MTH4100 Calculus I

Lecture notes for Week 6

Thomas' Calculus, Sections 3.5 to 4.1 except 3.7

Rainer Klages

School of Mathematical Sciences
Queen Mary University of London

Autumn 2009

Derivatives of trigonometric functions

(1) Differentiate $f(x) = \sin x$:

- Start with the **definition** of $f'(x)$:

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

- Use $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$f'(x) = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

- Collect terms and apply limit laws:

$$f'(x) = \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

- Use $\lim_{h\to 0} \dfrac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$ to conclude $f'(x) = \cos x$.

(2) A very similar derivation gives $\dfrac{d}{dx} \cos x = -\sin x$.

(3) We still need

$$\begin{aligned} \frac{\mathbf{d}}{\mathbf{dx}} \tan \mathbf{x} &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ \text{(quotient rule)} &= \frac{\frac{d}{dx}(\sin x)\cos x - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\mathbf{1}}{\mathbf{\cos^2 x}} \end{aligned}$$

Summary: Derivatives of trigonometric functions

$$\begin{aligned} \frac{d}{dx} \sin x &= \cos x \\ \frac{d}{dx} \cos x &= -\sin x \\ \frac{d}{dx} \tan x &= \frac{1}{\cos^2 x} = \sec^2 x \\ \frac{d}{dx} \sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x \\ \frac{d}{dx} \cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = -\csc^2 x \\ \frac{d}{dx} \csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x \end{aligned}$$

Derivative of composites

example: relating derivatives

$y = \frac{3}{2}x$ is the same as $y = \frac{1}{2}u$ and $u = 3x$. By differentiating

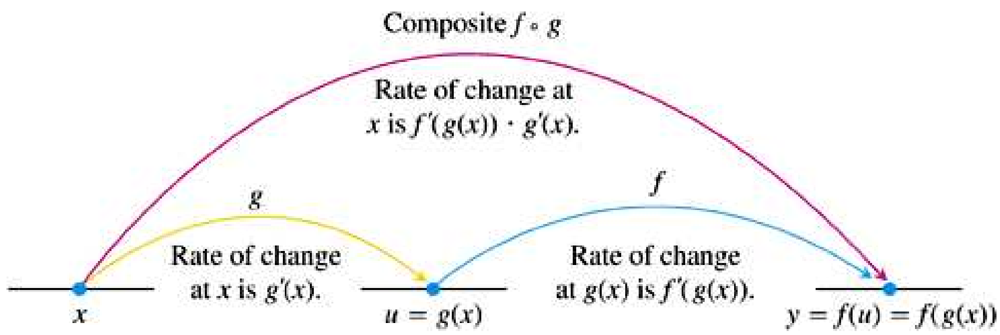
$$\frac{dy}{dx} = \frac{3}{2}\,, \quad \frac{dy}{du} = \frac{1}{2}\,, \quad \frac{du}{dx} = 3\,,$$

we find that

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}\,.$$

Coincidence or general formula: *Do rates of change multiply?*

The chain rule:



THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

examples:

(1) Differentiate $x(t) = \cos(t+1)$.
Here: Choose $x = \cos u$ and $u = t + 1$ and differentiate,

$$\frac{dx}{du} = -\sin u \quad \text{and} \quad \frac{du}{dt} = 1\,.$$

Then

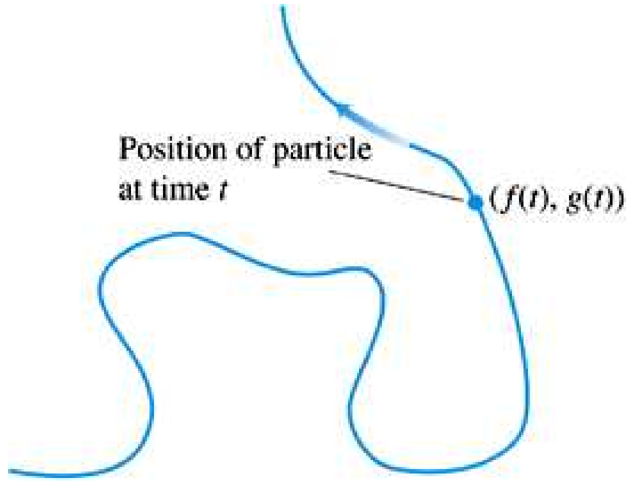
$$\frac{dx}{dt} = (-\sin u) \cdot 1 = -\sin(t+1)\,.$$

(2)

$$\frac{d}{dx} sin(x^2 + x) = \cos(x^2 + x)(2x + 1)$$

Parametric equations

example:



Describe a point moving in the xy-plane as a function of a **parameter** t ("time") by two functions

$$x = f(t) \ , \quad y = g(t) \ .$$

This *may* be the graph of a function, but it need not be.

DEFINITION **Parametric Curve**
If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

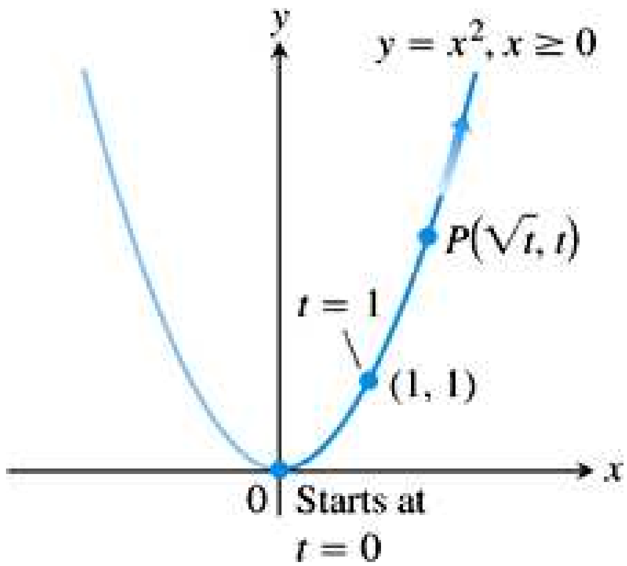
over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve. If $t \in [a, b]$, which is called a **parameter interval**, then $(f(a), g(a))$ is the **initial point**, and $(f(b), g(b))$ is the **terminal point**. Equations and interval constitute a **parametrisation** of the curve.

examples:

(1) Given is the parametrisation $x = \sqrt{t} \ , \ y = t \ , \ t \geq 0$. What is the path defined by these equations?

Solve for $y = f(x)$: $y = t \ , \ x^2 = t \ \Rightarrow y = x^2$. Note that the domain of f is *only* $[0, \infty)$!



(2) Find a parametrisation for the line segment from $(-2, 1)$ to $(3, 5)$.

- Start at $(-2, 1)$ for $t = 0$ by making the **ansatz** ("educated guess")

$$x = -2 + at \; , \quad y = 1 + bt \; .$$

- Implement the terminal point at $(3, 5)$ for $t = 1$:

$$3 = -2 + a \; , \quad 5 = 1 + b \; .$$

- We conclude that $a = 5$, $b = 4$.

- Therefore, the solution *based on our ansatz* is:

$$\boxed{x = -2 + 5t \; , \; y = 1 + 4t \; , \; 0 \leq t \leq 1} \; ,$$

  which indeed defines a straight line (why?).

A parametrised curve $x = f(t)$, $y = g(t)$ is **differentiable** at t if f and g are differentiable at t. At a point where y is a differentiable function of x, say $y = y(x)$, it is $y = y(x(t))$ and by the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \; .$$

Solving for dy/dx yields the

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \; .$$

example: Describe the motion of a particle whose position $P(x,y)$ at time t is given by

$$\boxed{x = a\cos t\ , \quad y = b\sin t\ , \quad 0 \le t \le 2\pi}$$

and compute the slope at P.

- Find the equation in (x,y) by eliminating t:
  Using $\cos t = x/a$, $\sin t = y/b$ and $\cos^2 t + \sin^2 t = 1$ we obtain
  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ ,$$
  which is the equation of an **ellipse**.
- With $\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$ the parametric formula yields
  $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t}{-a\sin t}\ .$$
  Eliminating t again we obtain $\quad \dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y}.$

Implicit differentiation

problem: We want to compute y' but do not have an **explicit relation** $y = f(x)$ available. Rather, we have an **implicit relation**

$$F(x,y) = 0$$

between x and y.

example:

$$F(x,y) = x^2 + y^2 - 1 = 0\ .$$

solutions:

1. Use *parametrisation*, for example, $x = \cos t$, $y = \sin t$ for the unit circle.
2. If no obvious parametrisation of $F(x,y) = 0$ is possible: use *implicit differentiation*.

example: Given $y^2 = x$, compute y'.

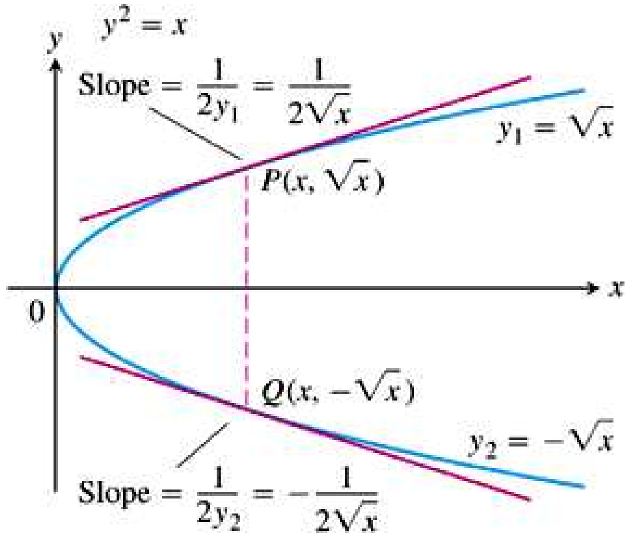
New method by differentiating *implicitly*:

- Differentiating *both sides* of the equation gives $2yy' = 1$.
- Solving for y' we get $\boxed{y' = \frac{1}{2y}}$.

Compare with differentiating *explicitly*:

- For $y^2 = x$ we have the two *explicit solutions* $|y| = \sqrt{x} \Rightarrow y_{1,2} = \pm\sqrt{x}$ with derivatives $\boxed{y'_{1,2} = \pm\frac{1}{2\sqrt{x}}}$.

- Compare with solution above: substituting $y = y_{1,2} = \pm\sqrt{x}$ therein reproduces the explicit result.



Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

example: Find dy/dx for the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

1. $\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$
2. $\frac{2yy'}{b^2} = -\frac{2x}{a^2}$
3. $y' = -\frac{b^2}{a^2}\frac{x}{y}$, as obtained via parametrisation in the previous lecture.

application: Motivate the power rule for rational powers by differentiating $y = x^{\frac{p}{q}}$ using implicit differentiation:

- write $$y^q = x^p$$
- differentiate: $$qy^{q-1}y' = px^{p-1}$$
- solve for y' as a function of x:

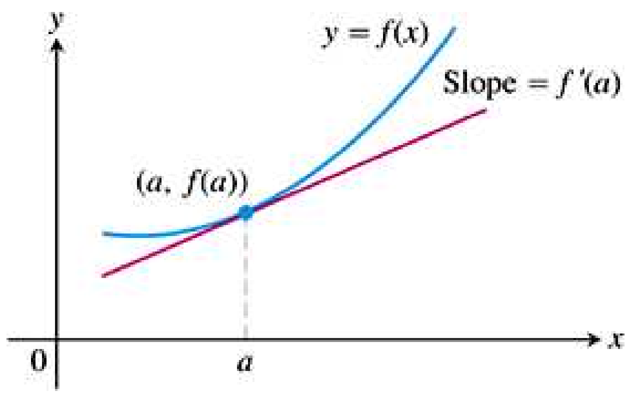
$$y' = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^p}{y^q}\frac{y}{x} = \frac{p}{q}\frac{y}{x} = \frac{p}{q}\frac{x^{\frac{p}{q}}}{x} = \frac{p}{q}x^{\frac{p}{q}-1}$$

THEOREM 4 Power Rule for Rational Powers
If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

note: Above we have silently assumed that y' exists! Therefore we have 'motivated' but not (yet) proved this theorem!

Linearisation



"Close to" the point $(a, f(a))$, the tangent $L(x) = f(a) + f'(a)(x - a)$ *(point-slope form)* is a "good" approximation for $y = f(x)$.

DEFINITIONS Linearization, Standard Linear Approximation
If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

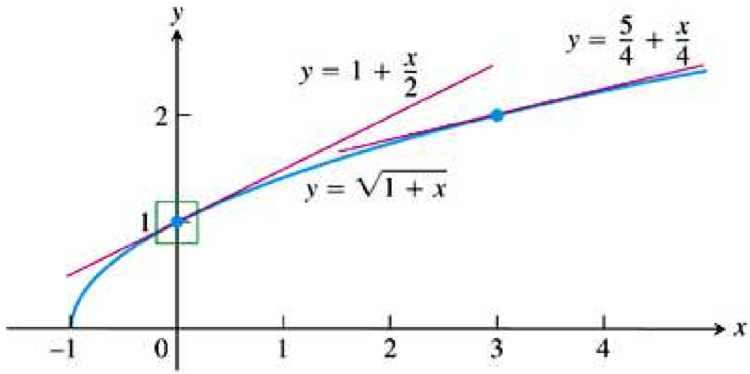
$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

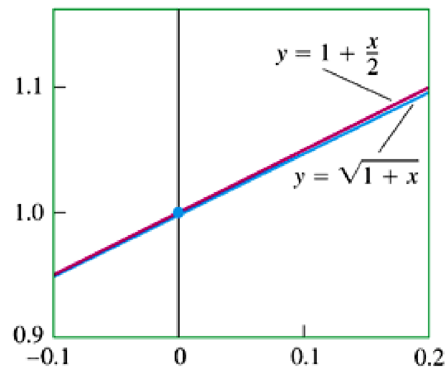
example: Compute the linearisation for $f(x) = \sqrt{1+x}$ at $x = a = 0$.

We have $f(0) = 1$ and with $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ we get $f'(0) = \frac{1}{2}$, so

$$L(x) = 1 + \frac{1}{2}x\ .$$



How accurate is this approximation? Magnify region around $x = 0$:



Approximation	True value	$\lvert$True value − approximation$\rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

Why are linearisations useful? Simplify problems, solve equations analytically, …many applications!

Make phrases like *"close to a point* $(a, f(a))$ *the linearisation is a good approximation"* mathematically precise in terms of **differentials**:

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ \underbrace{L(x) - f(a)}_{dy} &= f'(a)\underbrace{(x - a)}_{dx} \end{aligned}$$

Choose $x = a + dx$, $a = x$:

DEFINITION Differential

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Reading Assignment: read
Thomas' Calculus, p.225-228 about **Differentials**

Extreme values of functions

DEFINITIONS Absolute Maximum, Absolute Minimum

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

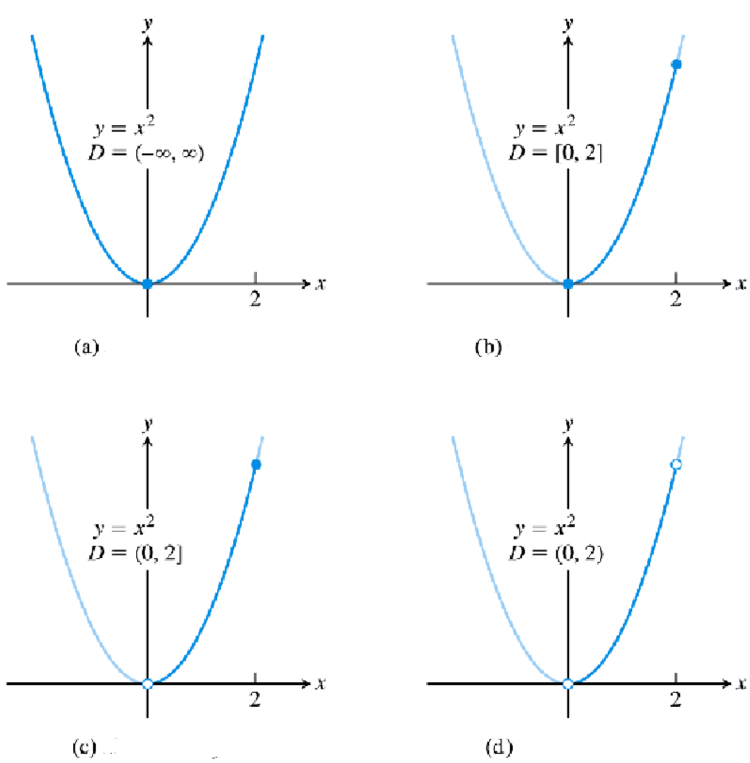
$$f(x) \le f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in } D.$$

These values are also called absolute **extrema**, or **global** extrema.

example:

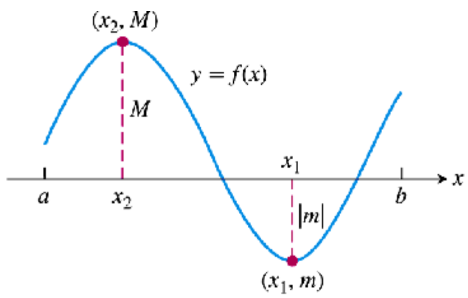


	Domain	abs. max.	abs. min.
(a)	$(-\infty, \infty)$	none	0, at 0
(b)	$[0, 2]$	4, at 2	0, at 0
(c)	$(0, 2]$	4, at 2	none
(d)	$(0, 2)$	none	none

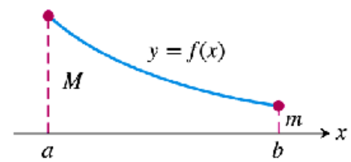
The existence of a global maximum and minimum is ensured by

THEOREM 1 The Extreme Value Theorem
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

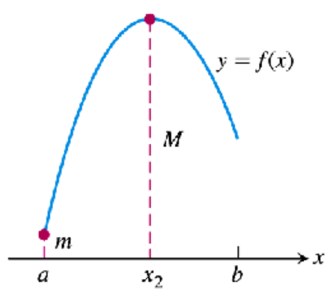
examples:



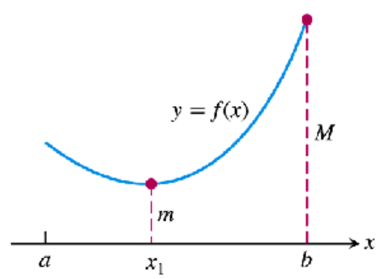
Maximum and minimum at interior points



Maximum and minimum at endpoints

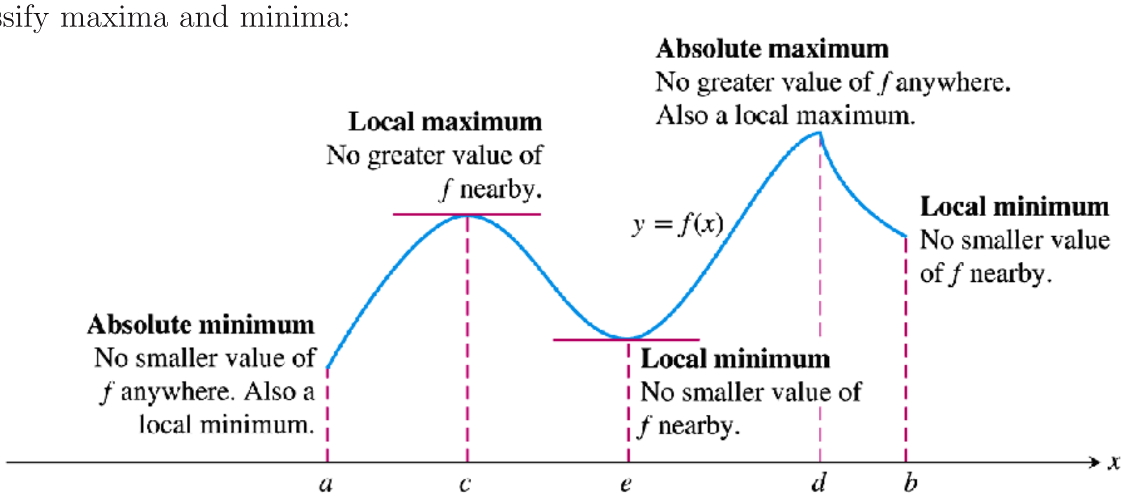


Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

Classify maxima and minima:



DEFINITIONS Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

...and the extension of this definition to endpoints via half-open intervals at endpoints.

note: *Absolute* extrema are automatically *local* extrema!