^2 + y^2$, (b) $f(x, y) = -x^2 - y^2$, (c) $f(x, y) = x^2 + y^2 + 5$ and (d) $f(x, y) = y^2 - x^2$.



The set of points in the x-y plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f (cf. what is plotted in geographic maps). Accordingly, the set of points (x, y, z) in space where a function $f(x, y, z) = c$ is called a **level surface** of f.

Example:

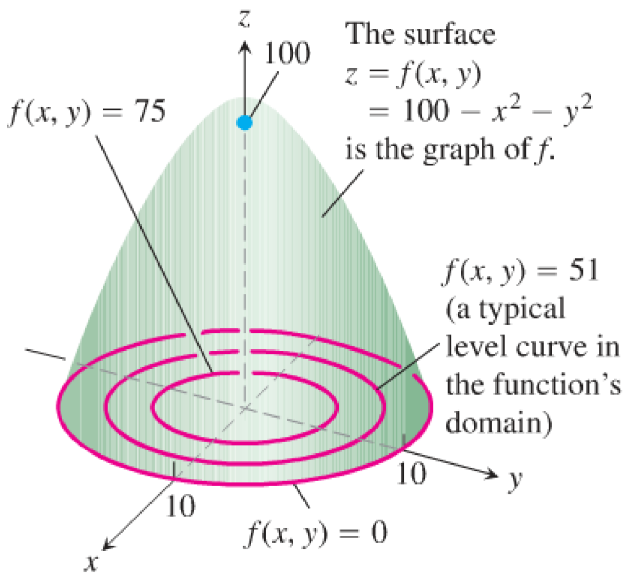
Graph the function $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$ and $f(x, y) = 75$ in the domain of f in the plane.

The domain is the entire x-y plane and the range is the set of real numbers ≤ 100. The graph is the paraboloid given by $z = 100 - x^2 - y^2$:

When $f(x, y) = 0$, we have $100 - x^2 - y^2 = 0$ or $x^2 + y^2 = 100$. This corresponds to a circle of radius 10.

When $f(x, y) = 51$, we have $100 - x^2 - y^2 = 51$ or $x^2 + y^2 = 49$. This corresponds to a circle of radius 7.

When $f(x, y) = 75$, we have $100 - x^2 - y^2 = 75$ or $x^2 + y^2 = 25$. This corresponds to a circle of radius 5.



z
100
The surface
$z = f(x, y)$
$= 100 - x^2 - y^2$
is the graph of f.
$f(x, y) = 75$
$f(x, y) = 51$
(a typical
level curve in
the function's
domain)
10
y
10
x
$f(x, y) = 0$

MTH4101 Calculus II

Lecture notes for Week 5
Derivatives IV

Thomas' Calculus, Sections 14.1 to 14.4

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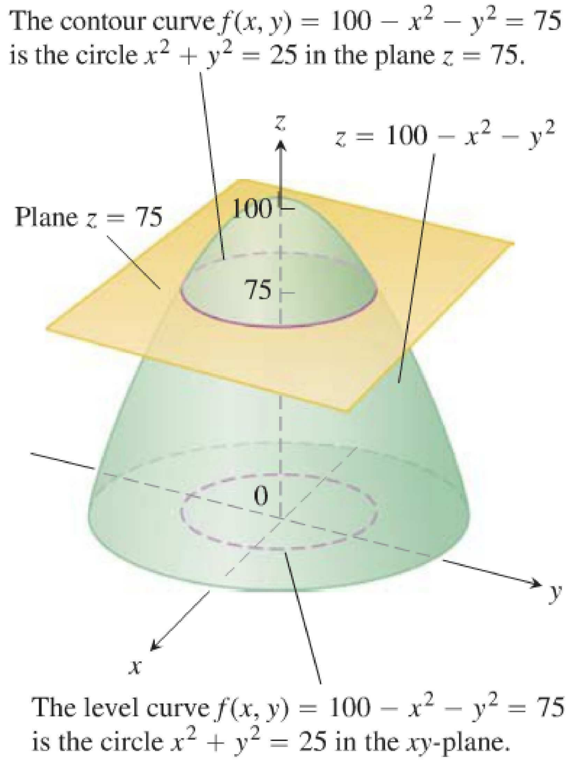
The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is called the **contour curve** $f(x, y) = c$. The following figure shows the contour curve produced where the plane $z = 75$ intersects the surface $z = f(x, y) = 100 - x^2 - y^2$.

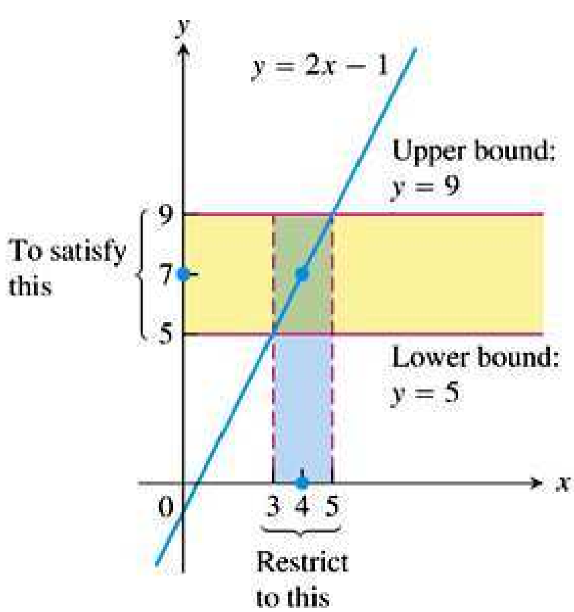


Limits and Continuity in Higher Dimensions

Reminder: Limits
For functions of one variable we say that $f(x)$ approaches the **limit** L whenever $f(x)$ is *arbitrarily close* to L for all x *sufficiently close* to a, written as

$$\lim_{x \to a} f(x) = L\,.$$

Example:
$\lim_{x \to 4}(2x - 1) = 7.$



Analogously, if the values of $f(x,y)$ lie *arbitrarily close* to a fixed real number L for all points (x,y) *sufficiently close* to a point (x_0,y_0), we say that f approaches the limit L as (x,y) approaches (x_0,y_0). More rigorously:[1]

DEFINITION Limit of a Function of Two Variables

We say that a function $f(x,y)$ approaches the **limit** L as (x,y) approaches (x_0,y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f,

$$|f(x,y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

It can be shown that this definition leads to the following properties (you have seen an analogous theorem for functions of one variable in Calculus 1):

Theorem *Properties of limits of functions of two variables*

If $L, M, k \in \mathbb{R}$, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$ then

1. $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm M$
2. $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
3. $\lim\limits_{(x,y)\to(x_0,y_0)} (kf(x,y)) = kL$
4. $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, $M \neq 0$
5. If r and s are integers with no common factors, and $s \neq 0$, then
   $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y))^{r/s} = L^{r/s}$ provided $L^{r/s}$ is a real number.

For polynomials and rational functions the limit as $(x,y) \to (x_0,y_0)$ can be calculated by evaluating the function at (x_0,y_0) (provided the rational function is defined at (x_0,y_0)).

Examples:

(1)

$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3\,.$$

(2) Find

$$\lim_{(x,y)\to(0,0)^+, x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}\,.$$

[1] see the footnote on p.9 of the week 3 lecture notes of Calculus 1 - you need to have read Thomas' Calculus Section 2.3 to fully appreciate this definition!

We need to avoid the whole path to the limit where $x = y$, hence the condition $x \neq y$. Accordingly, there is a problem with just setting $x = y = 0$ because $\sqrt{x} - \sqrt{y} \to 0$ as $(x, y) \to (0, 0)$. However, we can write

$$\begin{aligned}
\lim_{(x,y)\to(0,0)^+, x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)^+, x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \\
&= \lim_{(x,y)\to(0,0)^+, x\neq y} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{(x - y)} \\
&= \lim_{(x,y)\to(0,0)^+, x\neq y} x(\sqrt{x} + \sqrt{y}) = 0 \,.
\end{aligned}$$

Now we use limits to define continuity for a function of two variables.

Reminder: Continuity

For functions of one variable $f(x)$ is **continuous** at $x = a$ whenever $f(a)$ is defined, $\lim_{x\to a} f(x)$ exists and the limit L equals $f(a)$, that is, $\lim_{x\to a} f(x) = f(a)$. Analogously:

DEFINITION Continuous Function of Two Variables

A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

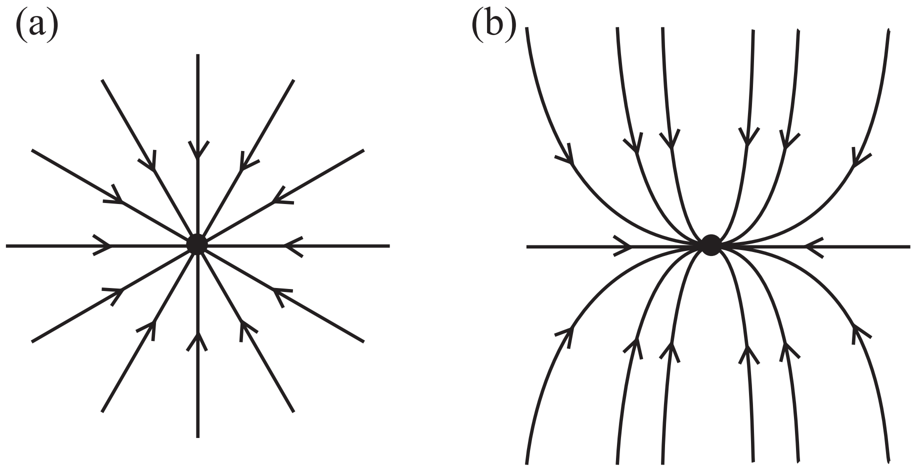
It follows from the previous Theorem that polynomials and rational functions of two variables are continuous on their domains.

Recall that for functions of one variable both the left- and the right-sided limits had to have *the same value* for a limit to exist at a point. For functions of two (or more) variables, this translates into the **Two-Path Test for Nonexistence of a Limit**: It states that if a function $f(x, y)$ has different limits along two different paths as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.

The following figure illustrates this concept for paths approaching a point in radial and tangential directions:



To have a limit at a point we have to have *the same limit* as the point is approached from all directions, including (a) radial directions and (b) tangential directions.

Example:

Show that the function

$$f(x,y) = \frac{2x^2y}{x^4+y^2}$$

has no limit as $(x,y) \to (0,0)$.

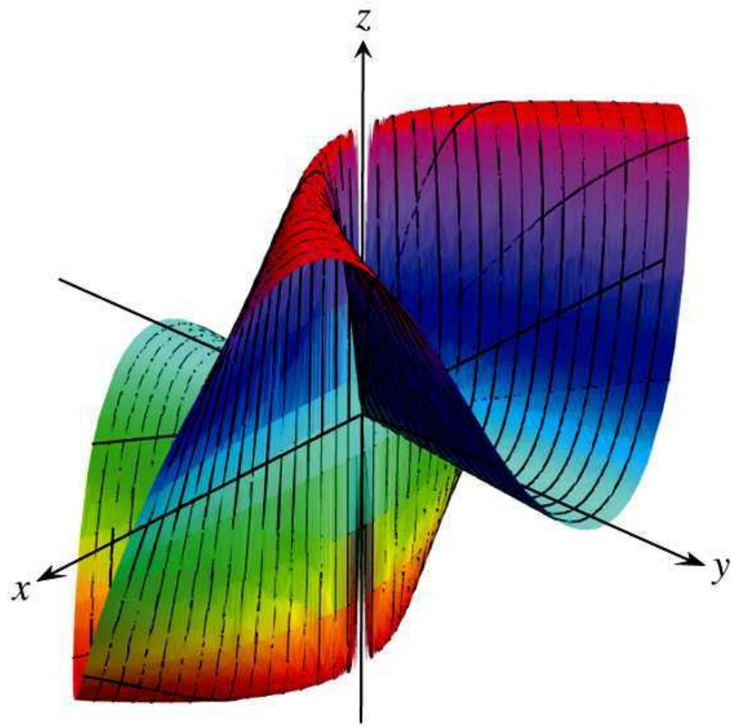
We cannot use substitution as it leads to $0/0$. However, we can consider what happens as we approach $(0,0)$ along a family of different curves. Remember, the choice of curves is up to us as the *Two-Path Test* does not specify what the path should be. You may wish to check, as an exercise, what happens for the family of paths $y = mx$ as $(x,y) \to (0,0)$. Here we consider the next more complicated case, which is the family of parabolas given by $y = kx^2$ $(x \neq 0)$. Along these curves the function is

$$f(x,y)\big|_{y=kx^2} = \left.\frac{2x^2y}{x^4+y^2}\right|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \frac{2kx^4}{x^4+k^2x^4} = \frac{2k}{1+k^2}\,.$$

Therefore, as we approach $(0,0)$ along any curve $y = kx^2$, we have

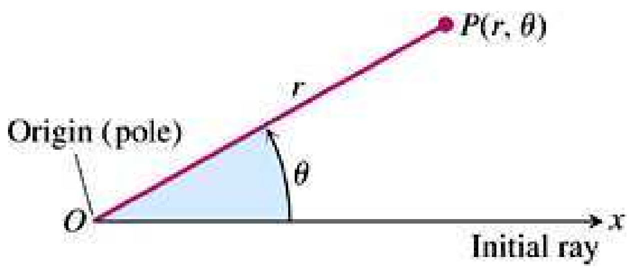
$$\lim_{(x,y)\to(0,0)} \left[f(x,y)\big|_{y=kx^2} \right] = \frac{2k}{1+k^2}\,.$$

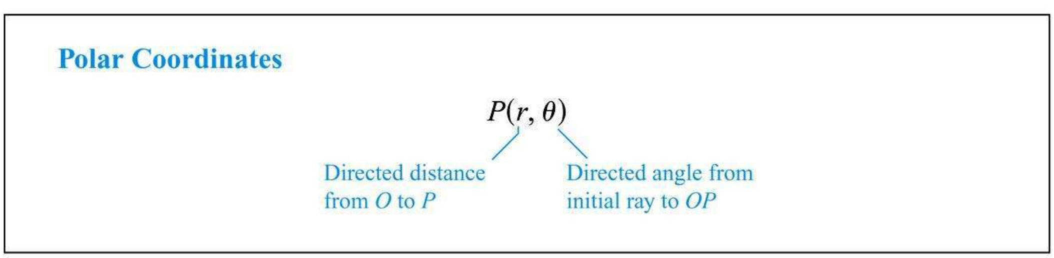
Consequently, the actual limit depends on which path of approach we take (i.e. which parabola we are on which is determined by the value of k). By the *Two-Path Test* there is hence *no limit* as $(x,y) \to (0,0)$. This is illustrated by looking at the surface of this function:



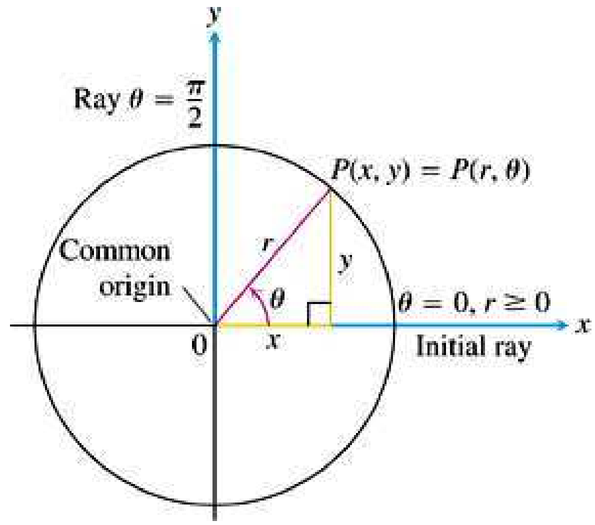
Sometimes it is useful to use polar coordinates.

Reminder (or perhaps not?): Polar coordinates
As an alternative to **Cartesian coordinates** (x, y), we can describe a point P in the plane by using **polar coordinates**:





These coordinates are particularly useful if a function, or a problem, has some circular symmetry. Typically, we restrict ourselves to $0 \leq r$ and $0 \leq \theta < 2\pi$ (why?). Polar and Cartesian coordinates can be converted into each other:



For the direction polar to Cartesian coordinates we easily derive

$$x = r\cos\theta \ , \quad y = r\sin\theta$$

That is, given (r, θ), we can compute (x, y). The direction Cartesian to polar coordinates is left to you as an exercise.[2]

Example:
Determine the continuity of the function defined by

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

In polar coordinates, i.e., by using $x = r\cos\theta$, $y = r\sin\theta$, the function can be written as

$$f(r, \theta) = \frac{2r^2 \cos\theta \sin\theta}{r^2(\cos^2\theta + \sin^2\theta)} = \sin 2\theta$$

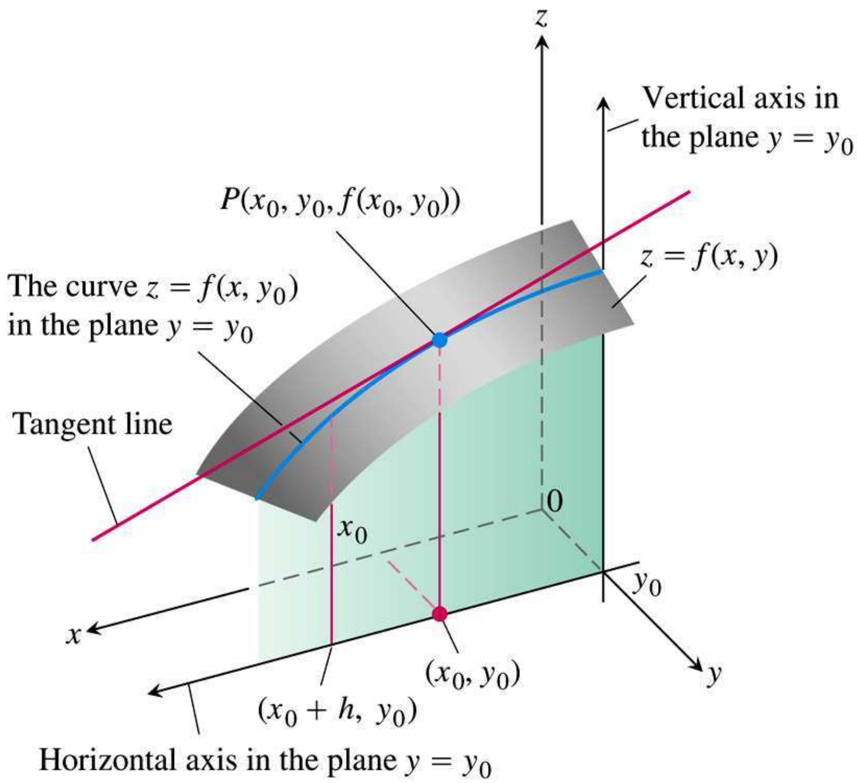
provided we are not at the origin (i.e. provided $r \neq 0$). Therefore, as $r \to 0$, the outcome depends on the angle θ. For example, along $\theta = \pi/4$, $f = \sin 2\theta = \sin \pi/2 = 1$ everywhere along the line. Therefore the function is not continuous.

Partial Derivatives

Reminder: Derivative
For functions of one variable, $y = f(x)$, the *derivative* at a point is the gradient of the tangent to the curve at that point.

But for functions of two variables, $z = f(x, y)$, an infinite number of tangents exist at a point. However, if we fix $y = y_0$ in $f(x, y)$ and let x vary, then $f(x, y_0)$ depends only on x:

[2] If you have not encountered polar coordinates before in sufficient detail, I highly recommend that you familiarize yourself with Thomas' Calculus, Section 11.3.



That is, we can reduce the problem of the many-variable derivative effectively to the one-variable case by holding all but one of the independent variables constant.

Definition
The **partial derivative** of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$$

provided the limit exists.

In complete analogy, the partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h} = f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

provided the limit exists.

For example, if $f(x, y) = x^2 + y^2$ then $f_x = 2x$, $f_y = 2y$.

Note how we treat the other variables as constants when we do partial differentiation!

We can extend this to three (or more) dimensions. For example, if $f(x, y, z) = xy^2z^3$ then $f_x = y^2z^3$, $f_y = 2xyz^3$, $f_z = 3xy^2z^2$.

Example:
Find $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ for the function $f(x, y) = x^2 + 3xy + y - 1$.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2+3xy+y-1) = 2x+3y \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2+3xy+y-1) = 3x+1\,.\end{aligned}$$

At the point $(4,-5)$ we have

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = -7\,, \qquad \left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 13\,.$$

Example:
Find $\partial z/\partial x$ if the equation $yz - \ln z = x + y$ (implicitly) defines $z = z(x,y)$.

$$\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x+y)\,.$$

Hence

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0\,.$$

This gives

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1\,; \qquad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{z}{yz-1}\,.$$

We can also obtain higher order derivatives.

Example:
If $f(x,y) = x\cos y + y\,e^x$, find

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}\,, \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}\,, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} \quad \text{and} \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}\,.$$

The first step is to find the first partial derivatives:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \cos y + y\,e^x \\ \frac{\partial f}{\partial y} &= -x\sin y + e^x\,.\end{aligned}$$

Now we take the partial derivatives of the first partial derivatives. This gives:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= y\,e^x \\ \frac{\partial^2 f}{\partial y \partial x} &= -\sin y + e^x \\ \frac{\partial^2 f}{\partial x \partial y} &= -\sin y + e^x \\ \frac{\partial^2 f}{\partial y^2} &= -x\cos y\,.\end{aligned}$$

This illustrates the following Theorem:

Theorem *Mixed Derivative Theorem*
If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are *defined* throughout an open region containing a point (a,b) and are *all continuous* at (a,b) then

$$f_{xy}(a,b) = f_{yx}(a,b)\,.$$

(An example where $f_{xy}(a,b) \neq f_{yx}(a,b)$ is provided by the function discussed on p.5/6 of the lecture notes of this week 5.)
The theorem can be extended to higher orders, provided the derivatives are continuous.

Queen Mary
University of London

MTH4101 Calculus II

Lecture notes for Week 6
Derivatives IV and V

Thomas' Calculus, Sections 14.3 to 14.6

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Spring 2015

Reminder:
For functions of a single variable it holds that if $y = f(x)$ is differentiable at $x = x_0$, then the change in the value of f that results from changing x from x_0 to $x_0 + \Delta x$ is given by the *differential approximation*

$$\Delta y = f'(x_0)\Delta x + \epsilon \Delta x$$

in which $\epsilon \to 0$ as $\Delta x \to 0$ (see Thomas' Calculus Section 3.9). For functions of two variables, the analogous property yields the *definition* of differentiability:

DEFINITION Differentiable Function
A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain.

Note in particular that for $z = f(x, y)$, *differentiability is more than the existence of the partial derivatives*, as becomes also clear from the following statement:
If f_x and f_y are *continuous* throughout an open region R, then f is *differentiable* at every point of R.
It also holds, in analogy to functions of a single variable:
If a function $f(x, y)$ is *differentiable* at a point (x_0, y_0) then f is *continuous* at (x_0, y_0).
If you are interested in the details underlying the above statements, like the *Increment Theorem*, please check out Thomas' Calculus p.771/772.

The Chain Rule

Reminder: Chain Rule for Functions of One Variable
If $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, then

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\,.$$

Similarly:

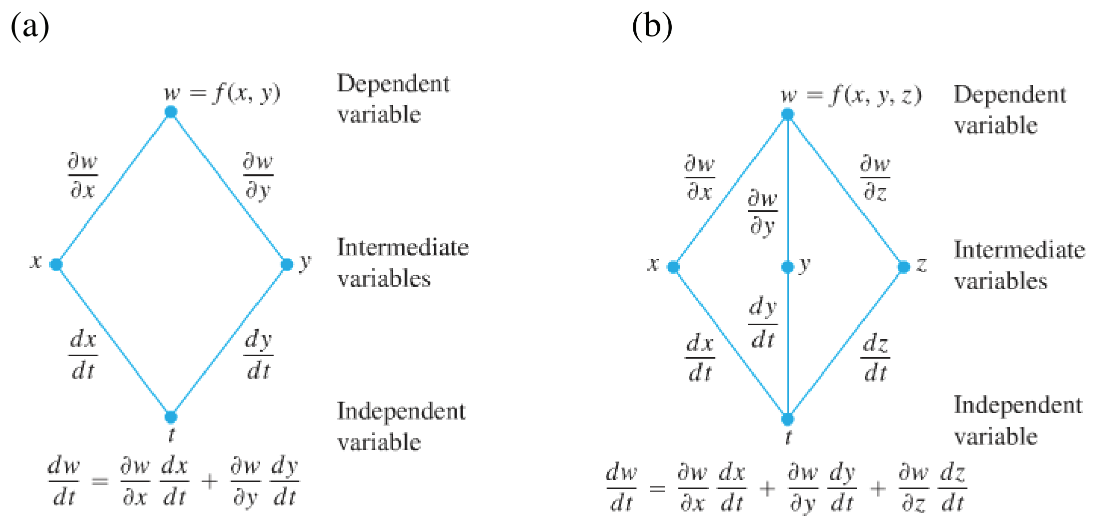
Theorem: Chain Rule for Functions of Two Variables
If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\,.$$

This straightforwardly follows from the above definition of differentiability.
We can easily extend this theorem to functions $w = f(x, y, z)$ of three variables:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\,.$$

We can use **tree diagrams** to illustrate the application of the Chain Rule:



(a) To find $\mathrm{d}w/\mathrm{d}t$, start at w and read down each route to t, multiplying derivatives along the way; then add the products. (b) For functions of three variables there are three routes from w to t instead of two, but finding $\mathrm{d}w/\mathrm{d}t$ is still the same: read down each route, multiplying derivatives along the way; then add.

Example:
Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$.

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t\,.$$

Note that we could have done this more directly by noting that

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t\,; \quad \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{2}\cdot 2\cos 2t = \cos 2t\,.$$

If $w = f(x, y)$ where $x = g(r, s)$ and $y = h(r, s)$ then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

and in analogy for functions $w = f(x, y, z)$. Also, if $w = f(x)$ and $x = g(r, s)$ then

$$\frac{\partial w}{\partial r} = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\partial x}{\partial s}\,.$$

Example:
For $u = w(x, y, z)$, express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2\,, \quad x = \frac{r}{s}\,, \quad y = r^2 + \ln s\,, \quad z = 2r\,.$$

We have

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ &= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) = \frac{1}{s} + 12r\end{aligned}$$

and

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (1)\left(\frac{-r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}\,.\end{aligned}$$

Suppose that $w = F(x, y)$ is differentiable and that $F(x, y) = 0$ defines y (implicitly) as a differentiable function of x. Then

$$0 = \frac{dw}{dx} = F_x\frac{dx}{dx} + F_y\frac{dy}{dx} = F_x + F_y\frac{dy}{dx}\,.$$

Hence, at any point where $F_y \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}\,.$$

This is the **Formula for Implicit Differentiation**.

Example:
Find $\mathrm{d}y/\mathrm{d}x$ if $y^2 - x^2 - \sin xy = 0$.

$$\begin{aligned}F(x,y) &= y^2 - x^2 - \sin xy \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{F_x}{F_y} = -\frac{(-2x - y\cos xy)}{(2y - x\cos xy)} = \frac{2x + y\cos xy}{2y - x\cos xy}\,.\end{aligned}$$

You may wish to compare this method with the one that you have learned in Calculus 1, i.e., differentiating the whole equation with respect to x and then solving for dy/dx.

Directional Derivatives and Gradient Vectors

We now investigate the derivative of a function $f(x, y)$ at a point *in a particular direction*:

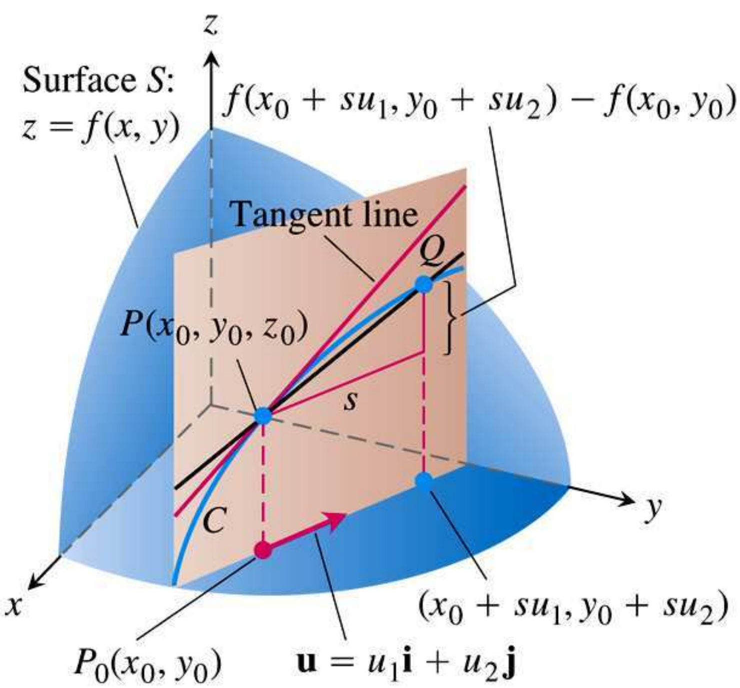
DEFINITION **Directional Derivative**

The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0}\frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.

It is also denoted by $(D_{\mathbf{u}}f)_{P_0}$ as the derivative of f at the point P_0 in the direction of the unit vector $\mathbf{u}$. The meaning is illustrated in the following figure:



We can develop a more efficient formula for the directional derivative by considering the line

$$x = x_0 + su_1\,, \qquad y = y_0 + su_2$$

through the point $P_0(x_0, y_0)$, parametrised with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then

$$\begin{aligned}\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)_{\mathbf{u},P_0} &= \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{\mathrm{d}x}{\mathrm{d}s} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{\mathrm{d}y}{\mathrm{d}s} \qquad \text{(via the Chain Rule)}\\ &= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2\\ &= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right]\cdot[u_1\mathbf{i} + u_2\mathbf{j}]\end{aligned}$$

DEFINITION Gradient Vector

The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

Note that for a function $f(x, y, z)$ we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \quad .$$

The expression $\nabla f = \text{grad } f$ is called "grad f", "gradient of f", "del f" or "nabla f".

We can now write the directional derivative using the gradient:

Theorem: Directional Derivative
If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$ then

$$\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}\,,$$

which is the scalar product of grad f at P_0 and $\mathbf{u}$.

Example:
Find the derivative of $f(x, y) = x\,e^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.
The unit vector is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\,.$$

Now

$$\begin{aligned} f_x(2, 0) &= (e^y - y\sin(xy))|_{(2,0)} = e^0 - 0 = 1 \\ f_y(2, 0) &= (xe^y - x\sin(xy))|_{(2,0)} = 2e^0 - 2 \cdot 0 = 2\,. \end{aligned}$$

Hence

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

and so

$$D_{\mathbf{u}}f|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1\,.$$

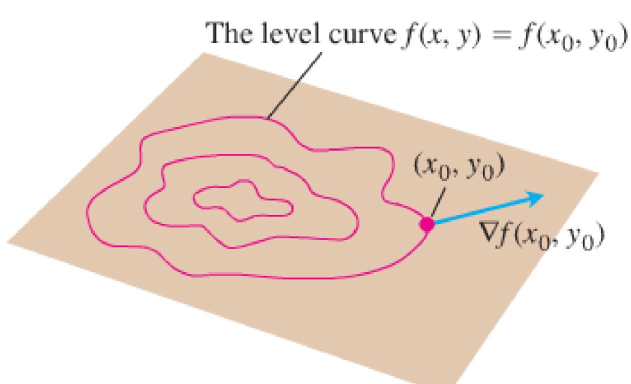
Note that

$$\boxed{D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta}$$

where θ is the angle between the vectors ∇f and $\mathbf{u}$. This implies the following:

1. f *increases most rapidly* when $\cos\theta = 1$ (i.e. $\mathbf{u}$ is parallel to ∇f)
2. f *decreases most rapidly* when $\cos\theta = -1$ (i.e. $\mathbf{u}$ is in opposite direction to ∇f)
3. f has *zero change* when $\cos\theta = 0$ (i.e. $\mathbf{u}$ is orthogonal to ∇f).

Point 1 implies (why?): ∇f points in the direction of *maximal increase* of f.
Point 3 implies (why?): At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$ the gradient of f is normal to the level curve through (x_0, y_0).



Tangent lines to level curves are always normal to the gradient. If (x, y) is a point on the tangent line through the point $P(x_0, y_0)$ then

$$\mathbf{T} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j}\,,$$

is a vector parallel to it. The *equation of the tangent* is then

$$\nabla f \cdot \mathbf{T} = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0\,.$$

Tangent Planes and Differentials

DEFINITIONS **Tangent Plane, Normal Line**

The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

It follows[1] that the equation of the tangent plane is

$$\nabla f|_{P_0} \cdot \vec{P_0P} = f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

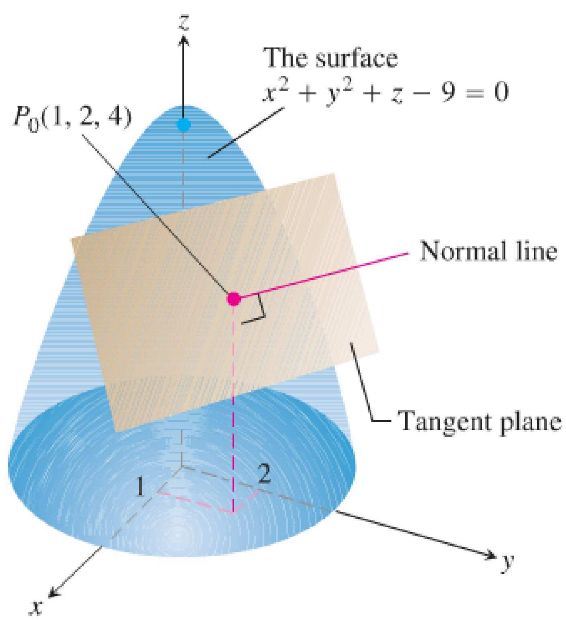
and the equation of the normal line is

$$x = x_0 + f_x(P_0)t\,, \qquad y = y_0 + f_y(P_0)t\,, \qquad z = z_0 + f_z(P_0)t\,.$$

Example:
Find the tangent plane and normal line of the surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0$$

(a circular paraboloid) at the point $P_0(1, 2, 4)$



[1]See Section 12.5 in Thomas' Calculus for details if you are in trouble with this.

$$\nabla f|_{P_0} = (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}$$

where at the point P_0 we have $f_x(P_0) = 2$, $f_y(P_0) = 4$ and $f_z(P_0) = 1$. Therefore the equation of the tangent plane is

$$2(x-1) + 4(y-2) + (z-4) = 0$$

which simplifies to

$$2x + 4y + z = 14\,.$$

The normal line to the surface at P_0 is

$$x = 1 + 2t\,, \qquad y = 2 + 4t\,, \qquad z = 4 + t\,.$$

MTH4101 Calculus II

Lecture notes for Week 8
Derivatives V and Integration III

Thomas' Calculus, Sections 14.6 to 14.8 and 15.1

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Spring 2015

We remark that the gradient has the following algebraic properties:

$$\begin{aligned}
\nabla(kf) &= k\,\nabla f \qquad \text{for any number } k \\
\nabla(f \pm g) &= \nabla f \pm \nabla g \\
\nabla(fg) &= f\,\nabla g + g\,\nabla f \\
\nabla\left(\frac{f}{g}\right) &= \frac{g\,\nabla f - f\,\nabla g}{g^2}
\end{aligned}$$

(the proof is straightforward and is left as an exercise)
Before we linearise a function of two variables, recall that a function $z = f(x, y)$ is *differentiable* at (x_0, y_0) if

$$\Delta z = f(x, y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

with $\epsilon_1, \epsilon_2 \to 0$ $(\Delta x, \Delta y \to 0)$. Solve for $f(x, y)$ and approximate:

DEFINITIONS Linearization, Standard Linear Approximation

The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \qquad (5)$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

Example:
Find the linearisation of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point $(3, 2)$.
We first evaluate f, f_x and f_y at the point $(x_0, y_0) = (3, 2)$:

$$\begin{aligned}
f(3, 2) &= \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Bigg|_{(3,2)} = 8 \\
f_x(3, 2) &= \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Bigg|_{(3,2)} = (2x - y)|_{(3,2)} = 4 \\
f_y(3, 2) &= \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Bigg|_{(3,2)} = (-x + y)|_{(3,2)} = -1
\end{aligned}$$

giving

$$\begin{aligned}
L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
&= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2\,.
\end{aligned}$$

Hence the linearisation of f at $(3, 2)$ is $L(x, y) = 4x - y - 2$.

Optional reading assignment:
Work yourself through the definition and example about the total differential (cf. p.796/797 of the little section in Thomas' Calculus).

Recall that for $y = f(x)$ we have defined the *differential* $dy = f'(x)dx$.

DEFINITION Total Differential

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

in the linearization of f is called the **total differential of f**.

Example:
The volume $V = \pi r^2 h$ of a cylinder is to be calculated from measured values of r (the radius) and h (the height). Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V.
First note that

$$\left|\frac{dr}{r}100\right| \leq 2\,, \ \left|\frac{dh}{h}100\right| \leq 0.5\,.$$

Then

$$dV = V_r\, dr + V_h\, dh = 2\pi r h\, dr + \pi r^2\, dh$$

and so

$$\frac{dV}{V} = \frac{2\pi r h\, dr + \pi r^2\, dh}{\pi r^2 h} = \frac{2\, dr}{r} + \frac{dh}{h}\,.$$

Hence

$$\left|\frac{dV}{V}\right| = \left|2\frac{dr}{r} + \frac{dh}{h}\right| \leq \left|2\frac{dr}{r}\right| + \left|\frac{dh}{h}\right| \leq 2(0.02) + 0.005 = 0.045\,.$$

Therefore the error is no more than 4.5%.

Extreme Values and Saddle Points

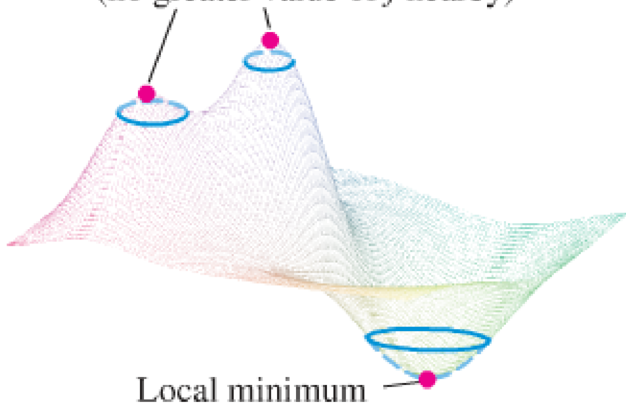
When we investigated extreme values for functions of one variable we looked for points where the graph had a horizontal tangent line. For functions of two variables we look for points where the *surface* defined by $z = f(x, y)$ has a *horizontal tangent plane*. This leads to the following definition:

DEFINITIONS Local Maximum, Local Minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

Local maxima correspond to "mountain peaks" on the surface $z = f(x, y)$ and local minima correspond to "valley bottoms":



Not too hard to show:

THEOREM 10—First Derivative Test for Local Extreme Values If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Define an important object:

DEFINITION Critical Point

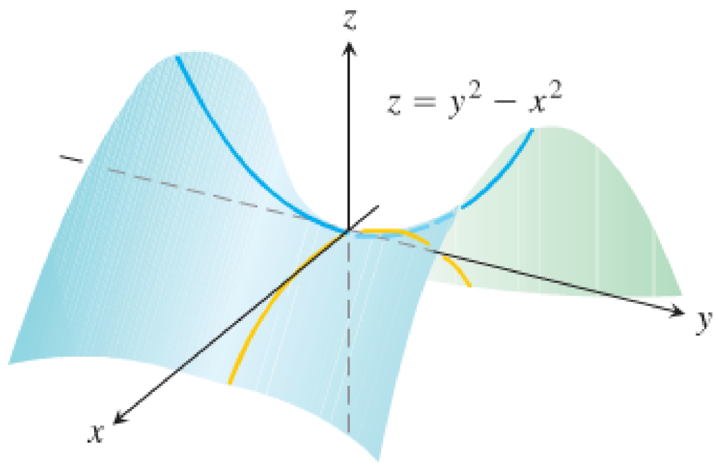
An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

Therefore local maxima and minima are critical points (why?) but critical points can also include **saddle points**:

DEFINITION Saddle Point

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.40).

An example of a saddle point is the origin in the following surface:



Therefore, finding critical points of a function is not sufficient to identify the type of critical point (local maximum, local minimum or saddle point). To do this we need to make use of second partial derivatives.

THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The quantity $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of the function f. Note that

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix},$$

i.e., the Hessian is the *determinant* (cf. Geometry 1) of the matrix of the second partial derivatives.[1]

Example:
Find the local extreme values of $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$ and determine the nature of each.
$f(x, y)$ is defined and differentiable for all points in its domain. Hence, at extreme values f_x and f_y are simultaneously zero. This gives the two equations

$$f_x = y - 2x - 2 = 0; \qquad f_y = x - 2y - 2 = 0.$$

[1]If you want to know why: check out Thomas' Calculus Section 14.9.

The solution of these equations is $x = y = -2$. Hence $(-2, -2)$ is the only point where f may take an extreme value. Now take the second derivatives:

$$f_{xx} = -2 < 0\,, \qquad f_{yy} = -2\,, \qquad f_{xy} = 1\,.$$

At the point $(-2, -2)$,

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 1^2 = 3 > 0\,.$$

So $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$. Therefore f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$.

In general the situation can be slightly more complicated:

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i. **boundary points** of the domain of f

ii. **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum**

ii. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum**

iii. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point**

iv. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive**.

Lagrange Multipliers

We now consider the problem to find extrema of a function $f(x, y, z)$ whose domain is constrained by another function $g(x, y, z) = 0$ to lie within some subset.

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the **local maximum and minimum values** of f **subject to the constraint** $g(x, y, z) = 0$,[2] we need to find the values of x, y, z and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0\,.$$

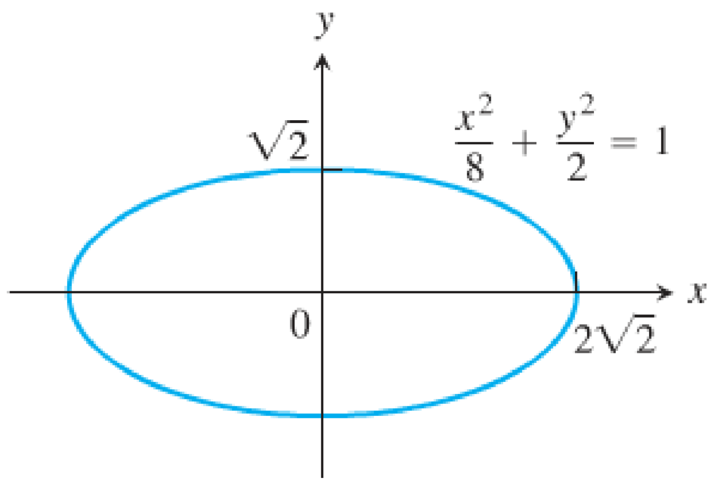
This is the **Method of Lagrange Multipliers**. For functions of two variables the condition is similar but without the variable z.

[2] If these exist - there is a small subtlety here, cf. Thomas' Calculus p.815.

We will see how the method works by considering two examples.[3]

Example:
Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1\,.$$



We need to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0\,.$$

First, find the values of x, y and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0\,.$$

$$\begin{aligned} \nabla f &= f_x\mathbf{i} + f_y\mathbf{j} = y\mathbf{i} + x\mathbf{j} \\ \nabla g &= g_x\mathbf{i} + g_y\mathbf{j} = \frac{x}{4}\mathbf{i} + y\mathbf{j}\,. \end{aligned}$$

Hence

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j}\,.$$

Comparing components gives

$$y = \frac{\lambda}{4}x\,, \qquad x = \lambda y\,.$$

Therefore

$$y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y\,.$$

Hence $y = 0$ or $\lambda = \pm 2$ and there are two cases to consider.

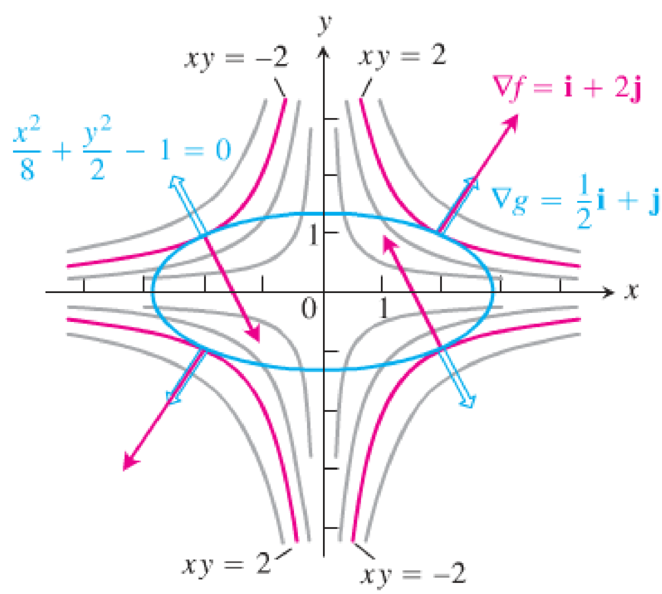
[3] A detailed motivation and a sketch of the proof are provided in Thomas' Calculus, beginning of Section 14.8.

1. If $y = 0$, then $x = y = 0$. But $(0, 0)$ does not lie on the ellipse, hence $y \neq 0$.

2. If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting in $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1 \qquad \Rightarrow 4y^2 + 4y^2 = 8 \qquad \Rightarrow y = \pm 1\,.$$

Therefore $f(x, y)$ has its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.
The level curves of $f(x, y) = xy$ are the hyperbolas $xy = c$. The extreme values are the points on the ellipse when ∇f (red) is a scalar multiple of ∇g (blue):



Example:
Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

$$f(x, y) = 3x + 4y\,, \qquad g(x, y) = x^2 + y^2 - 1\,.$$

The Lagrange multiplier condition states that $\nabla f = \lambda \nabla g$, hence

$$3\mathbf{i} + 4\mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j} \qquad \Rightarrow x = \frac{3}{2\lambda}\,, \quad y = \frac{2}{\lambda} \qquad (\lambda \neq 0\,;\ \text{why?})\,.$$

Therefore x and y have the same sign.
The condition $g(x, y) = 0$ gives

$$x^2 + y^2 - 1 = 0$$

and this gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0\,.$$

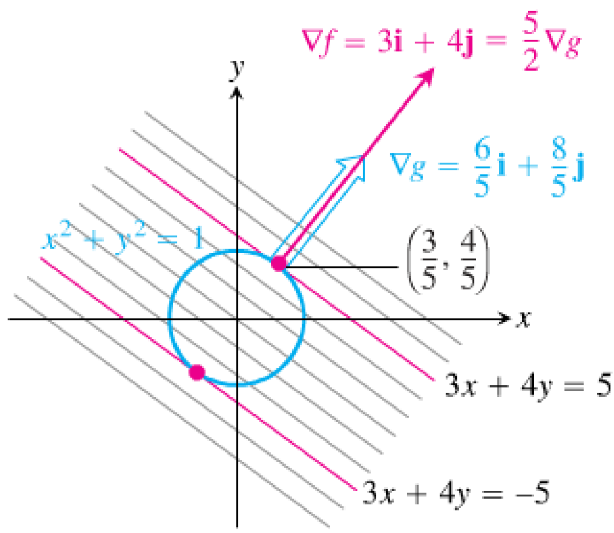
This gives

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1 \qquad \Rightarrow 9 + 16 = 4\lambda^2 \qquad \Rightarrow \lambda = \pm\frac{5}{2}\,.$$

Hence

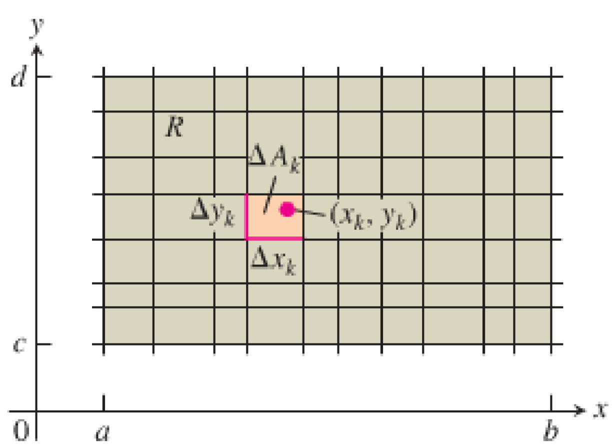
$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}\,, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5}\,.$$

Therefore the function $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.
The level curves of $f(x, y) = 3x + 4y$ are the lines $3x + 4y = c$. The further the lines lie from the origin, the larger the absolute value of f:



Double Integrals

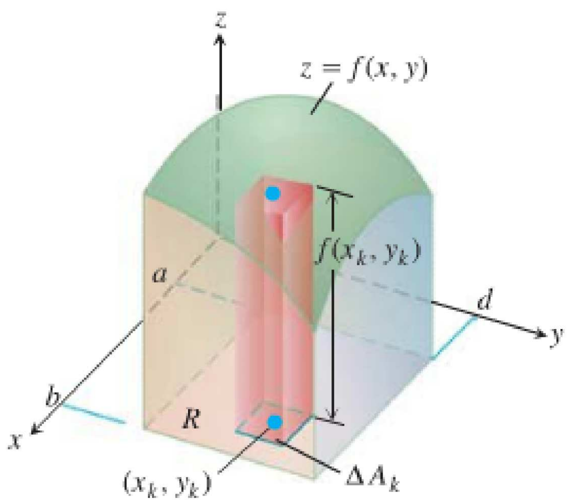
Consider a function $f(x, y)$ defined on a rectangular region $R: \ a \le x \le b,\ c \le y \le d$ partitioned into small rectangles A_k:



The area of a small rectangle with sides Δx_k and Δy_k is

$$\Delta A_k = \Delta x_k\, \Delta y_k\,.$$

Choose a point (x_k, y_k) in the (suitably numbered) kth rectangle with function value $f(x_k, y_k)$. We can consider $z = f(x, y)$ as defining the height z at the point (x, y). The product $f(x_k, y_k)\,\Delta A_k$ is then the *volume of a solid* with base area ΔA_k and height $f(x_k, y_k)$ (for which we assume that $f(x_k, y_k) > 0$):



The **Riemann sum** S_n of these solids over R is

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k\,.$$

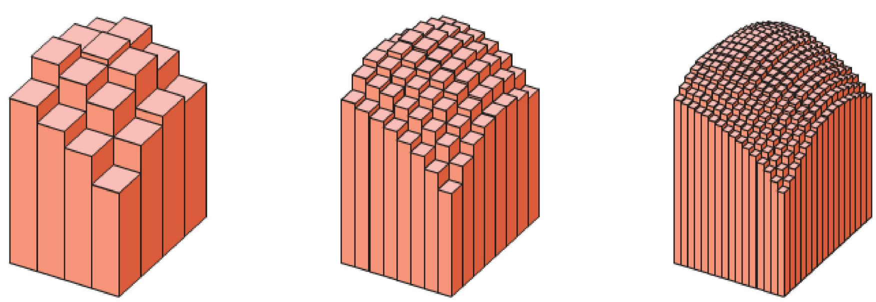
Now consider what happens as $\Delta A_k \to 0$ (as $n \to \infty$), i.e., we refine the partitioning. When the limit of these sums exists the function f is said to be **integrable** and the limit is called the **double integral** of f over R, written as

$$\int_R\!\int f(x, y)\,\mathrm{d}A \qquad \text{or} \qquad \int_R\!\int f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

The volume of the portion of the solid directly above the base ΔA_k is $f(x_k, y_k)\,\Delta A_k$. Hence the total volume above the region R is

$$\text{Volume } = \lim_{n\to\infty} S_n = \int_R\!\int f(x, y)\,\mathrm{d}A$$

where $\Delta A_k \to 0$ as $n \to \infty$. The following figure shows how the Riemann sum approximations of the volume become more accurate as the number n of boxes increases:



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Queen Mary
University of London

MTH4101 Calculus II

Lecture notes for Week 9
Integration III to IV

Thomas' Calculus, Sections 15.1 to 15.3

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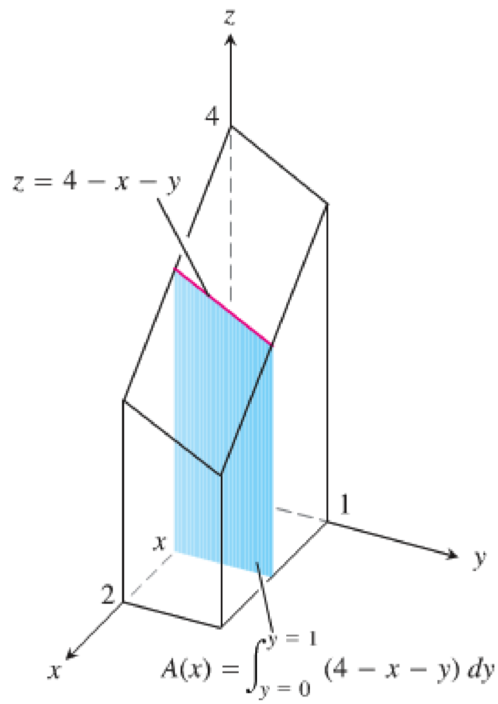
Spring 2015

Consider the calculation of the volume under the plane $z = 4 - x - y$ over the rectangular region $R: \ 0 \le x \le 2$ and $0 \le y \le 1$ in the x-y plane.
First consider a slice perpendicular to the x-axis:



The volume under the plane is

$$\int_{x=0}^{x=2} A(x)\,\mathrm{d}x$$

where $A(x)$ is the cross-sectional area at x. For each value of x we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\,\mathrm{d}y$$

which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y. Combining the above two equations we have

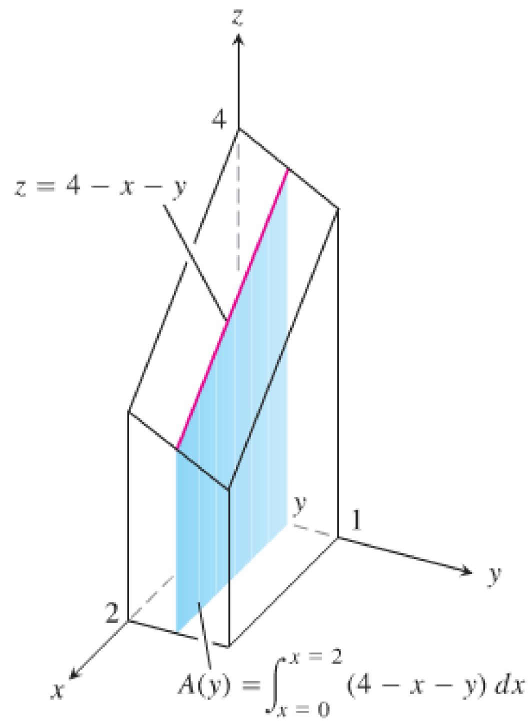
$$\begin{aligned} \text{Volume} &= \int_{x=0}^{x=2} A(x)\,\mathrm{d}x \\ &= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} \mathrm{d}x = \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) \mathrm{d}x \\ &= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = \left(\frac{7}{2} \cdot 2 - \frac{2^2}{2} \right) - (0 - 0) = 5\,. \end{aligned}$$

We can write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

This is an **iterated** or **repeated integral**. The expression states that we can get the volume under the plane by (i) integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$, holding x fixed, and then (ii) integrating the resulting expression in x from $x = 0$ to $x = 2$. In other words, *first do the* $\mathrm{d}y$ *integral and then do the* $\mathrm{d}x$ *integral.*

Now consider the plane perpendicular to the y-axis:



We have

$$A(y) = \int_{x=0}^{x=2} (4 - x - y)\,\mathrm{d}x = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2} = 6 - 2y\,.$$

The volume is then

$$\text{Volume} = \int_{y=0}^{y=1} A(y)\,\mathrm{d}y = \int_{y=0}^{y=1} (6 - 2y)\,\mathrm{d}y = \left[6y - y^2\right]_0^1 = 5$$

as before.

This illustrates

THEOREM 1 Fubini's Theorem (First Form)

If $f(x, y)$ is continuous throughout the rectangular region $R: a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx.$$

Example:
Calculate the volume V under $z = f(x,y) = x^2y$ over the rectangle R defined by $1 \le x \le 2$, $-3 \le y \le 4$.

$$\begin{aligned} V &= \iint_R x^2y\,\mathrm{d}A = \int_{x=1}^{x=2}\left(\int_{y=-3}^{y=4} x^2y\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{x=1}^{x=2}\left[\frac{x^2y^2}{2}\right]_{y=-3}^{y=4}\mathrm{d}x = \int_{x=1}^{x=2}\frac{7x^2}{2}\,\mathrm{d}x = \left[\frac{7x^3}{6}\right]_{x=1}^{x=2} = \frac{49}{6}\,. \end{aligned}$$

Changing the order gives the same result:

$$\begin{aligned} V &= \iint_R x^2y\,\mathrm{d}A = \int_{y=-3}^{y=4}\left(\int_{x=1}^{x=2} x^2y\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{y=-3}^{y=4}\left[\frac{x^3y}{3}\right]_{x=1}^{x=2}\mathrm{d}y = \int_{y=-3}^{y=4}\frac{7y}{3}\,\mathrm{d}y = \left[\frac{7y^2}{6}\right]_{y=-3}^{y=4} = \frac{49}{6}\,. \end{aligned}$$

In this example we could have separated the integrand into its x and y parts:

$$V = \int_{x=1}^{x=2}\left(\int_{y=-3}^{y=4} x^2y\,\mathrm{d}y\right)\mathrm{d}x = \left(\int_{x=1}^{x=2} x^2\,\mathrm{d}x\right)\left(\int_{y=-3}^{y=4} y\,\mathrm{d}y\right) = \frac{7}{3}\cdot\frac{7}{2} = \frac{49}{6}\,.$$

More generally, if $f(x,y) = g(x)\,h(y)$, (i.e. the function is **separable**) *and* the region is **rectangular** then

$$\begin{aligned} \iint_R g(x)\,h(y)\,\mathrm{d}A &= \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} g(x)\,h(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \left(\int_{x=a}^{x=b} g(x)\,\mathrm{d}x\right)\left(\int_{y=c}^{y=d} h(y)\,\mathrm{d}y\right)\,. \end{aligned}$$

Now consider the case where the region R is *not rectangular*:[1]

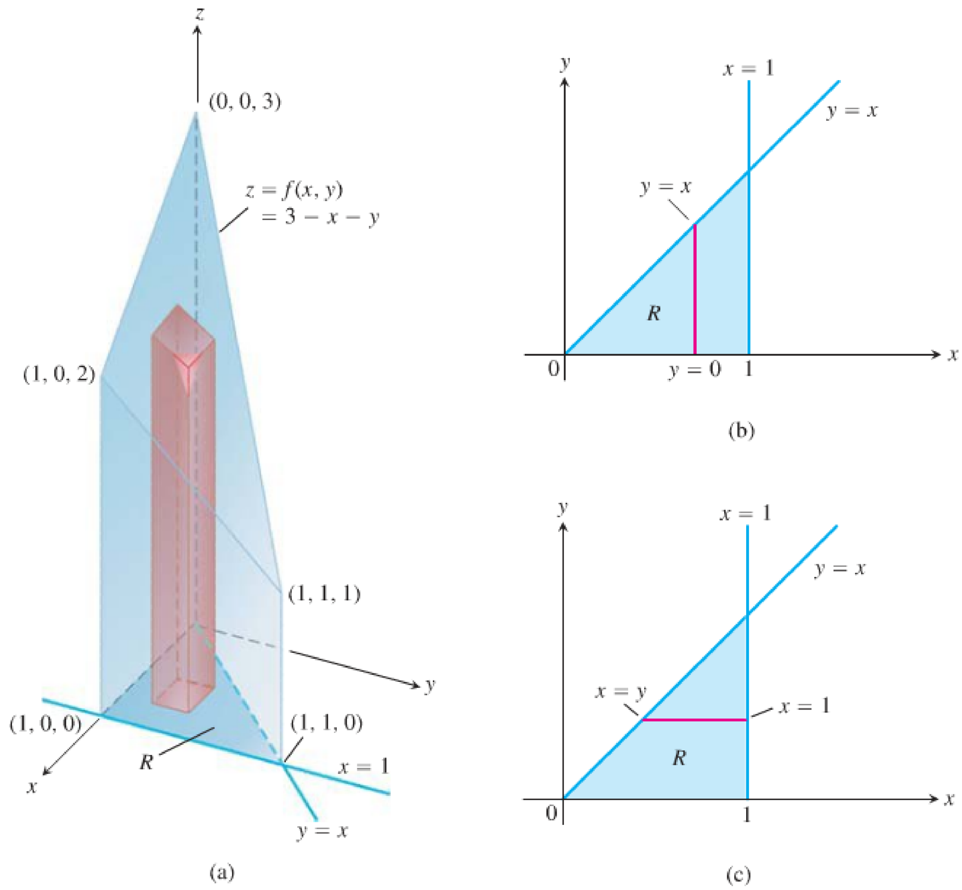
THEOREM 2 Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

[1] see Thomas' Calculus, beginning of Section 15.2 for details underlying this theorem

Example:
Find the volume of the prism $\int\int_R (3-x-y)\,\mathrm{d}A$ where R is the region bounded by the x-axis and the lines $x=1$ and $y=x$.



The region of integration in the x-y plane and the volume defined by $z=3-x-y$ are shown in the figure. In order to do the double integral we will first consider the approach where we fix the value of x and do the y integral. We have

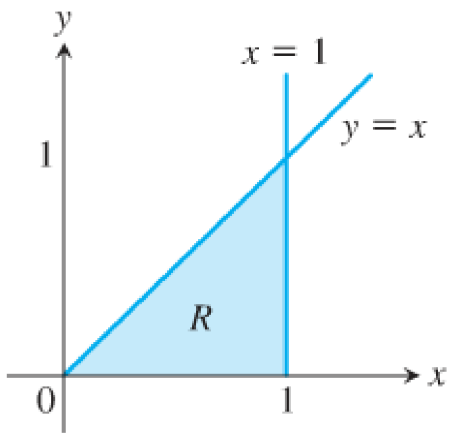
$$\begin{aligned}\int_R\int(3-x-y)\,\mathrm{d}A &= \int_{x=0}^{x=1}\int_{y=0}^{y=x}(3-x-y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\left[3y-xy-\frac{y^2}{2}\right]_{y=0}^{y=x}\mathrm{d}x\\ &= \int_0^1\left(3x-\frac{3x^2}{2}\right)\mathrm{d}x = \left[\frac{3x^2}{2}-\frac{x^3}{2}\right]_0^1 = 1\,.\end{aligned}$$

We can also change the order of the integration where we fix the value of y and do the x integral. We have

$$\begin{aligned}\int_R\int(3-x-y)\,\mathrm{d}A &= \int_{y=0}^{y=1}\int_{x=y}^{x=1}(3-x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left[3x-\frac{x^2}{2}-xy\right]_{x=y}^{x=1}\mathrm{d}y\\ &= \int_0^1\left(\left(3-\frac{1}{2}-y\right)-\left(3y-\frac{y^2}{2}-y^2\right)\right)\mathrm{d}y\\ &= \int_0^1\left(\frac{5}{2}-4y+\frac{3}{2}y^2\right)\mathrm{d}y = \left[\frac{5}{2}y-2y^2+\frac{y^3}{2}\right]_{y=0}^{y=1} = 1\,.\end{aligned}$$

In some cases the order of integration can be crucial to solving the problem.

Example:
Calculate $\iint_R (\sin x)/x \, \mathrm{d}A$ where R is the triangle in the x-y plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.



Taking vertical strips (i.e. keeping x fixed and allowing y to vary) gives

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x} \, \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left[y \frac{\sin x}{x} \right]_{y=0}^{y=x} \mathrm{d}x = \int_0^1 \sin x \, \mathrm{d}x$$
$$= [-\cos x]_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1 \, .$$

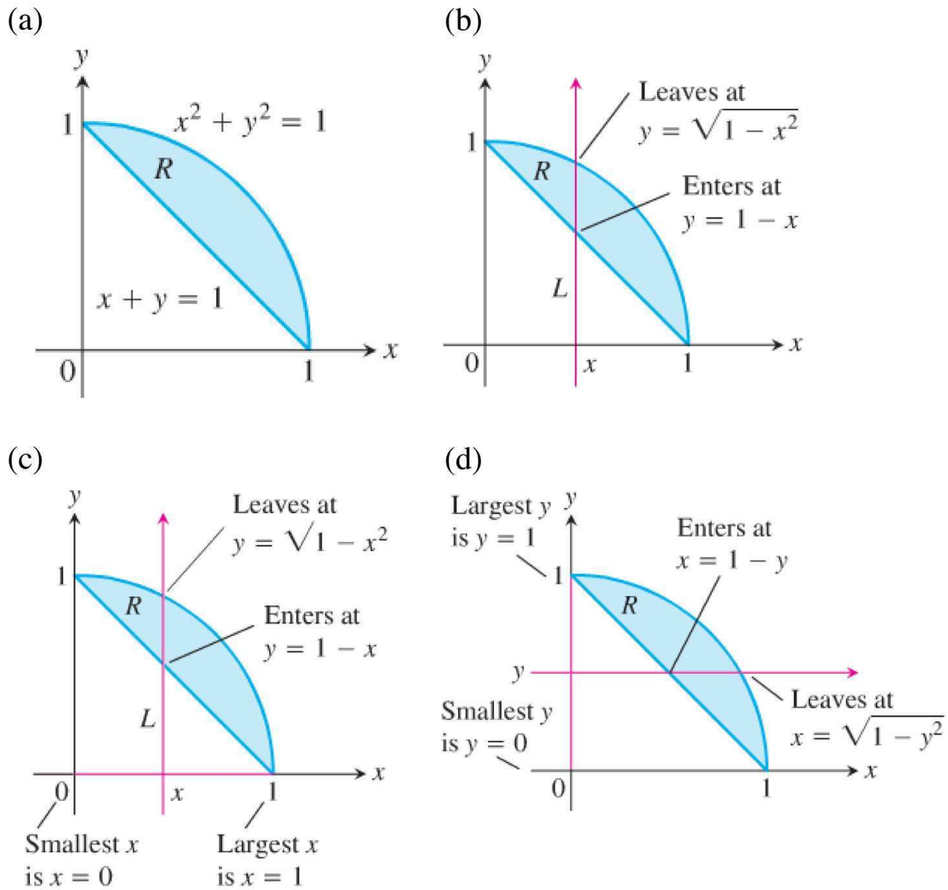
However, if we reverse the order of integration we get

$$\int_0^1 \int_y^1 \frac{\sin x}{x} \, \mathrm{d}x \, \mathrm{d}y$$

and $\int (\sin x)/x \, \mathrm{d}x$ cannot be expressed in terms of elementary functions making the integral difficult to do.
There are always two ways to do a double integral; choose the simpler because the other may be impossible!

A key part of the process of double (and multiple) integration over a region is to find the **limits of the integration**. We can illustrate the procedure by considering the double integral of a function over the region R given by the intersection of the line $x + y = 1$ with the circle $x^2 + y^2 = 1$ (see the picture next page).

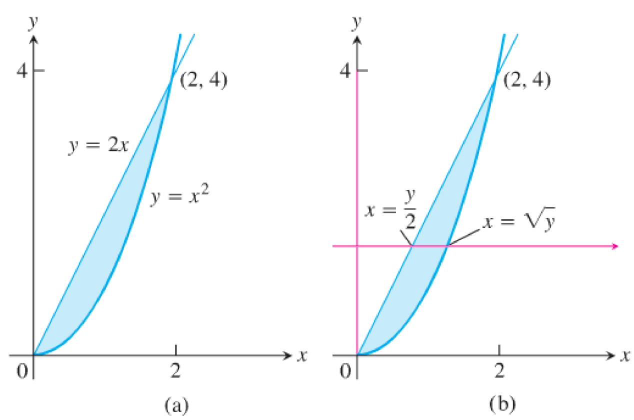
1. **Sketch the region of integration** and label its boundary curves.
2. If we decide to use vertical cross-sections first: **Find the y-limits of integration**. Imagine a vertical line through the region, R, and mark the points where it enters and leaves R. In this case such a line would enter at $y = 1 - x$ and leave at $y = \sqrt{1 - x^2}$.
3. **Find the x-limits of integration**: Choose the x-limits that include all vertical lines through R. In this case the lower limit is $x = 0$ and the upper limit is $x = 1$.
4. This step may not be necessary: **Reversing the order of integration**. Then the x-limits would be from $x = 1 - y$ to $x = \sqrt{1 - y^2}$ and the y-limits from $y = 0$ to $y = 1$.



Example:
Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, \mathrm{d}y\, \mathrm{d}x$$

and write an equivalent integral with the order of integration reversed. Evaluate the integral.



As written, the order of integration would imply that we do the y-integral first, from $y = x^2$ to $y = 2x$, followed by the x-integral from $x = 0$ to $x = 2$. However, we are told to reverse the order of integration. This means we do the x-integration first, from $x = y/2$ to $x = \sqrt{y}$, followed by the y-integral from $y = 0$ to $y = 4$. In other words,

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,\mathrm{d}x\,\mathrm{d}y$$

We can evaluate the integral using either ordering. Let us revert to the original:

$$\begin{aligned}
\int_0^2 \int_{x^2}^{2x} (4x+2)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^2 [4xy+2y]_{x^2}^{2x}\,\mathrm{d}x = \int_0^2 \left(8x^2+4x-4x^3-2x^2\right)\,\mathrm{d}x \\
&= \int_0^2 \left(-4x^3+6x^2+4x\right)\,\mathrm{d}x = \left[-x^4+2x^3+2x^2\right]_0^2 \\
&= -16+16+8 = 8\,.
\end{aligned}$$

Note that this example is *not* separable because it is a non-rectangular region (i.e. the limits on the x and y integrals now depend on the region of integration).

Double integrals can also be calculated over unbounded regions.

Example:
Evaluate the integral $\int_0^\infty \int_0^\infty x\,e^{-(x+2y)}\mathrm{d}x\,\mathrm{d}y$.
We have

$$\begin{aligned}
\int_0^\infty \int_0^\infty x\,e^{-(x+2y)}\mathrm{d}x\,\mathrm{d}y &= \int_0^\infty \int_0^\infty e^{-2y} x\,e^{-x}\mathrm{d}x\,\mathrm{d}y \\
&\qquad \text{(integrate by parts with } u = x,\ \mathrm{d}v = e^{-x}\mathrm{d}x\text{)} \\
&= \int_0^\infty e^{-2y} \left\{ \left[-x\,e^{-x}\right]_0^\infty - \int_0^\infty \left(-e^{-x}\right)\,\mathrm{d}x \right\}\,\mathrm{d}y \\
&= \int_0^\infty e^{-2y} \left((0-0) + \left[-e^{-x}\right]_0^\infty\right)\,\mathrm{d}y \\
&= \left[-\frac{1}{2}e^{-2y}\right]_0^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}\,.
\end{aligned}$$

Double integrals have the following properties:
Let $f(x,y), g(x,y)$ be continuous on the bounded region R. Then

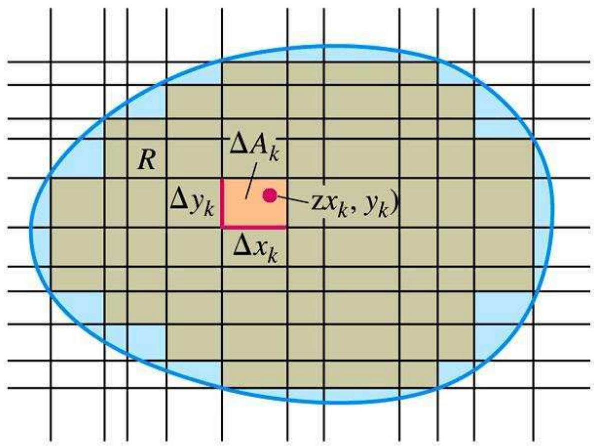
$$\begin{aligned}
\iint_R c\,f(x,y)\,\mathrm{d}A &= c \iint_R f(x,y)\,\mathrm{d}A \quad \text{for any number } c \\
\iint_R \left(f(x,y) \pm g(x,y)\right)\,\mathrm{d}A &= \iint_R f(x,y)\,\mathrm{d}A \pm \iint_R g(x,y)\,\mathrm{d}A \\
\iint_R f(x,y)\,\mathrm{d}A &\geq 0 \quad \text{if} \quad f(x,y) \geq 0 \text{ on } R \\
\iint_R f(x,y)\,\mathrm{d}A &\geq \iint_R g(x,y)\,\mathrm{d}A \quad \text{if} \quad f(x,y) \geq g(x,y) \text{ on } R \\
\iint_R f(x,y)\,\mathrm{d}A &= \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_2} f(x,y)\,\mathrm{d}A \\
&\qquad \text{if} \quad R = R_1 \cup R_2\,,\ R_1 \cap R_2 = \emptyset
\end{aligned}$$

Area by double integration

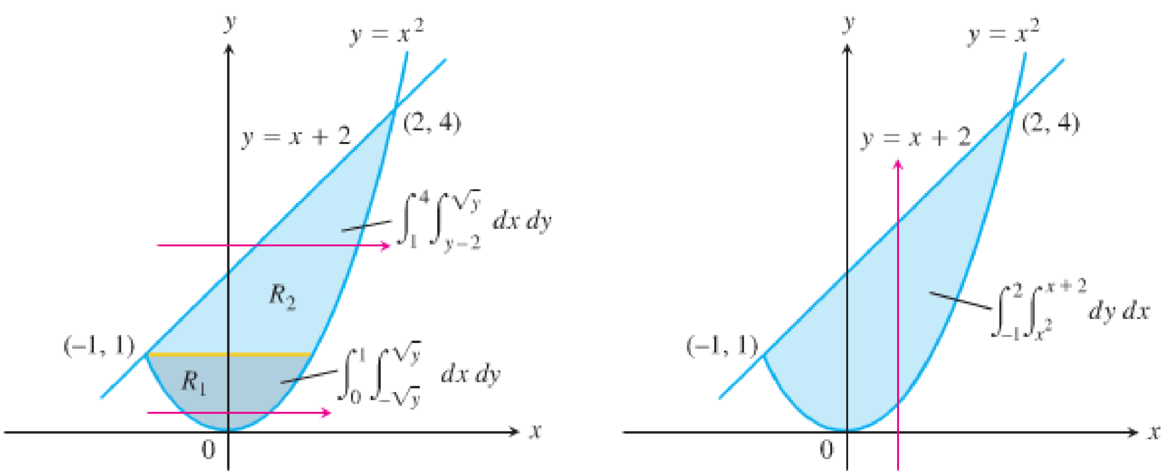
The **area** A of a closed, bounded plane region R is given by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k = \iint_R \mathrm{d}A\,,$$

which is equivalent to calculating $\iint_R f(x,y)\,\mathrm{d}A$ with $f(x,y) = 1$.



Example:
Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$. Determining the points of intersection is essential to determining the limits on the integrations. We can find the points by setting $x^2 = x + 2$ which gives $x^2 - x - 2 = (x+1)(x-2) = 0$, giving $x = -1$ and $x = 2$. The corresponding values of y are $y = 1$ and $y = 4$. So the points of intersection are $(-1, 1)$ and $(2, 4)$.



(a) (b)

If we use vertical strips (i.e. fix x and vary y) for the first integral we will not have to split up the region of integration. From the diagram we see that the lower and upper limits for the first integration are therefore $y = x^2$ and $y = x + 2$. This gives

$$\begin{aligned} A &= \int_{-1}^{2}\int_{x^2}^{x+2} \mathrm{d}y\,\mathrm{d}x = \int_{-1}^{2} [y]_{x^2}^{x+2}\,\mathrm{d}x \\ &= \int_{-1}^{2} \left(x + 2 - x^2\right)\,\mathrm{d}x = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}\,. \end{aligned}$$

Double integrals can also be used to find the **average value** of the function $f(x, y)$ over the region R, which is defined to be

$$\langle f \rangle = \frac{1}{\text{area of } R} \int\!\!\int_R f(x, y)\,\mathrm{d}A\,.$$

Example:
Find the average value of $f(x, y) = x\,\cos xy$ over the rectangle $R:\ 0 \le x \le \pi,\ 0 \le y \le 1$. The area of the region R is just π, the product of the length of the two sides of the rectangle. We just need to find $\int\!\!\int_R f(x, y)\,\mathrm{d}A$ and then divide by π.

$$\begin{aligned} \int_0^{\pi}\int_0^{1} x\,\cos xy\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{\pi} [\sin xy]_{y=0}^{y=1}\,\mathrm{d}x \\ &= \int_0^{\pi} (\sin x - 0)\,\mathrm{d}x = [-\cos x]_0^{\pi} = 1 + 1 = 2\,. \end{aligned}$$

Hence $\langle f \rangle = 2/\pi$.

MTH4101 Calculus II

Lecture notes for Week 10
Integration V

Thomas' Calculus, Sections 15.8 and 15.4

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Substitution in Double Integrals

For functions of one variable it is often useful to integrate by a change of variable, e.g. $x = x(u)$. Let us review **integration by substitution** in a slightly different way than you have learned in Calculus 1, namely *backwards*: Replace x by $x(u)$ and $\mathrm{d}x$ by $(\mathrm{d}x/\mathrm{d}u)\mathrm{d}u$. Then alter the x-limits to the u-limits with $a < b$ and $u_1 < u_2$. First, assume that $x(u)$ increases with u giving $a = x(u_1)$ and $b = x(u_2)$. Then

$$I = \int_{x=a}^{x=b} f(x)\,\mathrm{d}x = \int_{u=u_1}^{u=u_2} f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\,.$$

If $x(u)$ decreases with u we have $a = x(u_2)$ and $b = x(u_1)$, and the u-limits are reversed. With $u_1 < u_2$ we therefore have a change of sign:

$$I = \int_{x=a}^{x=b} f(x)\,\mathrm{d}x = -\int_{u=u_1}^{u=u_2} f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\,.$$

But $\mathrm{d}x/\mathrm{d}u < 0$ in this case, so we can combine both cases in one formula:

$$\int_{x=a}^{x=b} f(x)\,\mathrm{d}x = \int_{u=u_1}^{u=u_2} f(x(u))\left|\frac{\mathrm{d}x}{\mathrm{d}u}\right|\,\mathrm{d}u\,.$$

Note that on the right-hand side of this equation the function $f(x)$ is expressed as $f(x(u))$. Also, the right-hand side of the equation includes a *scaling factor* $|\mathrm{d}x/\mathrm{d}u|$, multiplying the $\mathrm{d}u$; this comes from transforming from $\mathrm{d}x$ to $\mathrm{d}u$.
For functions of two variables one would similarly expect that the change in variables

$$x = x(u,v), \quad y = y(u,v)$$

(for example, for polar coordinates $u = r$ and $v = \theta$) would result in a change in the area by a *scaling factor* S such that

$$\mathrm{d}x\,\mathrm{d}y = S\,\mathrm{d}u\,\mathrm{d}v\,.$$

As an example consider a *linear change* of coordinates:

$$x = x(u,v) = au + bv, \qquad y = y(u,v) = cu + dv$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

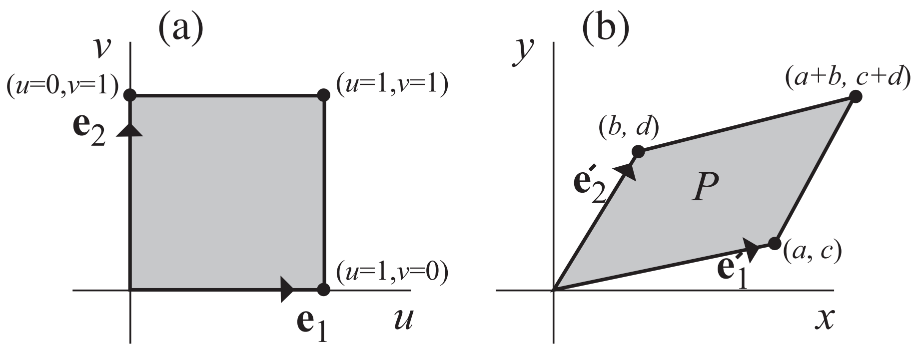
where a, b, c and d are constants.
Let us write $\mathbf{M}$ for the transformation matrix composed of a, b, c and d and recall that a unit square in (u,v) variables has sides

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{e}_2$$

To see what happens to this unit square under the transformation $\mathbf{M}$, just apply $\mathbf{M}$. This gives

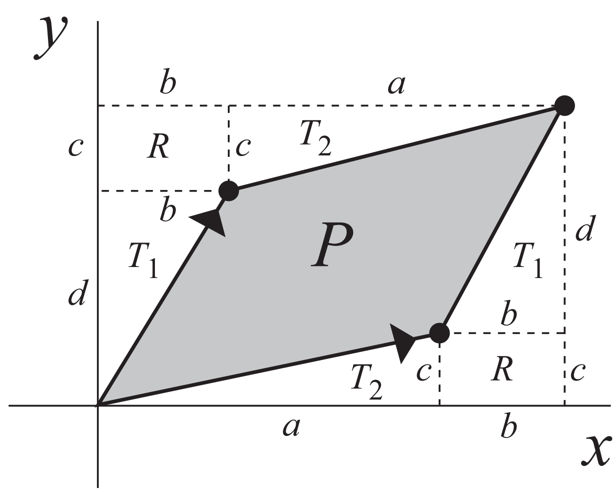
$$\begin{aligned} \mathbf{M}\,\mathbf{e}_1 &= \mathbf{e}_1' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \\ \mathbf{M}\,\mathbf{e}_2 &= \mathbf{e}_2' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \end{aligned}$$

where (a,c) and (b,d) represent the coordinates of the new corners in the (x,y) plane:



(note that the arrows are supposed to reach the respective points)

Therefore, under the transformation $\mathbf{M}$ we find that the unit square in (u, v) based on $\mathbf{e}_1$, $\mathbf{e}_2$ is transformed into the parallelogram in (x, y) based on $\mathbf{e}'_1$, $\mathbf{e}'_2$.
Note from the matrix and the diagram that the point $(1, 1)$ in (u, v) transforms to the point $(a + b, c + d)$ in (x, y).
Let us calculate the area of the parallelogram P:



We have

$$\begin{aligned} \text{Area } P \quad &= \quad [\text{Total area of rectangle}] \\ &\quad - [\text{Area of 2 pairs of equal triangles } T_1 \text{ and } T_2] \\ &\quad - [\text{Area of 2 rectangles } R] \ . \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Area } P \quad &= \quad (a + b)(c + d) - 2 \cdot \frac{1}{2}ac - 2 \cdot \frac{1}{2}bd - 2bc \\ &= \quad ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \mathbf{M} \end{aligned}$$

In view of the equation $\mathrm{d}x\,\mathrm{d}y = S\,\mathrm{d}u\,\mathrm{d}v$ one may understand this result such that the unit square of area $\mathrm{d}u\,\mathrm{d}v$ gets multiplied by a factor of $S = \det \mathbf{M}$. The same argument shows that a small rectangle of sides du and dv with area $du\,dv$ also gets multiplied by $S = \det \mathbf{M}$. Therefore, for a linear change of variables a small rectangular area $\mathrm{d}u\,\mathrm{d}v$ in the (u, v) plane is transformed into the parallelogram area $dx\,dy = \det \mathbf{M}\,du\,dv$ in the (x, y) plane.

Now let us consider a *nonlinear change* of coordinates. We take the transformation to have the form

$$x = x(u, v), \qquad y = y(u, v)\,,$$

where according to the total differential the increments in x and y are given by

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

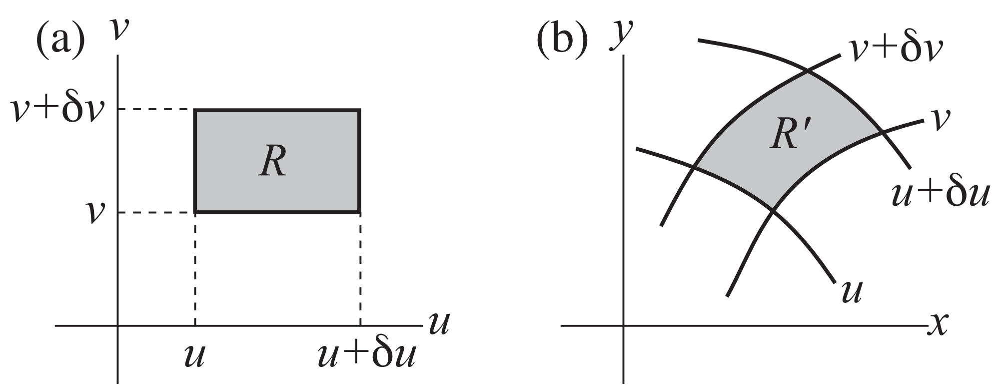
The **Jacobian matrix** is defined to be

$$\mathbf{M}(u, v) = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

and the **Jacobian determinant**, or **Jacobian**,

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \mathbf{M}(u, v)\,.$$

This suggests that for a nonlinear change of variables we also have that a rectangular area $\mathrm{d}u\,\mathrm{d}v$ in the (u, v) plane) is transformed into the (deformed) 'parallelogram' area $\det \mathbf{M} du\, dv$ in the (x, y) plane.



(with $du = \delta u$ $dv = \delta v$ and rather ignore the symbols in the right figure)

Therefore, the required transformation formula for double integrals under a change of variables is:

$$\int\!\!\int_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int\!\!\int_{R'} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v$$

where

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = |\det \mathbf{M}|$$

can be thought of as the scaling factor S.

Note that $|\cdot|$ denotes the absolute value of the determinant of the matrix, i.e., the modulus as in the one variable case. This may not be confused with the case of a matrix, where vertical lines on either side denote the determinant. For example, if we let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and

$$|\det \mathbf{A}| = |ad - bc| \,.$$

Example:
Evaluate the integral

$$I = \int\int_R (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$$

where R is a disk $x^2 + y^2 \leq a^2$, by changing to polar coordinates.
In polar coordinates we have

$$x = r\cos\theta, \qquad y = r\sin\theta\,.$$

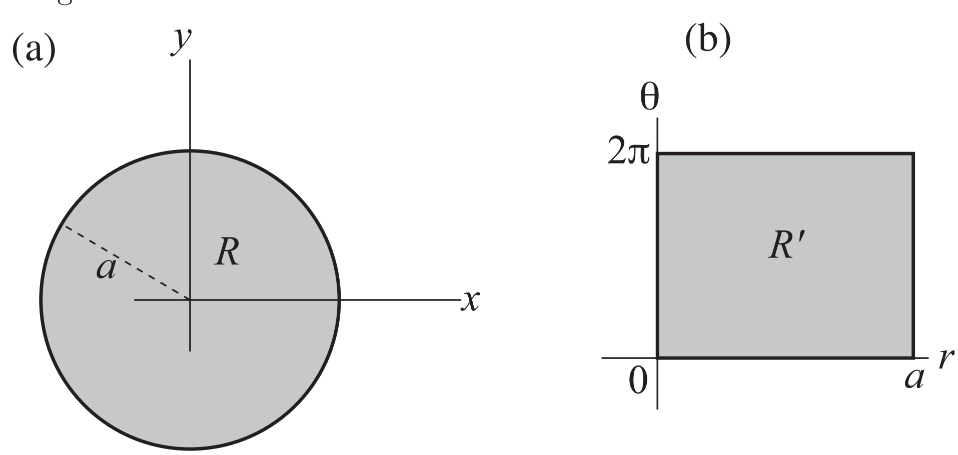
Therefore, taking $u = r$ and $v = \theta$, we can write the Jacobian matrix as

$$\mathbf{M} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and the Jacobian determinant is

$$\det \mathbf{M} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

where here and in the following we assume $r \geq 0$, so we do not need to take the absolute value. The original area R and the transformed area R' are shown below:



Note that the circle in the (x,y) plane transforms into a rectangle in the (r,θ) plane. Here R is the region given by $x^2 + y^2 \leq a^2$ and R' is the region given by $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$.

Therefore

$$I = \iint_R (x^2 + y^2)\, dx\, dy = \iint_{R'} (r^2)\,(r)\; dr\, d\theta$$

where the r^2 on the right-hand integral comes from the transformed $x^2 + y^2$ and the $r\, dr\, d\theta$ is from the transformed $dx\, dy$ with r coming from the Jacobian determinant $\det \mathbf{M}$. Hence

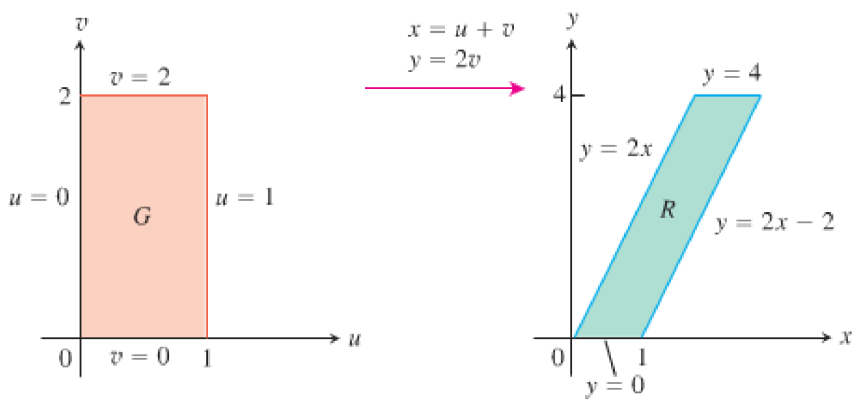
$$I = \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=2\pi} r^3\, d\theta\, dr = \left(\int_{r=0}^{r=a} r^3\, dr\right)\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) = \frac{\pi a^4}{2},$$

where we note that the integral is separable.

Example:
Evaluate the double integral

$$\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x - y}{2}\, dx\, dy$$

by applying the transformation $u = (2x - y)/2$, $v = y/2$ and integrating over an appropriate region of the u-v plane.
The region R in the x-y-plane looks as follows:



The corresponding region G in the u-v plane can be obtained by first writing x and y in terms of u and v as $x = u + v$ and $y = 2v$.
The boundaries of G are then found by substituting these equations for the boundaries of R:

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

The Jacobian of the transformation is

$$\det \mathbf{M}(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} \partial(u+v)/\partial u & \partial(u+v)/\partial v \\ \partial(2v)/\partial u & \partial(2v)/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2\,.$$

and we get

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} \mathrm{d}x\,\mathrm{d}y = \int_{v=0}^{v=2}\int_{u=0}^{u=1} u\,|\det \mathbf{M}(u,v)|\;\mathrm{d}u\,\mathrm{d}v = \int_{v=0}^{v=2}\int_{u=0}^{u=1} u\cdot 2\,\mathrm{d}u\,\mathrm{d}v = 2$$

Note that for invertible transformations

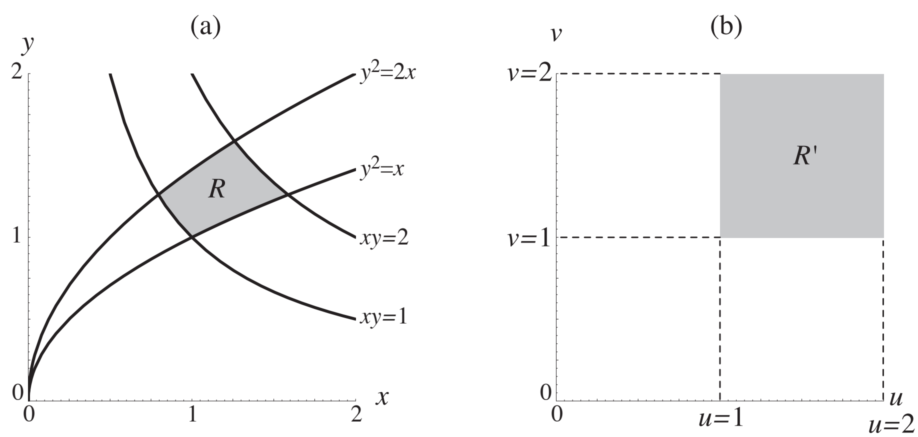
$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}, \tag{1}$$

as you have seen in Calculus 1 for a function of one variable. This can be useful in solving some problems.

Example:
Evaluate the integral

$$I = \int\int_R 1\cdot \mathrm{d}x\,\mathrm{d}y$$

(i.e. the area of the region R) where R is enclosed by $y^2 = x$, $y^2 = 2x$, $xy = 1$ and $xy = 2$.



To solve the integral consider the change of variables defined by

$$u = y^2/x, \qquad v = xy\,.$$

Then we can write the four bounding curves as

$$y^2 = x \Leftrightarrow u = 1, \quad y^2 = 2x \Leftrightarrow u = 2, \quad xy = 1 \Leftrightarrow v = 1, \quad xy = 2 \Leftrightarrow v = 2\,.$$

So the region becomes a square (the region R' in part (b) of the above figure).
Now, for the Jacobian determinant it is easier to use Eq. (1) above. So, to calculate $\partial(x,y)/\partial(u,v)$ we first calculate $\partial(u,v)/\partial(x,y)$ and then take the inverse. Using $u = y^2/x$ and $v = xy$ we have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} -y^2/x^2 & 2y/x \\ y & x \end{vmatrix} = -3\frac{y^2}{x} = -3u\,.$$

Therefore, using Eq. (1),

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = -\frac{1}{3u}\,.$$

Hence

$$\begin{aligned} I &= \iint_R 1 \cdot \mathrm{d}x\,\mathrm{d}y = \iint_{R'} 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \\ &= \iint_{R'} \left|-\frac{1}{3u}\right| \mathrm{d}u\,\mathrm{d}v = \frac{1}{3}\int_{u=1}^{u=2}\int_{v=1}^{v=2} \frac{1}{u}\,\mathrm{d}v\,\mathrm{d}u \\ &= \frac{1}{3}\int_{u=1}^{u=2} \left[\frac{v}{u}\right]_{v=1}^{v=2} \mathrm{d}u \\ &= \frac{1}{3}\int_{u=1}^{u=2} \frac{1}{u}\,\mathrm{d}u = \frac{1}{3}\left[\ln u\right]_{u=1}^{u=2} = \frac{\ln 2}{3} \end{aligned}$$

Optional reading assignment:
Work yourself through the following example
(not in Thomas' Calculus).

Example:
Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}\mathrm{d}x\,.$$

If we call this integral I, we can write

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\mathrm{d}y\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\mathrm{d}x\,\mathrm{d}y\,.$$

Now transform to polar coordinates with the limits $0 \le r < \infty$ and $-\pi \le \theta \le \pi$. This gives

$$\begin{aligned} I^2 &= \int_{-\pi}^{\pi}\int_0^{\infty} e^{-r^2/2}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}r\,\mathrm{d}\theta = \int_{-\pi}^{\pi}\int_0^{\infty} r\,e^{-r^2/2}\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi}\left[-e^{-r^2/2}\right]_0^{\infty} \mathrm{d}\theta = \int_{-\pi}^{\pi} ((0)-(-1))\,\mathrm{d}\theta = \int_{-\pi}^{\pi} \mathrm{d}\theta = 2\pi\,. \end{aligned}$$

Hence $I = \sqrt{2\pi}$.
Note that the probability density function for a normal (or Gaussian) distribution is

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

for mean μ and standard deviation σ. If we write $t = (x-\mu)/\sigma$ (i.e. express the displacement from the mean in terms of the standard deviation) then the total probability is

$$\begin{aligned} P &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\sigma\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,\mathrm{d}t = 1\,. \quad \text{(by our previous result)} \end{aligned}$$

MTH4101 Calculus II

Lecture notes for Week 11
Integration V and A First Look at Differential Equations

Thomas' Calculus, Sections 15.5, 15.8 and 7.4

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Triple Integrals

Triple integrals are integrations where the region of integration is a **volume**. The basic concepts are similar to those we introduced for two-dimensional (double) integrals, but now we have for the *Riemann sum*

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k\ ,$$

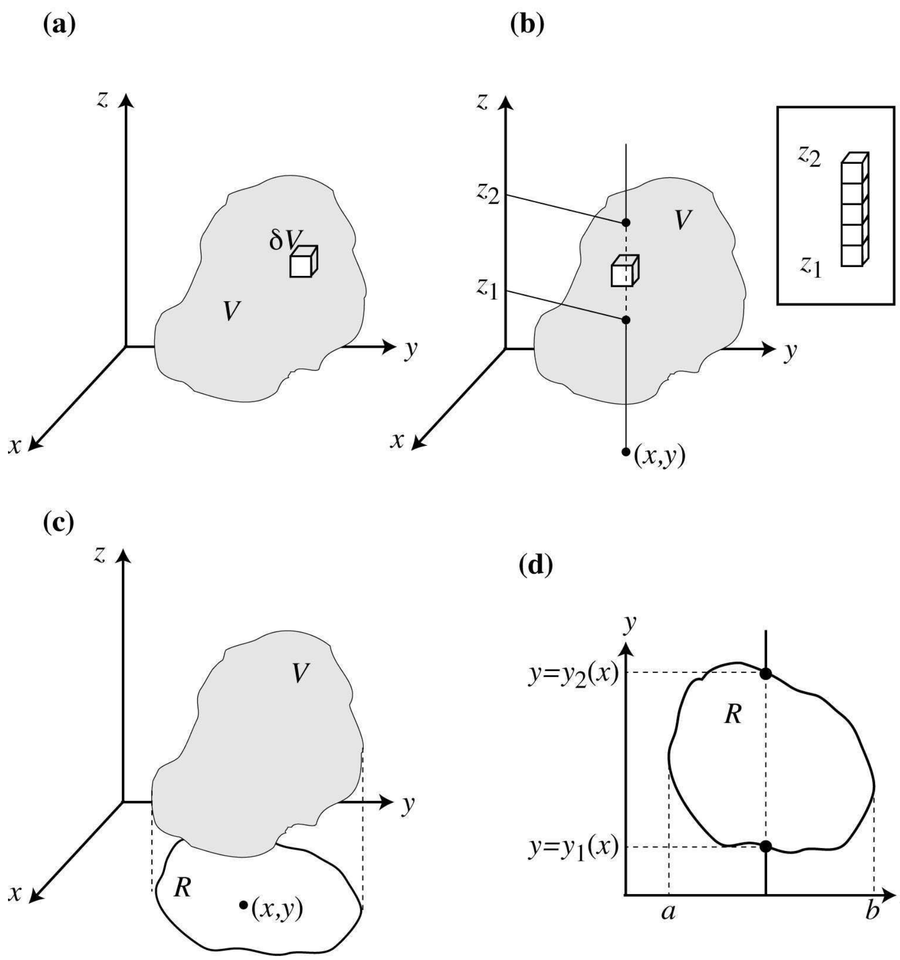
where $\Delta V_k = \Delta x_k\,\Delta y_k\,\Delta z_k$ are now small volumes at the point x_k, y_k, z_k, see (a) in the figure below (where it is $\Delta V_k = \delta V$).

The limit as the size of the volume element $\Delta V_k \to 0$ (as $n \to \infty$) is written as (if it exists)

$$\lim_{n\to\infty} S_n = \int\!\!\int\!\!\int_V f(x,y,z)\,\mathrm{d}V = \int\!\!\int\!\!\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ ,$$

where V is the three-dimensional region being integrated over.

The integrals are, as in the two-dimensional case, evaluated by repeated integration where we integrate over one variable at a time. For example, we could start by integrating over z first, see (b) in the figure below. The procedure is as follows:



1. **Sketch the region of integration** (if possible), see (a).
2. **Choose a direction of integration and integrate:** For example, fix a point (x, y) and integrate over the allowed values of z in the region V. The z-integral limits are the small, filled circles at the bottom and the top of the dashed line with, say, $z = z_1(x, y)$ at the bottom and $z = z_2(x, y)$ at the top as shown in (b). Therefore we are summing vertically over the boxes shown in (b).
3. This result depends on the choice of (x, y) and is defined in the region R of the (x, y) plane which is the projection of V onto this plane as shown in (c). This now **identifies the region in the (x, y) plane over which we must do the x and y integrations.**
4. Now we can **take the double integral** of the result of the z-integration **over the region R in the (x, y) plane**, see (d).

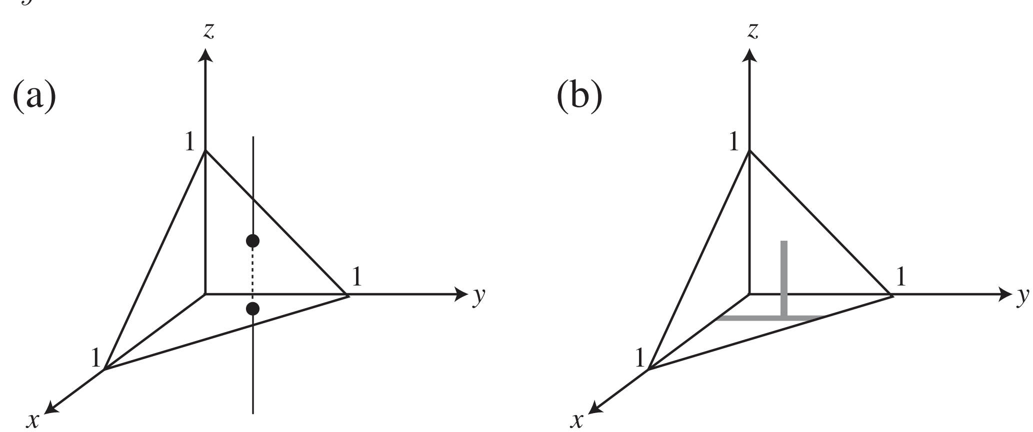
Therefore

$$\int\int\int_V f(x, y, z)\,\mathrm{d}V = \int_{x=a}^{x=b}\int_{y=y_1(x)}^{y=y_2(x)}\int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,.$$

Example:
Evaluate

$$\int\int\int_T f(x, y, z)\,\mathrm{d}V$$

over the tetrahedron T bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. Note that the plane $x + y + z = 1$ passes through $x = 1$ (putting $y = z = 0$) and similarly through $y = 1$ and $z = 1$ as shown below:



Now evidently for fixed (x, y) the z-limits are the heavy dots corresponding to $z = 0$ at the bottom and $z = 1 - x - y$ at the top. This gives our z-limits.
The projection R of T onto the (x, y) plane is the triangle on which the tetrahedron rests, i.e. the triangle given by $x = 0$, $y = 0$ and $x + y = 1$ (obtained by setting $z = 0$). So

$$I = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,.$$

For example, if $f(x, y, z) = 1$ then

$$I = \iiint_T 1\cdot\mathrm{d}V = \iiint_T \mathrm{d}V = \textbf{volume of } T\,.$$

Therefore, in this case

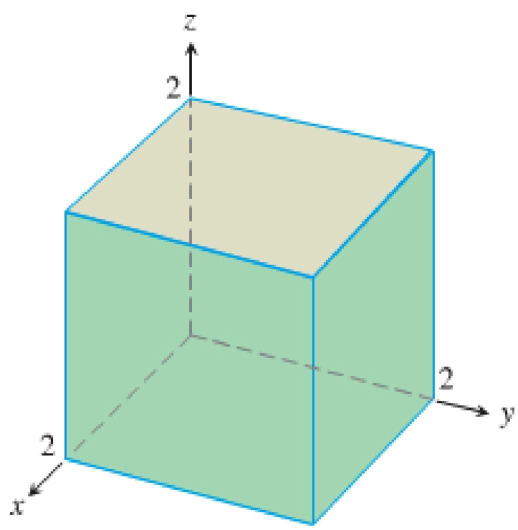
$$\begin{aligned} I &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} [z]_{z=0}^{z=1-x-y}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} (1-x-y)\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{x=1}\left[y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1-x}\mathrm{d}x \\ &= \int_{x=0}^{x=1}\frac{(1-x)^2}{2}\,\mathrm{d}x = \frac{1}{6} \end{aligned}$$

and this is the volume of the tetrahedron.

Triple integrals can be used to find the **average value of a function** $f(x, y, z)$ **over a volume** D defined as

$$\langle f(x,y,z)\rangle = \frac{1}{\text{volume of } D}\iiint_D f(x,y,z)\,\mathrm{d}V$$

Example:
Find the average value of $f(x, y, z) = xyz$ over the cube bounded by the planes $x = 2$, $y = 2$ and $z = 2$ in the first octant.



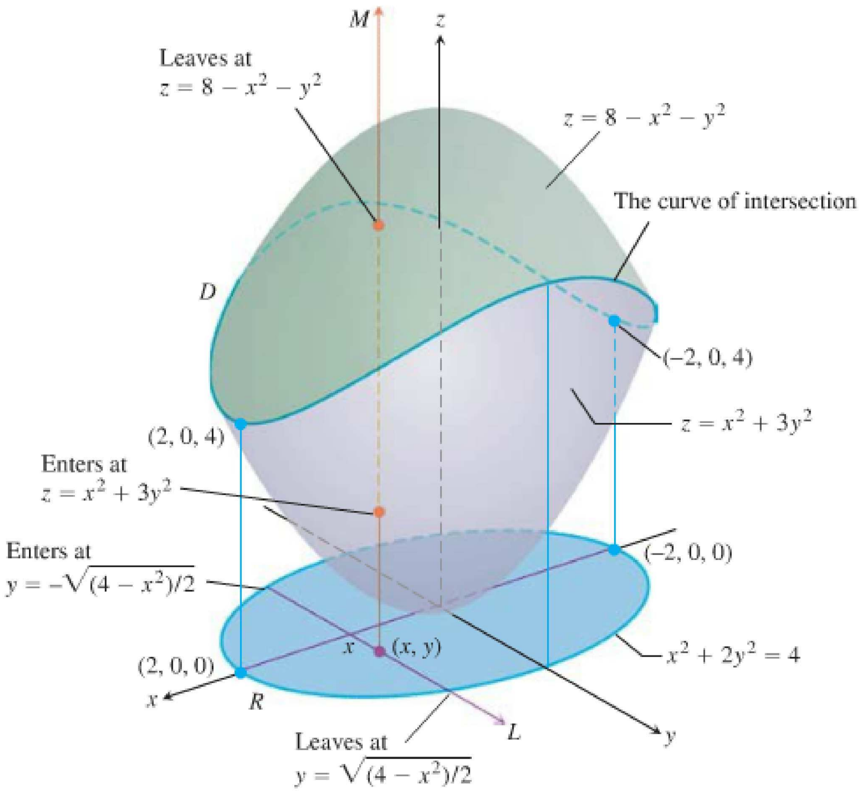
The volume of the cube is $2^3 = 8$. The integral is

$$\int_0^2\int_0^2\int_0^2 xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^2 x\,\mathrm{d}x\int_0^2 y\,\mathrm{d}y\int_0^2 z\,\mathrm{d}z = \left(\int_0^2 x\,\mathrm{d}x\right)^3 = \left(\left[\frac{x^2}{2}\right]_0^2\right)^3 = 8\,,$$

because the function is **separable** and the region is **cubic**. Therefore the average value of $f(x, y, z) = xyz$ over the cube is

$$\langle f(x,y,z)\rangle = \frac{1}{\text{volume of cube}}\iiint_{\text{cube}} xyz\,\mathrm{d}V = \frac{1}{8}\cdot 8 = 1\,.$$

Example:
Find the volume V of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$. The two surfaces intersect at $x^2 + 3y^2 = 8 - x^2 - y^2$. The equation $x^2 + 2y^2 = 4$ thus defines the boundary of the projection of D onto the x-y plane, which is the ellipse R:



We now have all the information necessary to do the integral:

$$
\begin{aligned}
V &= \iiint_D \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8 - 2x^2 - 4y^2\right)\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2} \left[(8-2x^2)y - \frac{4}{3}y^3\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \mathrm{d}x \\
&= \int_{-2}^{2} \left(2(8-2x^2)\sqrt{\frac{(4-x^2)}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right)\mathrm{d}x \\
&= \int_{-2}^{2} \left(8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right)\mathrm{d}x \\
&= \frac{4\sqrt{2}}{3}\int_{-2}^{2} \left(4-x^2\right)^{3/2}\mathrm{d}x \quad [\text{since } (8-8/3)/(2^{3/2}) = 4\sqrt{2}/3] \\
&= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} 4^{3/2}\left(\cos^2\theta\right)^{3/2}\cdot 2\,\cos\theta\,\mathrm{d}\theta \quad [\text{using subst. } x = 2\sin\theta]
\end{aligned}
$$

$$= \frac{4\sqrt{2}}{3} \cdot 16 \int_{-\pi/2}^{\pi/2} \cos^4\theta \,\mathrm{d}\theta = \frac{4\sqrt{2}}{3} \cdot 16 \int_{-\pi/2}^{\pi/2} \frac{1}{8}\left(3 + 4\cos 2\theta + \cos 4\theta\right)\,\mathrm{d}\theta$$
$$= \frac{4\sqrt{2}}{3} \cdot 2 \left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{4\sqrt{2}}{3} \cdot 2 \cdot 3 \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 8\sqrt{2}\,\pi\,.$$

Substitution in Triple Integrals

Changing variables in triple integrals is similar to the procedure used for double integrals. Suppose

$$x = x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w)\,.$$

We define the **Jacobian matrix** for change of variables from (x, y, z) to (u, v, w) to be

$$\mathbf{M}(u,v,w) = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{pmatrix}.$$

and the corresponding **Jacobian determinant** as

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \mathbf{M}$$

such that the transformation for volume is

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,.$$

As before, for invertible transformations we have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \left(\frac{\partial(u,v,w)}{\partial(x,y,z)}\right)^{-1}.$$

The integral under change of variables becomes

$$\iiint_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z =$$
$$\iiint_{V'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,,$$

where V' is the transformed volume in (u, v, w) coordinates.

Example:
A volume V in the first octant is bounded by the six surfaces $xy = 1$, $xy = 2$, $yz = 1$, $yz = 2$, $xz = 1$ and $xz = 2$. Using the change of variables

$$r = xy, \qquad s = yz, \qquad t = xz$$

and by assuming that this tranformation is invertible on V, evaluate the integral

$$\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

The new limits are $r = 1$ to $r = 2$, $s = 1$ to $s = 2$ and $t = 1$ to $t = 2$. The Jacobian determinant is

$$\begin{aligned} \frac{\partial(r,s,t)}{\partial(x,y,z)} &= \begin{vmatrix} \partial r/\partial x & \partial r/\partial y & \partial r/\partial z \\ \partial s/\partial x & \partial s/\partial y & \partial s/\partial z \\ \partial t/\partial x & \partial t/\partial y & \partial t/\partial z \end{vmatrix} = \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix} \\ &= y\begin{vmatrix} z & y \\ 0 & x \end{vmatrix} - x\begin{vmatrix} 0 & y \\ z & x \end{vmatrix} \\ &= y(xz) + x(yz) = 2xyz\,. \end{aligned}$$

But

$$\frac{\partial(x,y,z)}{\partial(r,s,t)} = \left(\frac{\partial(r,s,t)}{\partial(x,y,z)}\right)^{-1} = \frac{1}{2xyz}$$

and so

$$\begin{aligned} \iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint_{V'} xyz\left|\frac{1}{2xyz}\right|\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t = \int_{t=1}^{t=2}\int_{s=1}^{s=2}\int_{r=1}^{r=2}\frac{1}{2}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{2}\,[r]_1^2\,[s]_1^2\,[t]_1^2 = \frac{1}{2}\cdot 1\cdot 1\cdot 1 = \frac{1}{2}\,. \end{aligned}$$

First-order differential equations and their solutions

You have learned in Calculus 1 that a function y is an **antiderivative** of a function f if

$$\frac{dy}{dx} = f(x)\,.$$

Finding an antiderivative for a given function $f(x)$ means finding a function $y(x)$ that solves this equation. This is an example of a **differential equation**, an equation involving the derivative of an unknown function y.
Using $f = f(x)$ on the right hand side the above equation defines a special case of a differential equation, and you already know of how to solve it. More generally, a **first-order differential equation** is of the form

$$\frac{dy}{dx} = f(x,y)\,,$$

where $f = f(x,y)$ is a function of *both* the independent variable x and the dependent variable y defined on a region in the xy-plane. The equation is of *first-order*, because it involves only the first derivative dy/dx (and not higher-order derivatives).
A **solution** of this equation is a differentiable function $y = y(x)$ defined on an interval I of x-values such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on I. The **general solution** to such an equation is a solution that contains all possible solutions. As you will see in a moment (recall solving an indefinite integral), it always contains an arbitrary (integration) constant. This constant can be fixed by specifying an **initial condition**

$$y(x_0) = y_0\,.$$

The combination of a differential equation and an initial condition is called an **initial value problem**. The solution satisfying the initial condition $y(x_0) = y_0$ is the **particular solution** $y = y(x)$ whose graph passes through the point (x_0, y_0) in the xy-plane.

Example:
Show that

$$y = (x+1) - \frac{1}{3}e^x$$

solves the first-order initial value problem

$$\frac{dy}{dx} = y - x \ , \ y(0) = \frac{2}{3} \ .$$

Differentiate $y(x)$ to calculate the left hand side:

$$\frac{dy}{dx} = 1 - \frac{1}{3}e^x \ .$$

Now check for the right hand side:

$$y - x = (x+1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x \ .$$

Both are equal, hence y solves the given equation. Since

$$y(0) = 1 - \frac{1}{3} = \frac{2}{3}$$

it also satisifies the initial condition.

Separable differential equations

An important class of first-order differential equation can be motivated by an

Example:
Solve the first-order differential equation.

$$\frac{dy}{dx} = ky \ ,$$

where the function $f(y) = ky$ on the right hand side only depends on y and is furthermore *linear* in y with a constant $k \in \mathbb{R}$.

By assuming that $y \neq 0$ we can write

$$\frac{1}{y}\frac{dy}{dx} = k \ .$$

If we treat dy/dx as a quotient of differentials dy and dx (by which strictly speaking we modify the problem - it defines a derivative!), we obtain

$$\frac{1}{y}dy = kdx$$

Now we can integrate:

$$\begin{aligned}
\int \frac{1}{y} dy &= \int k dx \\
\ln|y| &= kx + C \ , \ C = \text{const.} \\
|y| &= e^{kx} e^{C} \\
y &= A e^{kx} \text{ with } A = \pm e^{C} \ .
\end{aligned}$$

We see that the solution of this differential equation undergoes *exponential change.*

Queen Mary
University of London

MTH4101 Calculus II

Lecture notes for Week 12
A First Look at Differential Equations

Thomas' Calculus, Sections 7.4, 9.1 and 9.2

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The example to the end of the previous lecture is a special case of what is called a **separable differential equation** $y' = dy/dx = f(x, y)$, where f can be expressed as a product of a function of x and a function of y. We can always try to solve such an equation by **separation of variables**:

$$\begin{aligned} y' &= g(x)h(y) \\ \frac{1}{h(y)}y' &= g(x) \end{aligned}$$

The detailed justification of what we have done in the previous example is integration by substitution

$$\int \frac{1}{h(y)}y'dx = \int g(x)dx$$

using $u = y(x)$,

$$\int \frac{1}{h(y)}dy = \int g(x)dx\,.$$

After completing the integrations on both sides (which may not always be possible), we obtain the solution y as a function of x in *implicit form*.

Example:
Solve the initial value problem

$$\frac{dy}{dx} = (1+y)e^x\,,\ y > -1\,,\ y(0) = 0\,.$$

By separation of variables and integration we obtain

$$\begin{aligned} \int \frac{dy}{1+y} &= \int e^x dx \\ \ln(1+y) &= e^x + C\,,\ C = \text{const.}\,, \end{aligned}$$

which gives the solution y in *implicit form*. The constant C is determined by using the initial condition $y(0) = 0$:

$$\ln(1+0) = 0 = e^0 + C = 1 + C$$

giving $C = -1$. The *explicit solution* of the initial value problem is obtained as

$$y(x) = e^{e^x - 1} - 1\,.$$

First-order linear differential equations and the integrating factor

A first-order **linear** differential equation is one that can be written in the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x)\,, \tag{1}$$

where $P = P(x)$ and $Q = Q(x)$ are continuous functions of x. It is linear (in y), because y and its derivative dy/dx occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function (such as $\sin y, \exp(y)$, etc.).

Example:
Put the equation

$$x\frac{dy}{dx} = x^2 + 3y\ ,\ x > 0\ ,$$

in standard form.

$$\begin{aligned} \frac{dy}{dx} &= x + \frac{3}{x}y \\ \frac{dy}{dx} - \frac{3}{x}y &= x\ . \end{aligned}$$

Hence, $P(x) = -3/x$ and $Q(x) = x$.
An equation in standard form can be solved as follows: Multiply it by a function $v = v(x)$,

$$v\frac{dy}{dx} + vPy = vQ\ .$$

Now let's play a little trick: *If we choose v such that it transforms the left-hand side into the derivative of the product vy*, that is,

$$v\frac{dy}{dx} + vPy = \frac{d}{dx}(vy)\ ,$$

we can write

$$\frac{d}{dx}(vy) = vQ$$

and easily solve by integration:

$$\begin{aligned} vy &= \int vQ\,dx \\ y &= \frac{1}{v}\int vQ\,dx\ . \end{aligned} \tag{2}$$

We call $v(x)$ an **integrating factor**, because it makes the linear differential equation integrable.
Now, did we mysteriously get rid of $P(x)$ by solving our differential equation? Not quite, because we still have to determine $v(x)$ by solving the previously imposed equation

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + vPy\ .$$

Apply the product rule and simplify:

$$\begin{aligned} \frac{dv}{dx}y + v\frac{dy}{dx} &= v\frac{dy}{dx} + Pvy \\ \frac{dv}{dx}y &= Pvy\ . \end{aligned}$$

But this equation will hold if

$$\frac{dv}{dx} = Pv\ ,$$

which is separable, $P = P(x)$:

$$\int \frac{dv}{v} = \int P dx$$

Without loss of generality we may assume that $v > 0$,

$$\begin{aligned} \ln v &= \int P dx \\ v &= e^{\int P dx} . \end{aligned} \tag{3}$$

The general solution to Eq.(1) is thus given by Eq.(2) together with Eq.(3). Note that any antiderivative of P works for Eq.(3).

Example:
Solve

$$x\frac{dy}{dx} = x^2 + 3y \ , \ x > 0 \, .$$

The **integrating factor method** consists of three steps:
1. Put the equation in **standard form**:

$$\frac{dy}{dx} - \frac{3}{x}y = x \ ,$$

hence $P(x) = -3/x$.
2. Calculate the **integrating factor**:

$$v = e^{\int P(x)dx} = e^{\int (-3/x)dx}$$

by choosing the simplest constant of integration, $C = 0$, and noting that $x > 0$:

$$v = e^{-3lnx} = e^{lnx^{-3}} = x^{-3} \, .$$

3. **Multiply and integrate:** Multiply both sides of the standard form by $v(x)$,

$$\frac{1}{x^3}\left(\frac{dy}{dx} - \frac{3}{x}y\right) = \frac{1}{x^3}x = \frac{1}{x^2} \ ,$$

and remember that the left hand side *always* integrates into the product vy, as we have designed it to be:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} \\ \frac{1}{x^3}y &= \int \frac{1}{x^2}dx \\ \frac{1}{x^3}y &= -\frac{1}{x} + C \, . \end{aligned}$$

Solving this equation for y gives the general solution

$$y(x) = -x^2 + Cx^3 \ , \ x > 0 \, .$$

THE END