# Chapter 3

# Determinants

We recall the Leibniz formula for the determinant of a square matrix, and show that the function it defines is the unique function on square matrices satisfying certain nice properties. This provides an axiomatic definition of the determinant, and demystifies, to a certain extent, why the determinant is defined the way it is. We examine methods of calculating the determinant and some of its properties. We study two polynomials associated with a matrix, the minimal and characteristic polynomials. Finally we prove the Cayley-Hamilton Theorem, that states that every matrix satisfies its own characteristic equation.

The determinant is a function defined on square matrices; its value is a scalar. It has some very important properties: perhaps most important is the fact that a matrix is invertible if and only if its determinant is not equal to zero.

We denote the determinant function by det, so that $\det(A)$ is the determinant of $A$. For a matrix written out as an array, the determinant is denoted by replacing the square brackets by vertical bars:

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}.$$

The formula for the determinant involves some background notation.

**Definition 3.1.** A *permutation* of $\{1, \dots, n\}$ is a bijection from the set $\{1, \dots, n\}$ to itself. The *symmetric group* $S_n$ consists of all permutations of the set $\{1, \dots, n\}$. (There are $n!$ such permutations.) For any permutation $\pi \in S_n$, there is a number $\operatorname{sign}(\pi) = \pm 1$, computed as follows: write $\pi$ as a product of disjoint cycles; if there are $k$ cycles (including cycles of length 1), then $\operatorname{sign}(\pi) = (-1)^{n-k}$. A *transposition* is a permutation which interchanges two symbols and leaves all the others fixed. Thus, if $\tau$ is a transposition, then $\operatorname{sign}(\tau) = -1$.

The last fact holds because a transposition has one cycle of size 2 and $n-2$ cycles of size 1, so $n-1$ altogether; so $\operatorname{sign}(\tau) = (-1)^{n-(n-1)} = -1$. We need one more fact about signs: if $\pi$ is any permutation and $\tau$ is a transposition, then $\operatorname{sign}(\pi\tau) = \operatorname{sign}(\tau\pi) = -\operatorname{sign}(\pi)$, where $\pi\tau$ denotes the composition of $\pi$ and $\tau$ (apply first $\tau$, then $\pi$).

**Definition 3.2.** Let $A = (a_{ij})$ be an $n \times n$ matrix over $\mathbb{K}$. The *determinant* of $A$ is defined by the Leibniz formula

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}.$$

This gives us a nice mathematical formula for the determinant of a matrix. Unfortunately, it is a terrible formula for practical computation, since it involves working out $n!$ terms, each a product of matrix entries, and adding them up with $+$ and $-$ signs. For $n$ of moderate size, this will take a very long time! (For example, $10! = 3628800$.)

Let's come at this from another direction. Consider the following three properties of a function $D$ defined on $n \times n$ matrices.

(D1) For every $1 \leq i \leq n$, $D(A)$ is linear in the $i$th row of the matrix $A$. (We'll spell out below what this means.)

(D2) If $A$ has two equal rows, then $D(A) = 0$.

(D3) $D(I_n) = 1$, where $I_n$ is the $n \times n$ identity matrix.

Some clarification of property (D1). Suppose we have any matrices $A$ and $B$ such that $B$ agrees with $A$, except that row $i$ is multiplied by some scalar $c$. Thus,

$$A = \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \\ v_i \\ v_{i+1} \\ \vdots \\ v_n \end{bmatrix} \qquad B = \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \\ cv_i \\ v_{i+1} \\ \vdots \\ v_n \end{bmatrix}, \tag{3.1}$$

where $v_1, \ldots, v_n$ are *row* vectors. Then (D1) legislates that $D(B) = cD(A)$. Furthermore, suppose we have three matrices $A$, $A'$ and $B$, such that $A$ and $A'$ agree except at the $i$th row, and such that the $i$th row of $B$ is the sum of the corresponding rows of $A$ and $A'$:

$$A = \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \\ v_i \\ v_{i+1} \\ \vdots \\ v_n \end{bmatrix} \qquad A' = \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \\ v_i' \\ v_{i+1} \\ \vdots \\ v_n \end{bmatrix} \qquad B = \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \\ v_i + v_i' \\ v_{i+1} \\ \vdots \\ v_n \end{bmatrix}. \tag{3.2}$$

Then (D1) legislates that $D(B) = D(A) + D(A')$.

Why are these natural? Well, condition (D1) says that if we fix all the entries of $A$ apart from those in the $i$th row, then $D$ is some linear function of the remaining entries $a_{1i}, \ldots, a_{ni}$. This is a *linear* algebra course, so this property seems reasonable enough. A matrix $A$ with two equal rows has rank less than $n$. Property (D2) says that the function $D(A)$ is zero on at least some (in fact all) matrices of rank less than $n$. If we are looking for a function that is non-zero exactly for invertible matrices, this is a reasonable condition to impose. The conditions (D1) and (D2) cannot define a unique function, since if $D$ satisfies (D1) and (D2) then so does any multiple of $D$. So if we want to pin down the function $D$ precisely, we need some condition like (D3) to fix the function at a particular point.

If we believe (D1)–(D3) are nice conditions, then the determinant is a nice function.

**Lemma 3.3.** *The function* det() *satisfies (D1)–(D3).*

*Proof.* (D1) Suppose $A = (a_{k\ell})$ is an $n \times n$ matrix, and $A'$ and $B$ are matrices agreeing with $A$ apart from in the $i$th row. Furthermore, suppose matrices $A$, $A'$ and $B$ are related as in (3.2): thus the $i$th row of $B$ is the sum of the $i$th rows of $A$ and $A'$. Then, denoting the $i$th row of $A'$ by $a'_{i1}, a'_{i2}, \ldots, a'_{in}$, we obtain, by the Leibniz formula,

$$\begin{aligned}\det(B) &= \sum_{\pi\in S_n} \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} \underbrace{(a_{i,\pi(i)} + a'_{i,\pi(i)})}_{b_{i,\pi(i)}} a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)} \\ &= \sum_{\pi\in S_n} \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} a_{i,\pi(i)} a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)} \\ &\qquad + \sum_{\pi\in S_n} \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} a'_{i,\pi(i)} a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)} \\ &= \det(A) + \det(A').\end{aligned}$$

The case (3.1), where $B$ is obtained from $A$ by multiplying the $i$th row of $A$ by $c$, is similar, but easier, and is left as an exercise. Thus (D1) holds for the determinant.

(D2) Suppose that the $i$th and $j$th rows of $A$ are equal. Let $\tau$ be the transposition that interchanges $i$ and $j$ and leaves the other numbers fixed. Then,

$$a_{i,\pi\tau(i)} = a_{i,\pi(j)} = a_{j,\pi(j)} \quad \text{and} \quad a_{j,\pi\tau(j)} = a_{j,\pi(i)} = a_{i,\pi(i)},$$

where the second equality in each case uses the fact that the $i$th and $j$th rows of $A$ are identical. For any $k \notin \{i, j\}$ we naturally have $a_{k,\pi\tau(k)} = a_{k,\pi(k)}$. Thus, we see that the products

$$a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)} \quad \text{and} \quad a_{1,\pi\tau(1)} a_{2,\pi\tau(2)} \cdots a_{n,\pi\tau(n)}$$

are equal. But $\operatorname{sign}(\pi\tau) = -\operatorname{sign}(\pi)$. So the corresponding terms in the formula for the determinant cancel one another. The elements of $S_n$ can be divided up into $n!/2$ pairs of the form $\{\pi, \pi\tau\}$. As we have seen, each pair of terms in the formula cancel out. We conclude that $\det(A) = 0$. Thus (D2) holds.

(D3) If $A = I_n$, then the only permutation $\pi$ which contributes to the sum is the identity permutation $\iota$; any other permutation $\pi$ satisfies $\pi(i) \neq i$ for some $i$, so that $a_{i\pi(i)} = 0$. The sign of $\iota$ is $+1$, and all the factors $a_{ii}$ are equal to 1, so $\det(A) = 1$, as required.

□

So there exists at least one function that satisfies (D1)–(D3). We now show, perhaps surprisingly, that there is only one.

**Theorem 3.4.** *There is a unique function $D$ on $n \times n$ matrices satisfying (D1)–(D3). That function is* det().

*Proof.* Suppose that $D$ is any function on square matrices satisfying (D1)–(D3). First, we show that applying elementary row operations to matrix $A$ has a well-defined effect on $D(A)$.

(a) If $B$ is obtained from $A$ by adding $c$ times the $j$th row to the $i$th, then $D(B) = D(A)$.

(b) If $B$ is obtained from $A$ by multiplying the $i$th row by a non-zero scalar $c$, then $D(B) = cD(A)$.

(c) If $B$ is obtained from $A$ by interchanging two rows $i$ and $j$, then $D(B) = -D(A)$.

For (a), let $A'$ be the matrix which agrees with $A$ in all rows except the $i$th, which is equal to the $j$th row of $A$. By rule (D2), $D(A') = 0$. By rule (D1),

$$D(B) = D(A) + cD(A') = D(A).$$

Part (b) follows immediately from condition (D1).

To prove part (c), we observe that we can interchange the $i$th and $j$th rows by the following sequence of operations:

- add the $i$th row to the $j$th;
- multiply the $i$th row by $-1$;
- add the $j$th row to the $i$th;
- subtract the $i$th row from the $j$th.

Symbolically,

$$\begin{bmatrix} \vdots \\ v_i \\ \vdots \\ v_j \\ \vdots \end{bmatrix} \xrightarrow{\mathrm{R}_j + \mathrm{R}_i} \begin{bmatrix} \vdots \\ v_i \\ \vdots \\ v_i + v_j \\ \vdots \end{bmatrix} \xrightarrow{-1 \times \mathrm{R}_i} \begin{bmatrix} \vdots \\ -v_i \\ \vdots \\ v_i + v_j \\ \vdots \end{bmatrix} \xrightarrow{\mathrm{R}_i + \mathrm{R}_j} \begin{bmatrix} \vdots \\ v_j \\ \vdots \\ v_i + v_j \\ \vdots \end{bmatrix} \xrightarrow{\mathrm{R}_j - \mathrm{R}_i} \begin{bmatrix} \vdots \\ v_j \\ \vdots \\ v_i \\ \vdots \end{bmatrix}$$

The first, third and fourth steps don't change the value of $D$, while the second multiplies it by $-1$.

We now understand how elementary row operations on the matrix $A$ affect the value of $D(A)$. The proof now proceeds in two cases, depending on whether $A$ is invertible.

- If $A$ is not invertible, then its row rank is less than $n$ (Corollary 2.18). So the rows of $A$ are linearly dependent, and one row can be written as a linear combination of the others. Suppose, without loss of generality that the first row $v_1$ can be written $v_1 = c_2v_2 + c_3v_3 + \cdots + c_nv_n$. Applying property (D1), we see that

$$D \begin{bmatrix} c_2v_2 + c_3v_3 + \cdots + c_nv_n \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = c_2 D \begin{bmatrix} v_2 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + c_3 D \begin{bmatrix} v_3 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \cdots + c_n D \begin{bmatrix} v_n \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = 0.$$

  Note that each of the terms in the above sum is zero by (D2), as each matrix has a repeated row. So, assuming (D1)–(D3) we have shown $D(A) = 0$. Since det satisfies (D1)–(D3) (Lemma 3.3), we know in particular that $\det(A) = 0$. So $D(A)$ and $\det(A)$ agree on non-invertible matrices $A$: they are both zero.

- If $A$ is invertible, then we can reduce it to the identity by applying elementary row operations (Corollary 2.20). Suppose that these row operations correspond to elementary matrices $R_1, R_2, \ldots, R_t$, so that $R_t \ldots R_2 R_1 A = I$. Each row operation $R_i$ multiplies $D()$ by a certain factor $c_i$, determined by (a)–(c). Thus, on the one hand, $D(R_t \ldots R_2 R_1 A) = c_1 c_2 \ldots c_t D(A) = cD(A)$, where $c = c_1 c_2 \ldots c_t$. On the other hand $R_t \ldots R_2 R_1 A = I$, and so $D(R_t \ldots R_2 R_1 A) = D(I) = 1$, by (D3). It follows that $D(A) = c^{-1}$. Again, we deduce in particular that $\det(A) = c^{-1}$. Thus, $D(A)$ and $\det(A)$ agree on invertible matrices.

Putting together the two cases, we see that if $D$ is any function satisfying (D1)–(D3), then $D(A) = \det(A)$ for all $A$. □

**Corollary 3.5.** *A square matrix is invertible if and only if* $\det(A) \neq 0$.

*Proof.* See the case division at the end of the proof of the theorem. □

Note that Theorem 3.4 immediately yields a result from *Linear Algebra I.*

**Lemma 3.6.** *(a) If $B$ is obtained from $A$ by adding $c$ times the $j$th row to the $i$th, then* $\det(B) = \det(A)$.

*(b) If $B$ is obtained from $A$ by multiplying the $i$th row by a scalar $c$, then* $\det(B) = c\det(A)$.

*(c) If $B$ is obtained from $A$ by interchanging two rows, then* $\det(B) = -\det(A)$.

One of the most important properties of the determinant is the following.

**Theorem 3.7.** *If $A$ and $B$ are $n \times n$ matrices over $\mathbb{K}$, then* $\det(AB) = \det(A)\det(B)$.

*Proof.* Suppose first that $A$ is not invertible. Then $\det(A) = 0$. Also, $AB$ is not invertible. (For, suppose that $(AB)^{-1} = X$, so that $(AB)X = I = A(BX)$. Then $BX$ is the inverse of $A$.) So $\det(AB) = 0$, and the theorem is true.

In the other case, $A$ is invertible, so we can apply a sequence of elementary row operations to $A$ to get to the identity matrix. Suppose those row operations correspond to elementary matrices $R_1, \ldots, R_t$ and that the effect of the operations is to multiply the determinant in turn by by $c_1, \ldots, c_t$. Then, for any matrix $X$, we have $\det(R_t \ldots R_2 R_1 X) = c\det(X)$, where $c = c_1 c_2 \ldots c_t$. Noting that $R_t \ldots R_2 R_1 = A^{-1}$ we can write our finding as $\det(A^{-1}X) = c\det(X)$.

Setting $X = A$, we find that

$$c\det(A) = \det(A^{-1}A) = \det(I) = 1,$$

and setting $X = AB$ that

$$c\det(AB) = \det(A^{-1}(AB)) = \det((A^{-1}A)B) = \det(B).$$

combining these equalities we obtain $\det(AB) = \det(A)\det(B)$, as required. □

Finally, we have defined determinants using rows, but we could have used columns instead:

**Corollary 3.8.** *The determinant is the unique function $D$ of $n \times n$ matrices which satisfies the conditions*

(D1′) *for $1 \leq i \leq n$, $D$ is a linear function of the $i$th column;*

(D2′) *if two columns of $A$ are equal, then $D(A) = 0$;*

(D3′) $D(I_n) = 1$.

*Proof.* Swapping the roles of rows and columns in the Proof of Theorem 3.4 shows that there is a unique function satisfying (D1′)–(D3′) given by the formula

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi) a_{\pi(1),1} a_{\pi(2),2} \cdots a_{\pi(n),n},$$

which is the usual formula, but with the role of rows and columns reversed. But this formula contains the same terms as the usual one, but in a different order. (Check this. The term corresponding to $\pi$ in the usual formula is equal, after rearrangement, to the term corresponding to $\pi^{-1}$ in the above formula. Furthermore, $\operatorname{sign}(\pi^{-1}) = \operatorname{sign}(\pi)$.) □

Since det() is the unique function on matrices satisfying (D1′)–(D3′) and (D1)–(D3), and since these properties are invariant under interchange of rows and columns, the same must be true of det() itself.

**Corollary 3.9.** *If $A^\top$ denotes the transpose of $A$, then* $\det(A^\top) = \det(A)$.

## 3.1 Calculating determinants

Here is a second formula, which is also theoretically important but very inefficient in practice.

**Definition 3.10.** Let $A$ be an $n \times n$ matrix. For $1 \leq i, j \leq n$, denote by $A_{i,j}$ the $(n-1) \times (n-1)$ matrix obtained by deleting the $i$th row and $j$th column of $A$. The $(i, j)$ *cofactor* of $A$ is defined to be $(-1)^{i+j} \det(A_{i,j})$. (These signs have a chessboard pattern, starting with sign + in the top left corner.) We denote the $(i, j)$ cofactor of $A$ by $K_{ij}(A)$. Note that the cofactor is a scalar, even though we've denoted it by an upper case latter! Finally, the *adjugate* of $A$ is the $n \times n$ matrix $\operatorname{Adj}(A)$ whose $(i, j)$ entry is the $(j, i)$ cofactor $K_{ji}(A)$ of $A$. (Note the transposition!)

**Theorem 3.11.** (a) *For $1 \leq i \leq n$, we have*

$$\det(A) = \sum_{j=1}^{n} a_{ij} K_{ij}(A).$$

(b) *For $1 \leq j \leq n$, we have*

$$\det(A) = \sum_{i=1}^{n} a_{ij} K_{ij}(A).$$

This theorem says that, if we take any column or row of $A$, multiply each element by the corresponding cofactor, and add the results, we get the determinant of $A$. The expressions (a) and (b) appearing in Theorem 3.11 are the *cofactor* or *Laplace expansion* along row $i$ and column $j$ respectively.

**Example 3.12.** Using a cofactor expansion along the first column, we see that

$$\begin{aligned}\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 10\end{vmatrix} &= \begin{vmatrix}5 & 6\\ 8 & 10\end{vmatrix} - 4\begin{vmatrix}2 & 3\\ 8 & 10\end{vmatrix} + 7\begin{vmatrix}2 & 3\\ 5 & 6\end{vmatrix}\\ &= (5\cdot 10 - 6\cdot 8) - 4(2\cdot 10 - 3\cdot 8) + 7(2\cdot 6 - 3\cdot 5)\\ &= 2 + 16 - 21\\ &= -3\end{aligned}$$

using the standard formula for a $2\times 2$ determinant.

Theorem 3.11 looks plausible. Note that $K_{ij}(A)$ is an $(n-1)\times(n-1)$ determinant. Expanding $a_{ij}K_{ij}(A)$ by the Leibniz formula yields $(n-1)!$ terms that *ought* to correspond to those $(n-1)!$ terms in the Leibniz formula for $A$ that satisfy $\pi(i) = j$. But keeping track of the subscripts and the signs is fiddly and not very edifying, so we won't go into that here. The details can be found on the Wikipedia page on the "Laplace expansion".

Another way of going about proving Theorem 3.11 is to show that the expression $\sum_{j=1}^{n} a_{ij}K_{ij}(A)$ satisfies (D1)–(D3). The issue here is how to show (D2) when one of the two equal rows is row $i$. Again, there is no essential problem but we won't go into the details here.

At first sight, the Laplace expansion looks like a simple formula for the determinant, since it is just the sum of $n$ terms, rather than $n!$ as in the Leibniz formula. But each term is an $(n-1)\times(n-1)$ determinant. Working down the chain we find that this method is just as labour-intensive as the other one. But the cofactor expansion has further nice properties:

**Theorem 3.13.** *For any $n\times n$ matrix $A$, we have*

$$A\cdot \mathrm{Adj}(A) = \mathrm{Adj}(A)\cdot A = \det(A)\, I.$$

**Remark 3.14.** In the above identity, the $A\cdot\mathrm{Adj}(A)$ and $\mathrm{Adj}(A)\cdot A$ are *matrix* products, while $\det(A)\, I$ is the product of a *scalar* with a matrix. We can get into big trouble by ignoring this distinction and using matrices where scalars should go. Just in this section, we'll use dots to emphasise *matrix* multiplication.

*Proof of Theorem 3.13.* We calculate the matrix product. Recall that the $(i, j)$ entry of $\mathrm{Adj}(A)$ is $K_{ji}(A)$.

Now the $(i, i)$ entry of the product $A\cdot\mathrm{Adj}(A)$ is

$$\sum_{k=1}^{n} a_{ik}(\mathrm{Adj}(A))_{ki} = \sum_{k=1}^{n} a_{ik}K_{ik}(A) = \det(A),$$

by the cofactor expansion. On the other hand, if $i \neq j$, then the $(i, j)$ entry of the product is

$$\sum_{k=1}^{n} a_{ik}(\mathrm{Adj}(A))_{kj} = \sum_{k=1}^{n} a_{ik}K_{jk}(A).$$

This last expression is the cofactor expansion of the matrix $A'$ which is the same of $A$ except for the $j$th row, which has been replaced by the $i$th row of $A$. (Note that changing the $j$th row of a matrix has no effect on the cofactors of elements in this row.) So the sum is $\det(A')$. But $A'$ has two equal rows, so its determinant is zero.

Thus $A \cdot \mathrm{Adj}(A)$ has entries $\det(A)$ on the diagonal and 0 everywhere else; so it is equal to $\det(A)\,I$.

The proof for the product the other way around is the same, using columns instead of rows. □

**Corollary 3.15.** *If the $n \times n$ matrix $A$ is invertible, then its inverse is equal to*

$$(\det(A))^{-1}\,\mathrm{Adj}(A).$$

So how can you work out a determinant efficiently? The best method in practice is to use elementary operations.

Apply elementary operations to the matrix, keeping track of the factor by which the determinant is multiplied by each operation. If you want, you can reduce all the way to the identity, and then use the fact that $\det(I) = 1$. Often it is simpler to stop at an earlier stage when you can recognise what the determinant is. For example, if the matrix $A$ has diagonal entries $a_{11}, \ldots, a_{nn}$, and all off-diagonal entries are zero, then $\det(A)$ is just the product $a_{11} \cdots a_{nn}$.

**Example 3.16.** Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

Subtracting twice the first column from the second, and three times the second column from the third (these operations don't change the determinant) gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -11 \end{bmatrix}.$$

Now the cofactor expansion along the first row gives

$$\det(A) = \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix} = 33 - 36 = -3.$$

(At the last step, it is easiest to use the formula for the determinant of a $2 \times 2$ matrix rather than do any further reduction.)

## 3.2 The Cayley-Hamilton Theorem

Since we can add and multiply matrices, we can substitute them into a polynomial. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix},$$

then the result of substituting $A$ into the polynomial $x^2 - 3x + 2$ is

$$A^2 - 3A + 2I = \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 6 & -9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We say that the matrix $A$ *satisfies the equation* $x^2 - 3x + 2 = 0$. (Notice that for the constant term 2 we substituted $2I$.)

It turns out that, for every $n \times n$ matrix $A$, we can calculate a polynomial equation of degree $n$ satisfied by $A$.

**Definition 3.17.** Let $A$ be a $n \times n$ matrix. The *characteristic polynomial* of $A$ is the polynomial

$$p_A(x) = \det(xI - A).$$

This is a polynomial in $x$ of degree $n$.

For example, if

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix},$$

then

$$p_A(x) = \begin{vmatrix} x & -1 \\ 2 & x-3 \end{vmatrix} = x(x-3) + 2 = x^2 - 3x + 2.$$

Indeed, it turns out that this is the polynomial we want in general:

**Theorem 3.18** (Cayley–Hamilton Theorem)**.** *Let $A$ be an $n \times n$ matrix with characteristic polynomial $p_A(x)$. Then $p_A(A) = O$.*

**Example 3.19.** Let us just check the theorem for $2 \times 2$ matrices. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$p_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad - bc),$$

and so

$$p_A(A) = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad - bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O,$$

after a small amount of calculation.

*Proof.* We use the theorem

$$A \cdot \mathrm{Adj}(A) = \det(A)\, I.$$

In place of $A$, we put the matrix $xI - A$ into this formula:

$$(xI - A) \cdot \mathrm{Adj}(xI - A) = \det(xI - A)\, I = p_A(x)\, I.$$

Now it is very tempting (for lesser beings than the MTH6140 class) just to substitute $x = A$ into this formula: on the right we have $p_A(A)\, I = p_A(A)$, while on the left there is a factor $AI - A = O$. Unfortunately this is not valid, and the reason is connected to the remark following the statement of Theorem 3.13. The expression $p_A(A)$ is a matrix, and not valid in this context, where a scalar is expected. (A similar problem exists on the left side of the incorrect identity.)

Instead, we argue as follows. $\mathrm{Adj}(xI - A)$ is a matrix whose entries are polynomials, so we can write it as a sum of powers of $x$ times matrices, that is, as a polynomial whose coefficients are matrices. For example,

$$\begin{bmatrix} x^2+1 & 2x \\ 3x-4 & x+2 \end{bmatrix} = x^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}.$$

The entries in $\mathrm{Adj}(xI - A)$ are $(n-1) \times (n-1)$ determinants, so the highest power of $x$ that can arise is $x^{n-1}$. So we can write

$$\mathrm{Adj}(xI - A) = x^{n-1}B_{n-1} + x^{n-2}B_{n-2} + \cdots + xB_1 + B_0,$$

for suitable $n \times n$ matrices $B_0, \ldots, B_{n-1}$. Hence

$$\begin{aligned} p_A(x)I &= (xI - A) \cdot \mathrm{Adj}(xI - A) \\ &= (xI - A) \cdot (x^{n-1}B_{n-1} + x^{n-2}B_{n-2} + \cdots + xB_1 + B_0) \\ &= x^n B_{n-1} + x^{n-1}(-AB_{n-1} + B_{n-2}) + \cdots + x(-AB_1 + B_0) - AB_0. \end{aligned}$$

So, if we let

$$p_A(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

then we read off that

$$\begin{array}{rcrcl} & & B_{n-1} & = & I, \\ -AB_{n-1} & + & B_{n-2} & = & c_{n-1}I, \\ & & & \vdots & \\ -AB_1 & + & B_0 & = & c_1 I, \\ -AB_0 & & & = & c_0 I. \end{array}$$

We take this system of equations, and multiply the first by $A^n$, the second by $A^{n-1}$, ..., and the last by $A^0 = I$. What happens? On the left, all the terms cancel in pairs: we have

$$A^n B_{n-1} + A^{n-1}(-AB_{n-1} + B_{n-2}) + \cdots + A(-AB_1 + B_0) + I(-AB_0) = O.$$

On the right, we have

$$A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = p_A(A).$$

So $p_A(A) = O$, as claimed. □