Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 11

Thomas' Calculus, Sections 5.5 and 7.1 to 7.8 (except Sections 7.5, 7.6)

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example: Evaluate

$$\int \frac{2z}{\sqrt[3]{z^2+5}}dz \ :$$

1. Substitute $u = z^2 + 5$, $du = 2z\,dz$:

$$\int \frac{2z}{\sqrt[3]{z^2+5}}dz = \int u^{-1/3}du$$

2. Integrate:

$$\int u^{-1/3}du = \frac{3}{2}u^{2/3} + C$$

3. Replace $u = z^2 + 5$:

$$\int \frac{2z}{\sqrt[3]{z^2+5}}dz = \frac{3}{2}(z^2+5)^{2/3} + C$$

Transform integrals by using trigonometric identities.

example: Evaluate $\int \sin^2 x\,dx$:

Use half-angle formula $\sin^2 x = (1 - \cos 2x)/2$ to write

$$\begin{aligned} \int \sin^2 x\,dx &= \int \frac{1}{2}(1-\cos 2x)dx \\ &= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C \end{aligned}$$

Move on to substitution in definite integrals:

Theorem 1 *If g' is continuous on $[a,b]$ and f is continuous on the range of g, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du\ .$$

(note that $u = g(x)$! proof straightforward, see book p.377)

example: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.

Substitute $u = x^3 + 1$, $du = 3x^2dx$.

$x = -1$ gives $u = (-1)^3 + 1 = 0$; $x = 1$ gives $u = 1^3 + 1 = 2$, and we obtain

$$\begin{aligned} \int_{-1}^{1} 3x^2\sqrt{x^3+1}dx &= \int_0^2 \sqrt{u}du \\ &= \frac{2}{3}u^{3/2}\Big|_0^2 \\ &= \frac{2}{3}2^{3/2} - 0 \\ &= \frac{4\sqrt{2}}{3} \end{aligned}$$

Definite integrals of symmetric functions

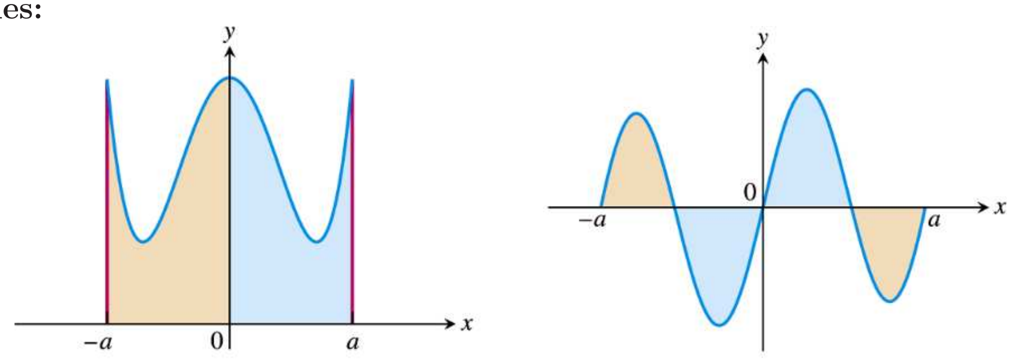
Theorem 2 *Let f be continuous on the symmetric interval $[-a, a]$.*

(a) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

(proof by splitting the integrals and straightforward formal manipulations, see book p.379 for part (a))

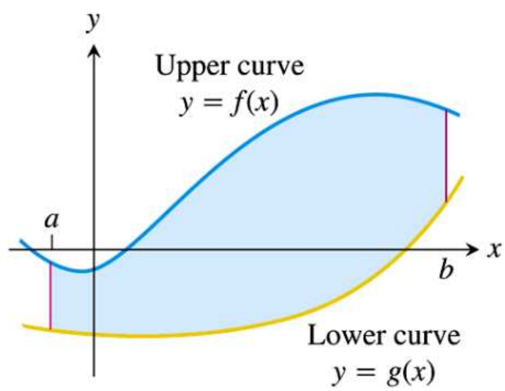
examples:



(a) (b)

Areas between curves

example:

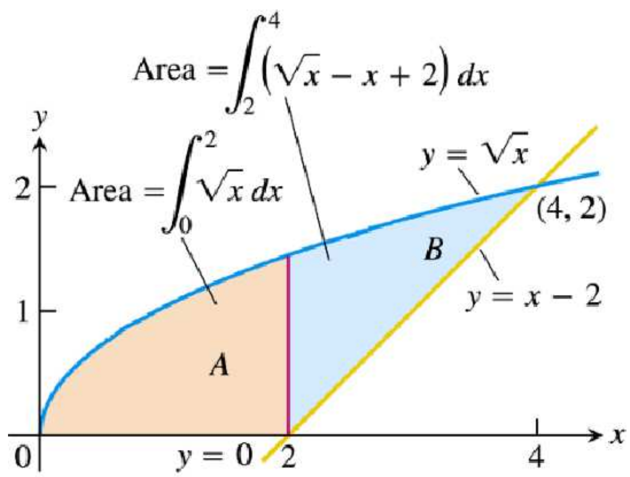


DEFINITION Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

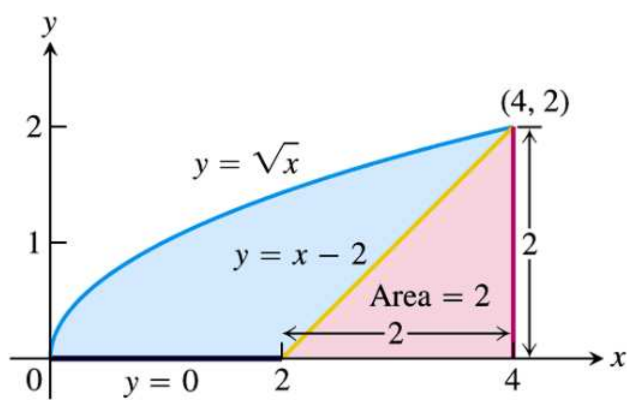
example: Find the area that is enclosed above by $y = \sqrt{x}$ and below by $y = 0$ and $y = x - 2$.
Two solutions:
(a) by definition:



Split total area into area A + area B.
Find right-hand limit for B by solving $\sqrt{x} = x - 2 \Rightarrow x = 4$.

$$\begin{aligned} \text{total area} &= \int_0^2 \sqrt{x} - 0dx + \int_2^4 \sqrt{x} - (x-2)dx \\ &= \frac{2}{3}x^{3/2}\Big|_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right)\Big|_2^4 \\ &= \frac{10}{3} \end{aligned}$$

(b) the clever way:



The area below the parabola is

$$A_1 = \int_0^4 \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3}\,.$$

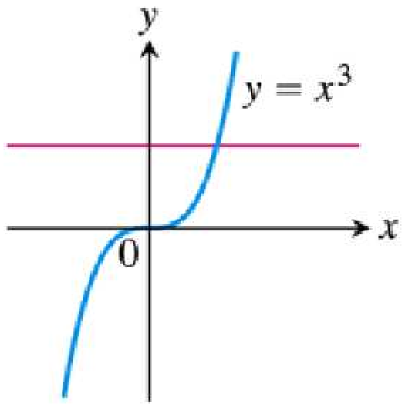
The area of the triangle is $A_2 = 2 \cdot 2/2 = 2$ so that

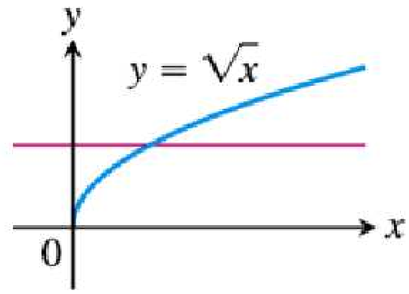
$$\text{total area} = A_1 - A_2 = \frac{16}{3} - 2 = \frac{10}{3}\,.$$

Inverse functions and their derivatives

DEFINITION One-to-One Function
A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

These functions take on any value in their range *exactly once*.
examples:

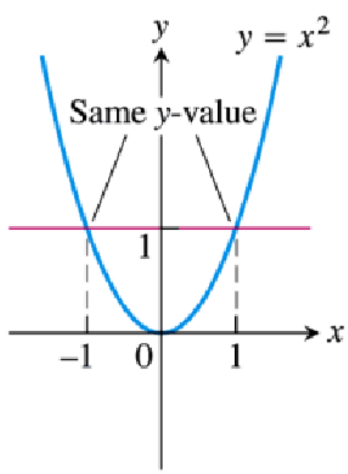


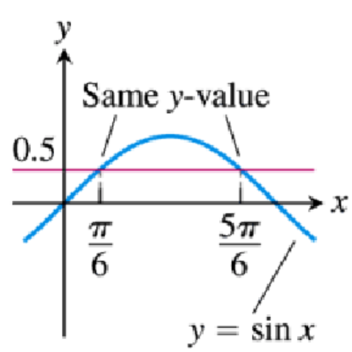


Both functions are one-to-one on $\mathbb{R}$, respectively on $\mathbb{R}_0^+$.

The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

examples:





$y = x^2$ is one-to-one on, e.g., $\mathbb{R}_0^+$ but not $\mathbb{R}$.
$y = \sin x$ is one-to-one on, e.g., $[0, \pi/2]$ but not $\mathbb{R}$.

DEFINITION **Inverse Function**

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

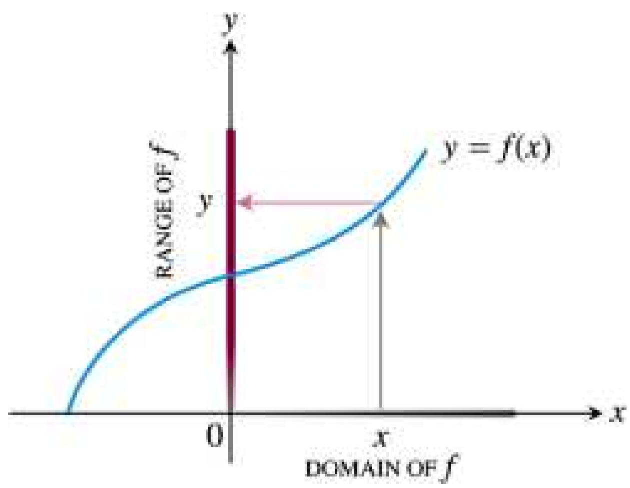
$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

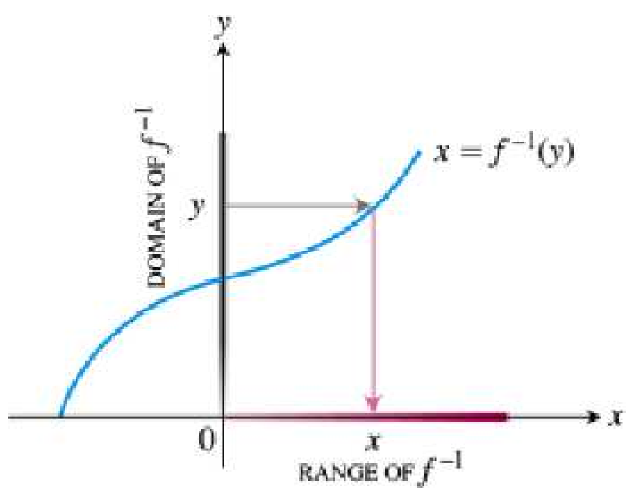
note:

- f^{-1} reads f *inverse*
- $f^{-1}(x) \neq (f(x))^{-1} = 1/f(x)$! (not an exponent)
- $(f^{-1} \circ f)(x) = x$ for all $x \in D(f)$
- $(f \circ f^{-1})(x) = x$ for all $x \in R(f)$

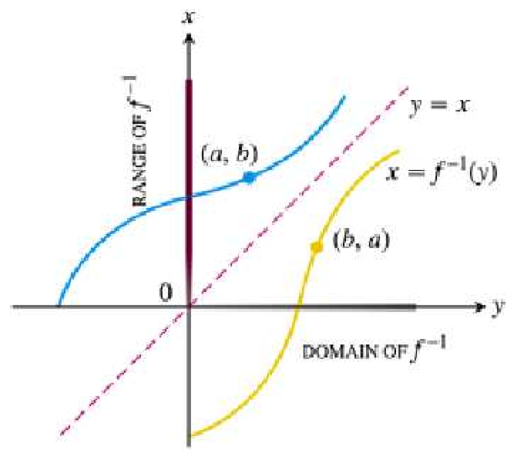
Read off inverse from graph of $f(x)$, as follows:
usual procedure $x \mapsto y = f(x)$:



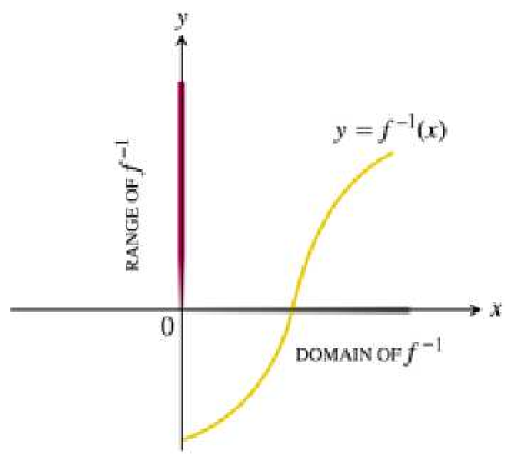
for inverse $y \mapsto x = f^{-1}(y)$:



Note that $D(f) = R(f^{-1})$ and $R(f) = D(f^{-1})$, which suggests to *reflect* $x = f^{-1}(y)$ along $y = x$:



After reflection, x and y have changed places. Therefore, swap x and y...



...and we have found $y = f^{-1}(x)$ *graphically*.
method for finding inverses *algebraically*:

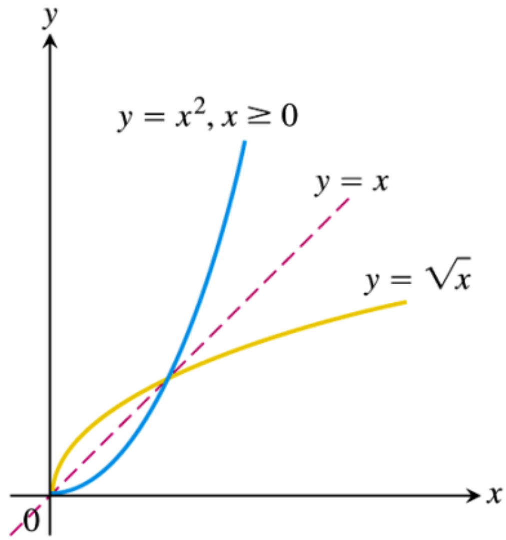
1. solve $y = f(x)$ for x: $x = f^{-1}(y)$

2. interchange x and y: $y = f^{-1}(x)$

example: Find the inverse of $y = x^2, x \geq 0$.

1. solve $y = f(x)$ for x: $\sqrt{y} = \sqrt{x^2} = |x| = x$, as $x \geq 0$.

2. interchange x and y: $y = \sqrt{x}$.



Calculate derivatives of inverse functions.
Differentiate $y = f^{-1}(x)$, or $x = f(y)$:

$$\frac{dx}{dx} = 1 = \frac{d}{dx} f(y) = f'(y) \frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\frac{dx}{dy}}$$

The derivatives are reciprocals of one another.

Be precise: $x = f(y)$ means $y = f^{-1}(x)$ so that

$$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$$

Be more precise:

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

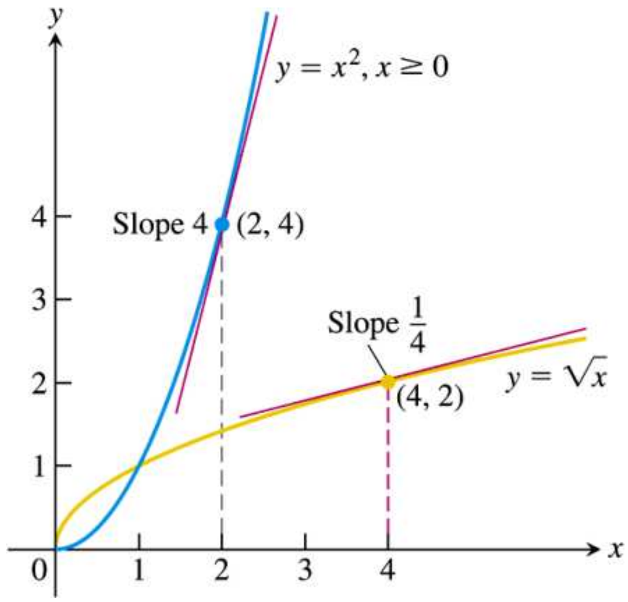
$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

example: $f(x) = x^2, x \geq 0$ continued.
$f^{-1}(x) = \sqrt{x}$ and $f'(x) = 2x$ so that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2f^{-1}(x)} = \frac{1}{2\sqrt{x}}$$



note: The theorem can be used pointwise to find a value of the inverse derivative without calculating any formula for the inverse (see the book p.472 for an example). Otherwise, simply differentiate the inverse.

Natural Logarithms

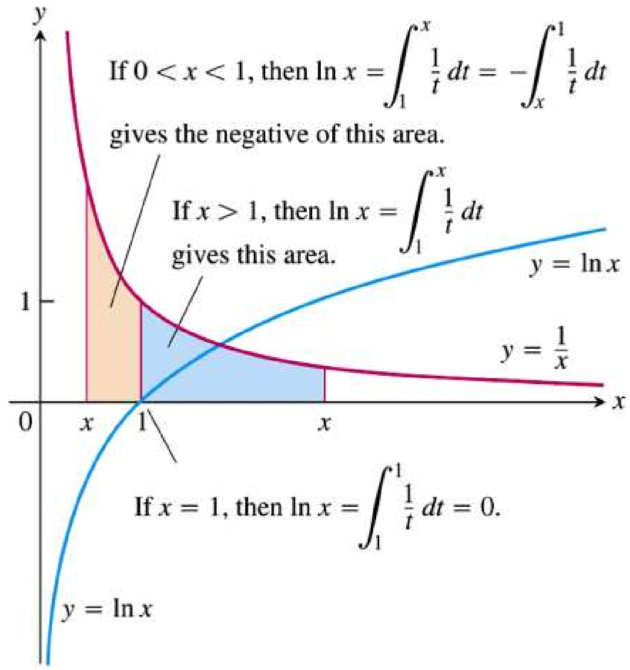
For $a \in \mathbb{Q} \setminus \{-1\}$ we know that

$$\int_1^x t^a dt = \frac{1}{a+1}\left(x^{a+1} - 1\right)$$

(Fundamental Theorem of Calculus part 2).
What happens if $a = -1$? $\int_1^x \frac{1}{t} dt$ is well defined for $x > 0$:

DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0$$



The range of $\ln x$ is $\mathbb{R}$.
A special value: the number $\boxed{e = 2.718281828459\ldots}$ (sometimes called *Euler's number*), satisfying

$$\ln e = 1 \,.$$

Differentiate $\ln x$ (according to the fundamental theorem of calculus part 1):

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x} \,.$$

If $u(x) > 0$, by the chain rule

$$\frac{d}{dx} \ln u = \frac{1}{u} u' \,.$$

If $u(x) = ax$ with $a > 0$,

$$\frac{d}{dx} \ln ax = \frac{1}{ax} a = \frac{1}{x}$$

Since $\ln ax$ and $\ln x$ have the same derivative (!),

$$\ln ax = \ln x + C \,.$$

For $x = 1$ we get $C = \ln a1 - \ln 1 = \ln a$ and therefore

$$\ln ax = \ln a + \ln x \,.$$

We have shown rule 1 in the following table:

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ — Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ — r rational

(For the proof of rule 4 see book p.480.)
examples: Apply the logarithm properties to function formulas by replacing $a \to f(x), x \to g(x)$.

1. $\ln 8 + \ln \cos x = \ln(8 \cos x)$
2. $\ln \dfrac{z^2+3}{2z-1} = \ln(z^2+3) - \ln(2z-1)$
3. $\ln \cot x = \ln \dfrac{1}{\tan x} = -\ln \tan x$
4. $\ln \sqrt[5]{x-3} = \ln(x-3)^{1/5} = \dfrac{1}{5} \ln(x-3)$

For $t > 0$, the Fundamental Theorem of Calculus tells us that

$$\int \frac{1}{t} dt = \ln t + C\,.$$

For $t < 0$, $(-t)$ is positive, and we find analogously

$$\int \frac{1}{(-t)} d(-t) = \ln(-t) + C\,.$$

For $t \neq 0$, together this gives

$$\int \frac{1}{t} dt = \ln|t| + C$$

Substituting $t = f(x)$, $dt = f'(x)dx$ leads to

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C}$$

(for all $f(x)$ that maintain a constant sign on the range of integration).
example:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Substitute $t = \cos x > 0$, $dt = -\sin x\, dx$ on $(-\pi/2, \pi/2)$:

$$\int \tan x\, dx = -\int \frac{1}{t} dt = -\ln|t| + C = -\ln|\cos x| + C$$

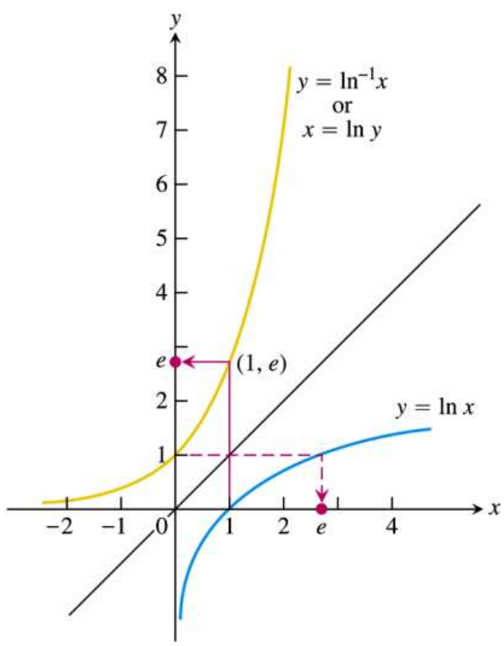
Analogously for $\cot x$:

$$\int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\, du = \ln|\sin u| + C = -\ln|\csc x| + C$$

The exponential function

$\ln x$ is strictly increasing, therefore invertible:



Definition 1 (Exponential function) *For every $x \in \mathbb{R}$, $\exp x = \ln^{-1} x$.*

Recall that $1 = \ln e$ so that $\exp 1 = e$.
Apply the power rule:

$$\ln e^r = r \ln e = r$$

so that

$$e^r = \exp r\ , \ r \in \mathbb{Q}\ .$$

But $\exp x$ is defined for *any real* x, which suggests to define *real* exponents for base e via $\exp x$:

Definition 2 *For every $x \in \mathbb{R}$, $e^x = \exp x$.*

It is

$$\ln(e^a) = a \,, \; a \in \mathbb{R}$$

and

$$e^{\ln a} = a \,, \; a > 0 \,.$$

With

$$\left(e^{\ln a}\right)^x = e^{x \ln a} = a^x$$

we can define *real powers of positive real numbers a:*

Definition 3 (General exponential functions) *For every $x \in \mathbb{R}$ and $a > 0$, the exponential function with base a is*

$$a^x = e^{x \ln a} \,.$$

note: By using $x^n = e^{n \ln x}$, it can be proved that

$$\frac{d}{dx} x^n = n x^{n-1} \,, x > 0,$$

for all real n. (see book p.492)
We have

THEOREM 3 Laws of Exponents for e^x
For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Proof of 1.:

$$\begin{aligned} \exp(x_1) \cdot \exp(x_2) &= \exp \ln(\exp(x_1) \cdot \exp(x_2)) \\ \text{(product rule for } \ln x) &= \exp(\ln \exp(x_1) + \ln \exp(x_2)) \\ &= \exp(x_1 + x_2) \end{aligned}$$

(2. and 3. follow from 1., 4. is proved similarly to 1.)

As $e^x = f^{-1}(x)$ with $f(x) = \ln x$ and $f'(x) = 1/x$, we find (by using the derivative rule for inverses)

$$\frac{d}{dx} e^x = \frac{1}{f'(f^{-1}(x))} = f^{-1}(x) = e^x$$

implying

$$\int e^x dx = e^x + C \,.$$

By the chain rule,

$$\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$$

so that

$$\int e^{f(x)}f'(x)dx = e^{f(x)} + C$$

or

$$\int e^{u}du = e^{u} + C$$

by substituting $u = f(x)$.

examples:

1.

$$\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}\sin x = e^{\sin x}\cos x$$

2.

$$\begin{aligned}\int_0^{\ln 2} e^{3x}dx &= \int_0^{\ln 8} e^{u}\frac{1}{3}du \\ &= \frac{1}{3}e^{u}\Big|_0^{\ln 8} \\ &= \frac{7}{3}\end{aligned}$$

We defined e via $\ln e = 1$ and stated $e = 2.718281828459\ldots$.

Theorem 3 (The number e as a limit)

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

Proof:

$$\begin{aligned}\ln\left(\lim_{x\to 0}(1+x)^{1/x}\right) &= \\ \text{(continuity of } \ln x \text{)} &= \lim_{x\to 0}\left(\ln(1+x)^{1/x}\right) \\ \text{(power rule)} &= \lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right) \\ (\ln 1 = 0 \text{ and l'Hôpital}) &= \lim_{x\to 0}\frac{1}{1+x} \\ &= 1 \\ &= \ln(e)\end{aligned}$$

q.e.d.

Differentiate general exponential functions of base $a > 0$:

$$\frac{d}{dx}a^{x} = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\ln a = a^{x}\ln a$$

implying

$$\int a^x dx = \frac{a^x}{\ln a} + C \ , \ a \neq 1$$

example:

$$\frac{d}{dx}x^x = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\frac{d}{dx}(x\ln x) = x^x(1+\ln x)$$

Definition 4 $(\log_a x)$ *The inverse of* $y = a^x$ *is*

$$\log_a x \text{ , the logarithm of } x \text{ with base } a,$$

provided $a > 0$ *and* $a \neq 1$ *(why?).*

It is

$$\log_a(a^x) = x \ , \ x \in \mathbb{R}$$

and

$$x = a^{\log_a x} \ , \ x > 0 \ .$$

Furthermore,

$$\ln x = \ln\left(a^{\log_a x}\right) = \log_a x \cdot \ln a \ .$$

yielding

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

note: The algebra for $\log_a x$ is precisely the same as that for $\ln x$.

Read

Thomas' Calculus:

Section 7.7 Inverse trigonometric functions,

and Section 7.8, Hyperbolic functions

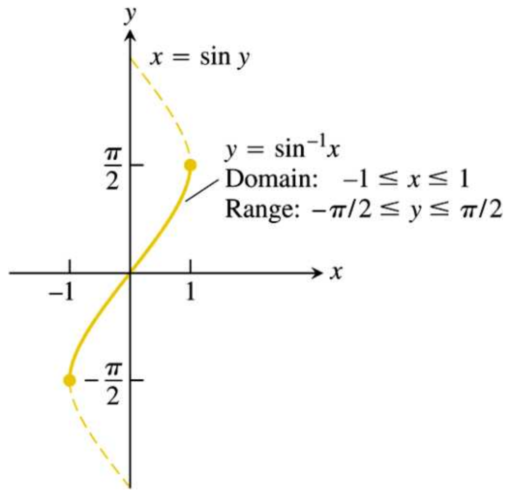
You will need this information for coursework 10!

In the following two sections I explain some very bare essentials that can be found on these pages.

Inverse trigonometric functions

note: sin, cos, sec, csc, tan, cot are not one-to-one *unless* the domain is restricted.

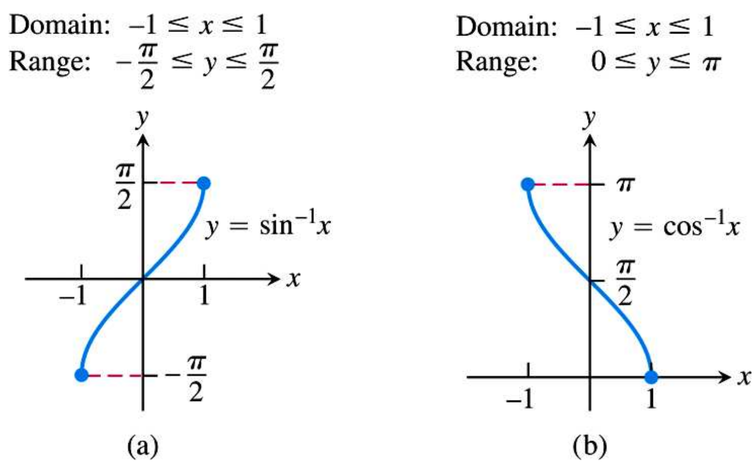
example:



Once the domains are suitably restricted, we can define:

$$\arcsin x = \sin^{-1} x \qquad \text{arccsc}\, x = \csc^{-1} x$$
$$\arccos x = \cos^{-1} x \qquad \text{arcsec}\, x = \sec^{-1} x$$
$$\arctan x = \tan^{-1} x \qquad \text{arccot}\, x = \cot^{-1} x$$

examples:



. . . and so on.

caution: $$\sin^{-1} x \neq (\sin x)^{-1}$$

Unfortunately this is inconsistent, since $\sin^2 x = (\sin x)^2$. Best to avoid $\sin^{-1} x$ and use $\arcsin x$ etc. instead.

How to differentiate inverse trigonometric functions?

example: Differentiate $y = \arcsin x$.

Start with implicit differentiation of $\sin y = x$,

$$\cos y \frac{dy}{dx} = 1 \,.$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

for $-\pi/2 < y < \pi/2$ ($\cos x = 0$ for $x = \pm\pi/2$). Therefore, for $|x| < 1$,

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

and, conversely,

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C\ .$$

example: Evaluate

$$\int \frac{dx}{\sqrt{4x-x^2}}\ .$$

Trick: complete the square!

$$4x - x^2 = 4 - (x-2)^2$$

Now integrate

$$\begin{aligned} \int \frac{dx}{\sqrt{4x-x^2}} &= \int \frac{dx}{\sqrt{4-(x-2)^2}} \\ (u = x-2) &= \int \frac{du}{\sqrt{4-u^2}} \\ &= \arcsin\frac{u}{2} + C \\ &= \arcsin\left(\frac{x}{2}-1\right) + C \end{aligned}$$

Hyperbolic functions

Every function f on $[-a, a]$ can be decomposed into

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even function}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd function}}$$

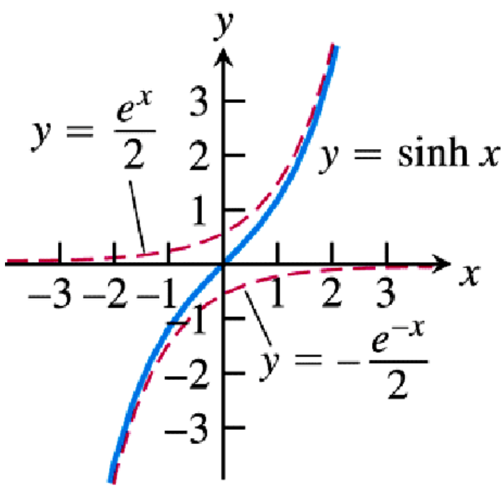
For $f(x) = e^x$:

$$e^x = \underbrace{\frac{e^x+e^{-x}}{2}}_{=\cosh x} + \underbrace{\frac{e^x-e^{-x}}{2}}_{=\sinh x}\ ,$$

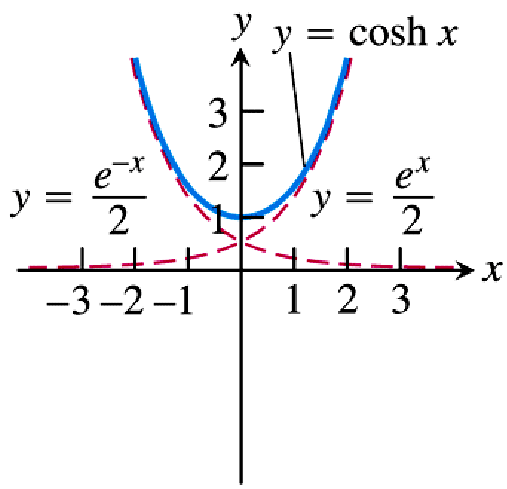
called *hyperbolic sine* and *hyperbolic cosine*.

Define tanh, coth, sech, and csch in analogy to trigonometric functions.

examples:



$$\sinh x = \frac{e^x - e^{-x}}{2}$$



$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Compare the following with trigonometric functions:

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

How to differentiate hyperbolic functions?
example:

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$
$$\frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

Inverse hyperbolic functions defined in analogy to trigonometric functions.