MTH4100 Calculus I

Lecture notes for Week 8

Thomas' Calculus, Sections 4.1 to 4.4

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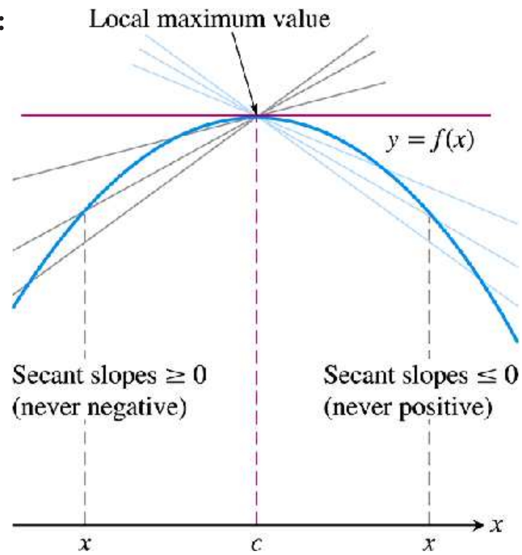
Theorem 1 (First Derivative Theorem for Local Extrema) *If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then* $f'(c) = 0$.

basic idea of the proof:



note: the converse is false! *(counterexample?)*

Where can a function f possibly have an extreme value according to this theorem?

answer:

1. at interior points where $f' = 0$
2. at interior points where f' is not defined
3. at endpoints of the domain of f.

combine 1 and 2:

DEFINITION Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Why the above assumptions? Because then we have the *extreme value theorem*, which ensures the *existence* of such values!

examples: (1) Find the absolute extrema of $f(x) = x^2$ on $[-1, 1]$.

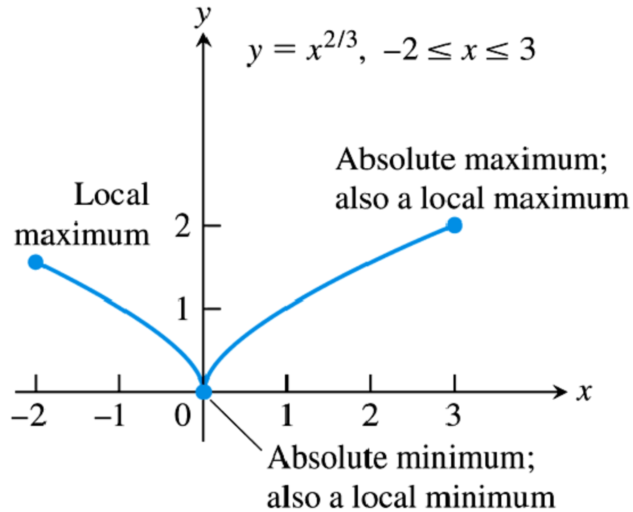
- f is differentiable on $[-1, 1]$ with $f'(x) = 2x$
- critical point: $f'(x) = 0 \quad \Rightarrow \quad x = 0$
- endpoints: $x = -1$ and $x = 1$
- $f(0) = 0,\ f(-1) = 1,\ f(1) = 1$

Therefore f has an *absolute maximum value* of 1 *twice* at $x = -1$ and an *absolute minimum value* of 0 *once* at $x = 0$.

(2) Find the absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3]$.

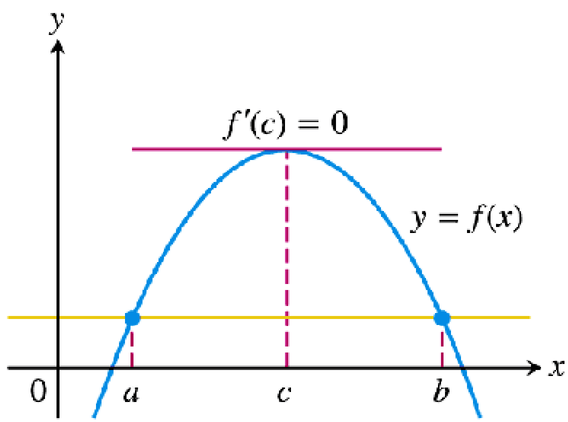
- f is differentiable with $f'(x) = \frac{2}{3}x^{-1/3}$ *except* at $x = 0$
- critical point: $f'(x) = 0$ or $f'(x)$ undefined $\Rightarrow \quad x = 0$
- endpoints: $x = -2$ and $x = 3$
- $f(-2) = \sqrt[3]{4},\ f(0) = 0,\ f(3) = \sqrt[3]{9}$

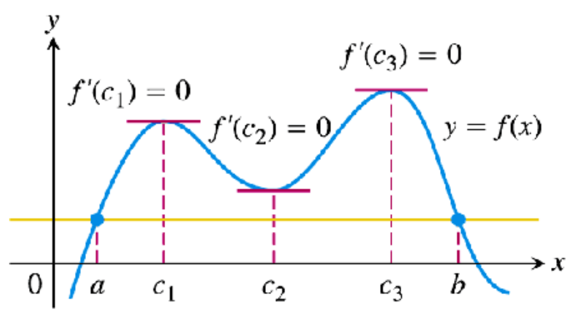
Therefore f has an *absolute maximum value* of $\sqrt[3]{9}$ at $x = 3$ and an *absolute minimum value* of 0 at $x = 0$.



Rolle's theorem

motivation:



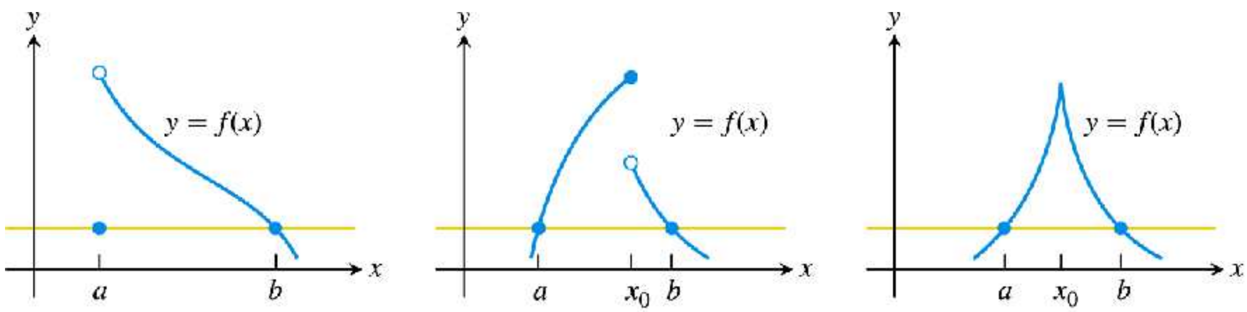


Theorem 2 *Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with $f'(c) = 0$.*

basic idea of the proof:
Apply extreme value theorem and first derivative theorem for extrema to interior points and consider endpoints separately; for details see the textbook Section 4.2.

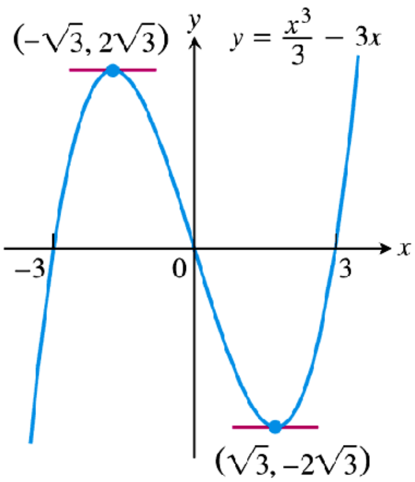
note: It is *essential* that all of the **hypotheses** in the theorem are fulfilled!

examples:



(a) Discontinuous at an endpoint of $[a, b]$ (b) Discontinuous at an interior point of $[a, b]$ (c) Continuous on $[a, b]$ but not differentiable at an interior point

example: Apply Rolle's theorem to $f(x) = \frac{x^3}{3} - 3x$ on $[-3, 3]$.

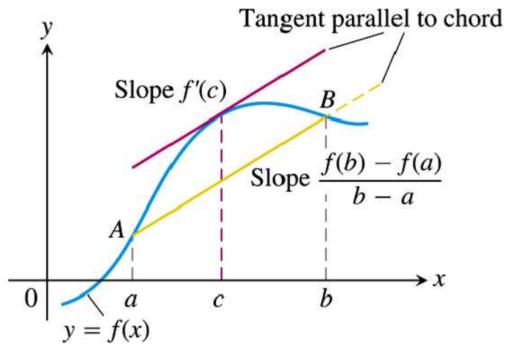


- The polynomial f is continuous on $[-3, 3]$ and differentiable on $(-3, 3)$.
- $f(-3) = f(3) = 0$
- By Rolle's theorem there exists (at least!) one $c \in [-3, 3]$ with $f'(c) = 0$.

From $f'(x) = x^2 - 3 = 0$ we find that indeed $x = \pm\sqrt{3}$.

The Mean Value Theorem

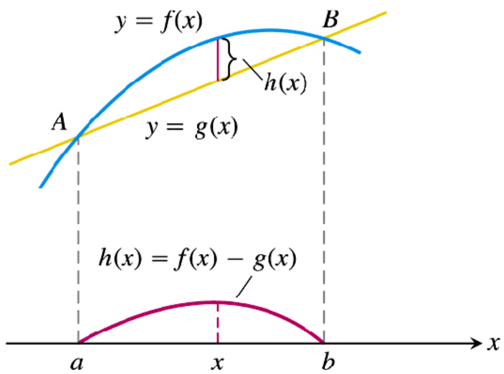
motivation: "slanted version of Rolle's theorem"



Theorem 3 (Mean Value Theorem) *Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ with*

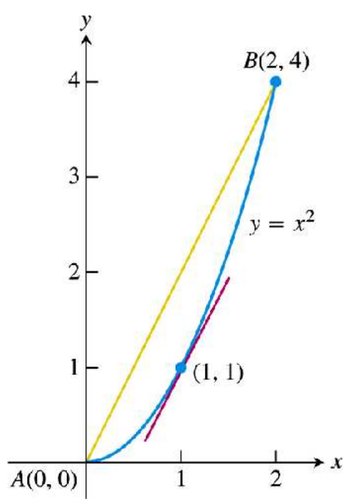
$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

basic idea of the proof:



Define $g(x)$ and $h(x)$ and apply Rolle's theorem.

example: Consider $f(x) = x^2$ on $[0,2]$.



- $f(x)$ is continuous and differentiable on $[0,2]$.
- Therefore there is a $c \in (0,2)$ with $f'(c) = \dfrac{f(2) - f(0)}{2-0} = 2$.
- Since $f'(x) = 2x$ we find that $c = 1$.

Know $f'(x) \Rightarrow$ know $f(x)$? **special case:**

Corollary 1 (Functions with zero derivatives are constant) *If* $f'(x) = 0$ *on* (a, b) *then* $f(x) = C$ *for all* $x \in (a, b)$.

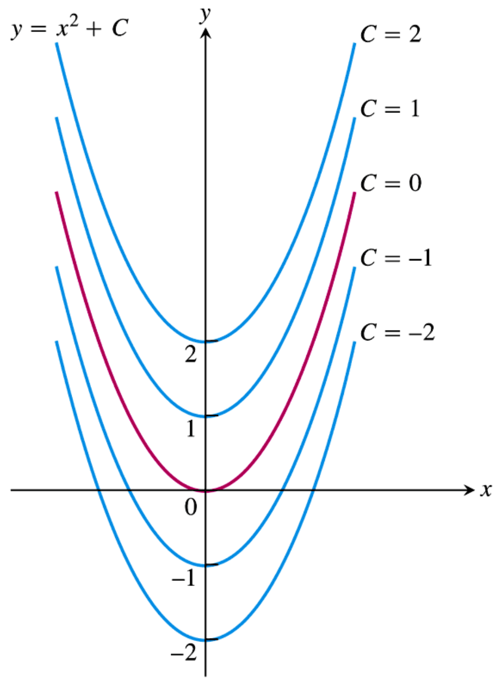
basic idea of the proof:
Apply the Mean Value Theorem *to all* $x_1, x_2 \in (a, b)$!

Know $f'(x) = g'(x) \Rightarrow$ know relation between f and g?

Corollary 2 (Functions with the same derivative differ by a constant) *If* $f'(x) = g'(x)$ *for all* $x \in (a, b)$, *then* $f(x) = g(x) + C$.

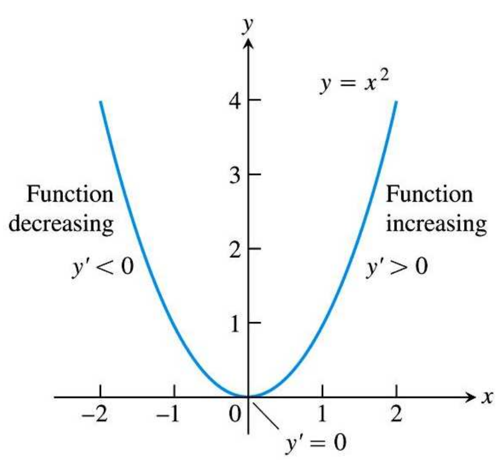
Proof: Consider $h(x) = f(x) - g(x)$. As $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$, $h(x) = C$ by the previous corollary and so $f(x) = g(x) + C$. q.e.d.

example:



Increasing and decreasing functions

motivation:



- make **increasing/decreasing** mathematically precise
- clarify relation to **positive/negative derivative**

> **DEFINITIONS Increasing, Decreasing Function**
> Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.
> 1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
> 2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.
>
> A function that is increasing or decreasing on I is called **monotonic** on I.

example: $f(x) = x^2$ *decreases* on $(-\infty, 0]$ and *increases* on $[0, \infty)$. It is *monotonic* on $(-\infty, 0]$ and $[0, \infty)$ but *not monotonic* on $(-\infty, \infty)$.

Corollary 3 (First derivative test for monotonic functions) *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

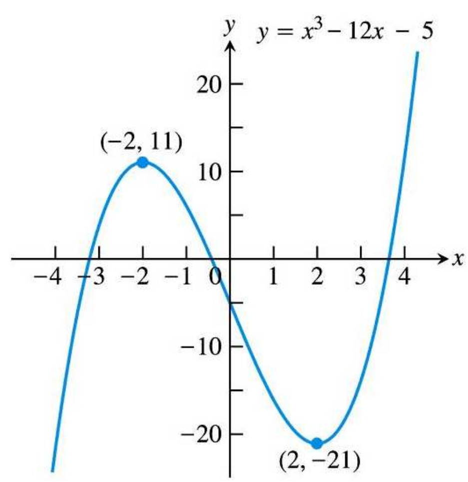
sketch of the proof:
The Mean Value theorem states that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for *any* $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. Hence, the sign of $f'(c)$ determines whether $f(x_2) < f(x_1)$ or the other way around, which in turn determines the type of monotonicity.

example: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2) \Rightarrow x_1 = -2, x_2 = 2$$

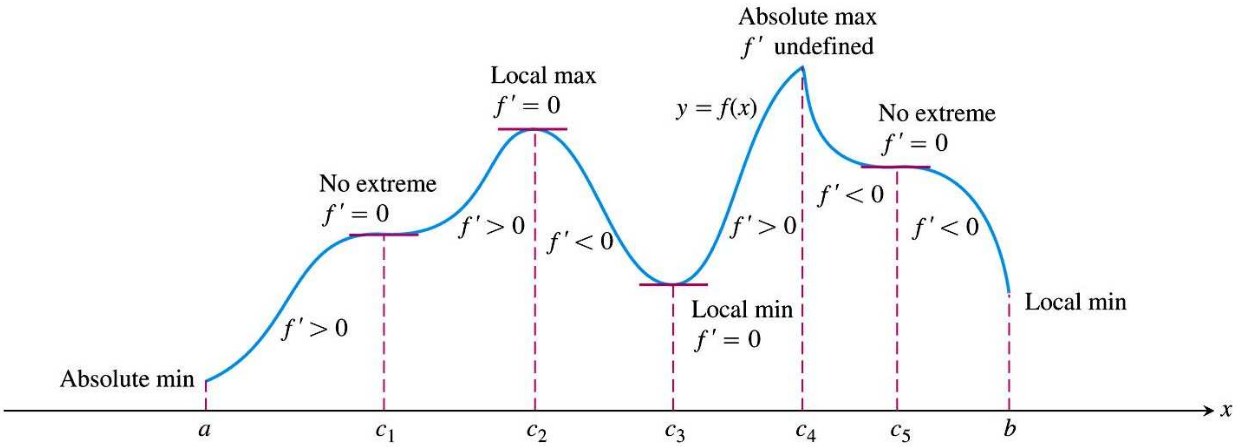
These critical points *subdivide the natural domain* into $(-\infty, -2), (-2, 2), (2, \infty)$.
rule: If $a < b$ are two nearby critical points for f, then f' must be positive on (a, b) *or* negative there. (proof relies on continuity of f'). This implies that **for finding the sign of f' it suffices to compute $f'(x)$ at *one* $x \in (a, b)$!**
Here: $f'(-3) = 15$, $f'(0) = -12$, $f'(3) = 15$.

intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
sign of f’	+	-	+
behaviour of f	increasing	decreasing	increasing



First derivatives and local extrema

example:



- Whenever f has a minimum, $f' < 0$ to the left and $f' > 0$ to the right.
- Whenever f has a maximum, $f' > 0$ to the left and $f' < 0$ to the right.

$\Rightarrow$ **At local extrema, the sign of $f'(x)$ changes!**

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

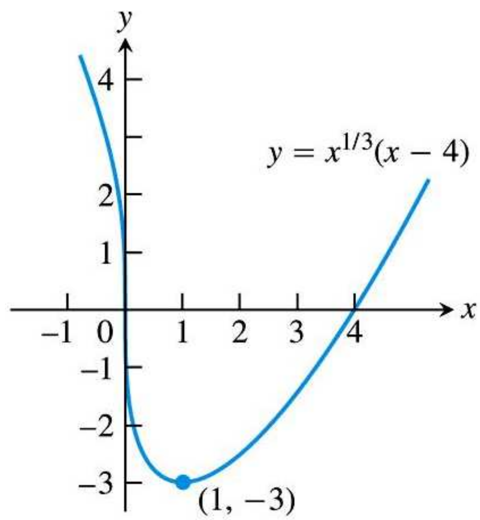
example: Find the critical points of $f(x) = x^{4/3} - 4x^{1/3}$. Identify the intervals on which f is increasing and decreasing. Find the function's extrema.

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}\frac{x-1}{x^{2/3}} \Rightarrow x_1 = 1\ ,\ x_2 = 0$$

intervals	$x < 0$	$0 < x < 1$	$1 < x$
sign of f'	-	-	+
behaviour of f	decreasing	decreasing	increasing

Apply the first derivative test to identify local extrema:

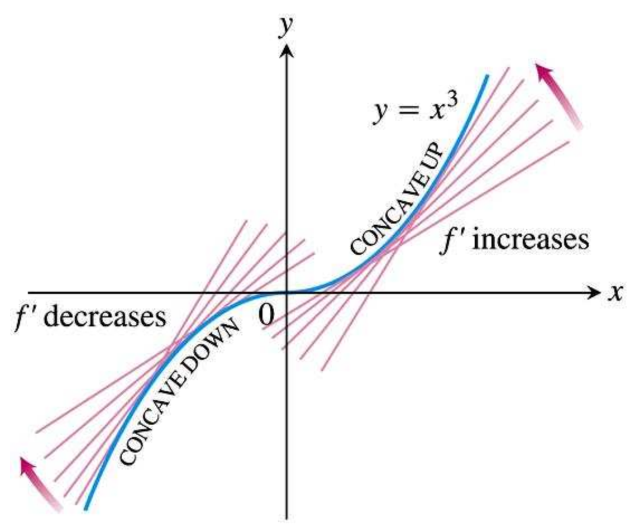
- f' does not change sign at $x = 0 \Rightarrow$ no extremum
- f' changes from $-$ to $+$ at $x = 1 \Rightarrow$ local minimum



Since $\lim_{x\to\pm\infty} = \infty$, the minimum at $x = 1$ with $f(1) = -3$ is also an *absolute minimum*. Note that $f'(0) = -\infty$!

Concavity and curve sketching

example:



intervals	$x < 0$	$0 < x$
turning of curve	turns to the *right*	turns to the *left*
tangent slopes	decreasing	increasing

The turning or bending behaviour defines the **concavity** of the curve.

DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

(a) concave up on an open interval I if f' is increasing on I

(b) concave down on an open interval I if f' is decreasing on I.

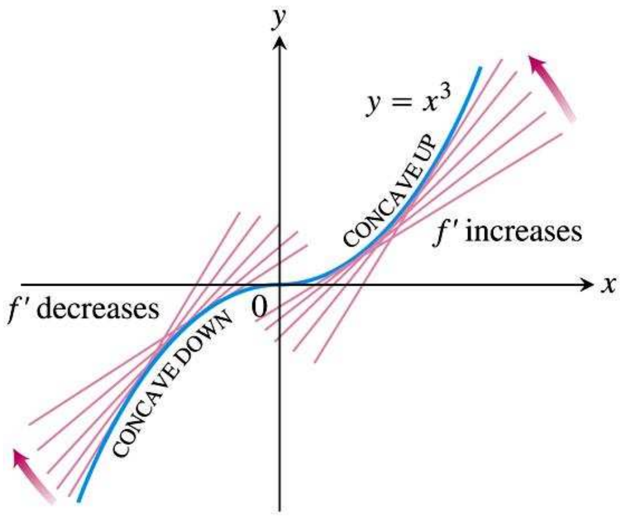
In the literature you often find that ‘concave up’ is denoted as *convex*, and ‘concave down’ is simply called *concave*.

If f'' exists, the last corollary of the mean value theorem implies that f' *increases* if $f'' > 0$ on I and *decreases* if $f'' < 0$:

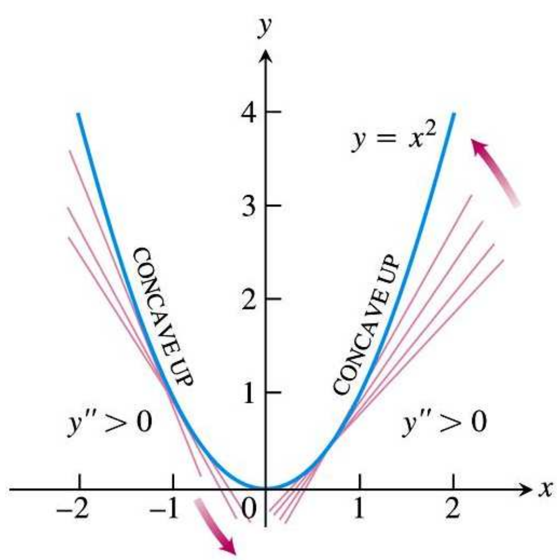
The Second Derivative Test for Concavity
Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

examples: (1) $y = x^3 \Rightarrow y'' = 6x$: For $(-\infty, 0)$ it is $y'' < 0$ and graph **concave down**. For $(0, \infty)$ it is $y'' < 0$ and graph **concave up**.



(2) $y = x^2 \Rightarrow y'' = 2 > 0$: graph is **concave up everywhere**.



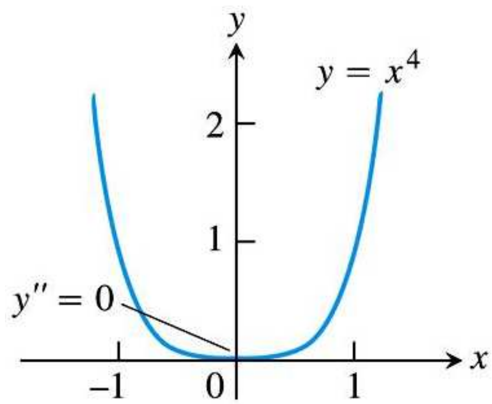
$y = x^3$ **changes concavity** at the point $(0,0)$; specify:

DEFINITION Point of Inflection
A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

At a point of inflection it is $y'' > 0$ on one, $y'' < 0$ on the other side, and either $y'' = 0$ or undefined at such point.

If y'' exists at an inflection point it is $y'' = 0$ and y' has a local maximum or minimum.

examples: (1) $y = x^4 \Rightarrow y'' = 12x^2$: $y''(0) = 0$ but y'' does not change sign – **no inflection point** at $x = 0$.



(2) $y = x^{1/3} \Rightarrow y'' = \left(\frac{1}{3}x^{-\frac{2}{3}}\right)' = -\frac{2}{9}x^{-\frac{5}{3}}$: y'' *does* change sign – **inflection point** at $x = 0$ *but* $y''(0)$ does not exist.