Alternative Proof of Lemma 2.9. This proof follows the same lines as the first one, but is expressed in terms of elementary matrices rather than row and column operations.

First note that if A is an $m \times n$ matrix and $u = [c_1, c_2, \dots, c_n]^\top$ an n-vector, then Au can be thought of as expressing a linear combination of the columns of A. Specifically, $Au = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$, where m-vectors $v_1, v_2, \dots, v_n$ are the n columns of A, taken in order.

- Suppose A' is obtained from matrix A by some elementary column operation. Equivalently, $A' = AC$ for some elementary matrix C. Consider any vector v in the column space of A'; as we observed, the condition for v to be in the column space is that there exists a vector u such that $v = Au$. Then $v = A'u = (AC)u = A(Cu) = Au'$, where $u' = Cu$. Thus v also is in the column space of A. It follows that the column space of A' is contained in the column space of A. Finally, elementary column operations are invertible, so the inclusion holds also in the other direction. We deduce that the row spaces of A and A' are equal.

(b) Follows by symmetry from (a).

(c) Suppose A' is obtained from matrix A by some elementary row operation. Equivalently, $A' = RA$ for some elementary matrix R. Consider any linear dependancy among the columns of A, say $Au = \mathbf{0}$. Then $A'u = (RA)u = R(Au) = R\,\mathbf{0} = \mathbf{0}$ and so the same linear dependency exists among the columns of A'. Let $S \subseteq \{1, \dots, n\}$ be a subset of columns. If the columns of A indexed by S are linearly dependent then the same columns in A' are linearly dependent. Equivalently, if the S-indexed columns of A' are linearly independent, then so are the S-indexed columns of A. Let k be the column rank of A'. Choose a basis for the column space among the columns of A'. These columns are linearly independent in A' and hence also in A. It follows that the rank of the column space of A is at least k. Thus the column rank of A is at least as large as that of A'. As before, row operations are invertible, so the inequality holds also in the other direction.

(d) Follows from (c) by symmetry.

$\square$