Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 5

Thomas' Calculus, Sections 2.6 to 3.5 except 3.3

Rainer Klages

School of Mathematical Sciences
Queen Mary University of London

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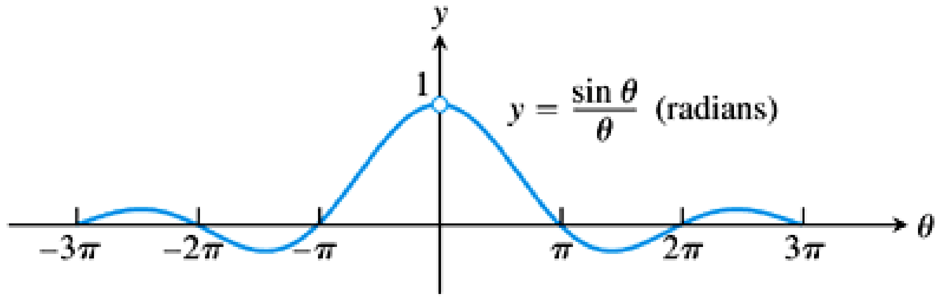
Continuous extension to a point
example:

$$f(x) = \frac{\sin x}{x}$$



NOT TO SCALE

is defined and continuous for all $x \neq 0$. As $\lim_{x \to 0} \frac{\sin x}{x} = 1$, it makes sense to **define a new function**

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Definition 1 *If* $\lim_{x \to c} f(x) = L$ *exists, but* $f(c)$ *is not defined, we define a new function*

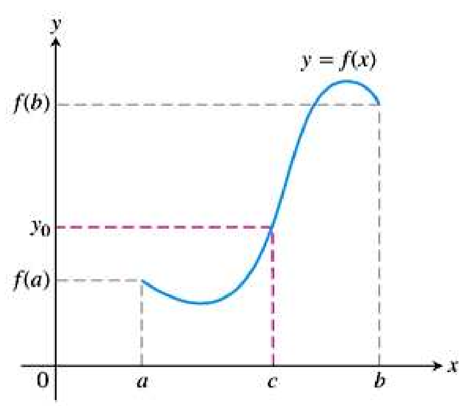
$$F(x) = \begin{cases} f(x) & \text{for } x \neq c \\ L & \text{for } x = c \end{cases},$$

which is continuous at c*. It is called the* **continuous extension** *of* $f(x)$ *to* c*.*

A function has the *intermediate value property* if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Geometrical interpretation of this theorem: Any horizontal line crossing the y-axis between $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

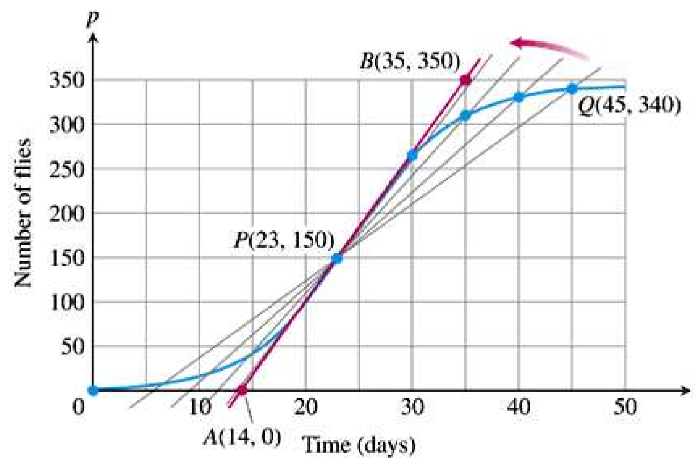
Continuity is essential: if f is discontinuous at any point of the interval, then the function may "jump" and miss some values.

Differentiation

Recall our discussion of **average and instantaneous rates of change**.

example: revisit growth of fruit fly population

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$



basic idea:

- Investigate the **limit of the secant slopes** as Q approaches P.
- Take it to be the **slope of the tangent** at P.

Now we can use **limits** to make this idea precise...

example: Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$.

- Choose a point Q a **horizontal distance** $h \neq 0$ away from P,

$$Q(2 + h, (2 + h)^2) .$$

- The **secant** through P and Q has the slope

$$\frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{(2 + h) - 2} = \frac{4 + 4h + h^2 - 4}{h} = 4 + h .$$

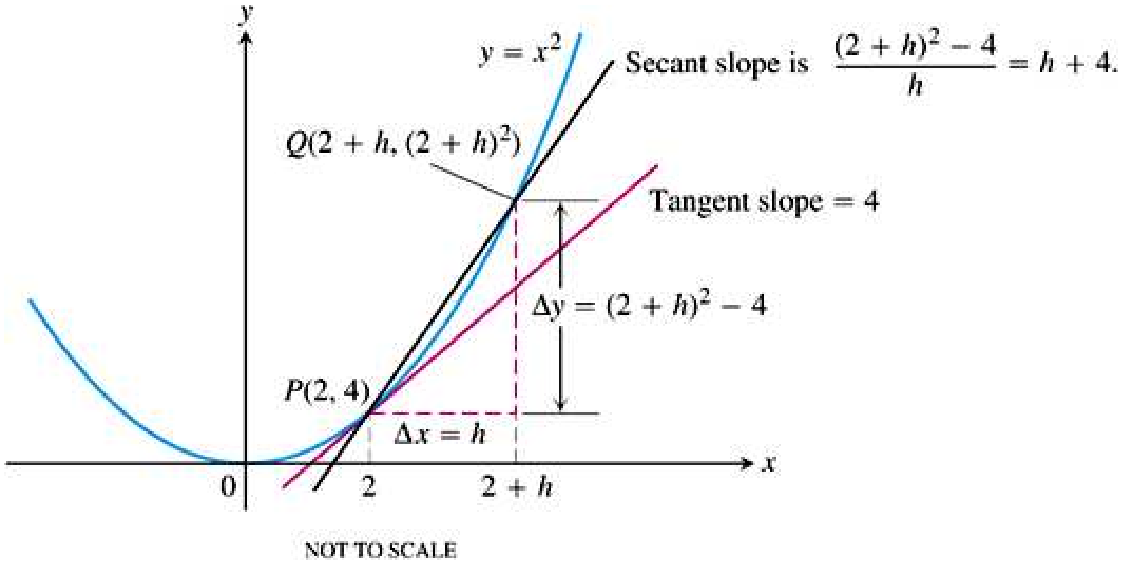
- As Q approaches P h approaches 0, hence

$$m = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0}(4 + h) = 4$$

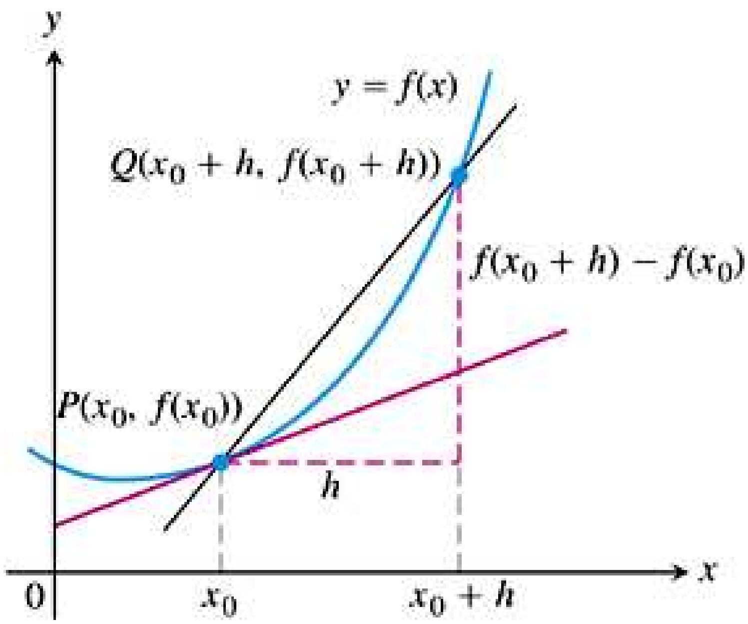
  must be the **parabola's slope at** P.

- The **equation of the tangent** through $P(2,4)$ is $y = y_1 + m(x - x_1)$;
  here: $y = 4 + 4(x - 2)$ or $y = 4x - 4$.

summary:



Now **generalise** to arbitrary curves and arbitrary points:



DEFINITIONS Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

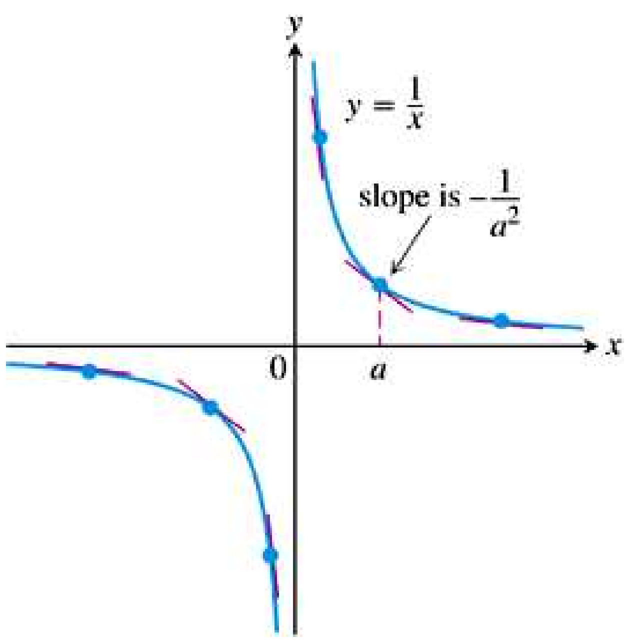
The **tangent line** to the curve at P is the line through P with this slope.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

example: Find slope and tangent to $y = 1/x$ at $x_0 = a \neq 0$

1. $f(a) = \dfrac{1}{a}$, $f(a+h) = \dfrac{1}{a+h}$
2. slope:
$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\ &= \lim_{h \to 0} \frac{a - (a+h)}{h \cdot a(a+h)} \\ &= \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2} \end{aligned}$$
3. tangent line at $(a, 1/a)$: $y = \dfrac{1}{a} + \left(-\dfrac{1}{a^2}\right)(x - a)$ or $y = \dfrac{2}{a} - \dfrac{x}{a^2}$.



The expression $\dfrac{f(x_0 + h) - f(x_0)}{h}$ is called the **difference quotient** of f at x_0 with increment h. The limit as h approaches 0, if it exists, is called the **derivative** of f at x_0.

Let $x \in D(f)$.

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

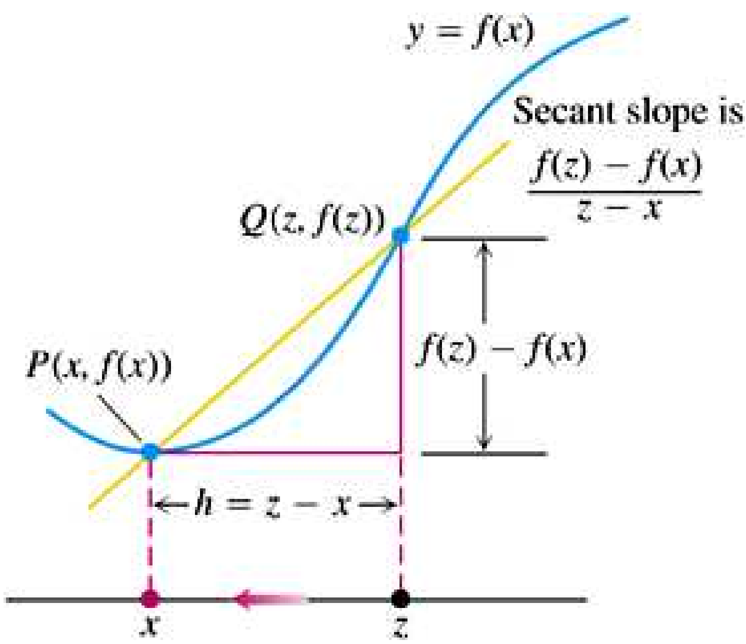
provided the limit exists.

If $f'(x)$ exists, we say that f is **differentiable** at x.

Choose $z = x + h$: $h = z - x$ approaches 0 if and only if $z \to x$.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



Equivalent notation: If $y = f(x)$, $y' = f'(x) = \frac{d}{dx} f(x) = \frac{dy}{dx}$.

Calculating a derivative is called **differentiation** ("derivation" is something else!).

example: Differentiate from first principles $f(x) = \frac{x}{x-1}$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)}\\
&= \lim_{h\to 0}\frac{1}{h}\frac{-h}{(x+h-1)(x-1)}\\
&= -\frac{1}{(x-1)^2}
\end{aligned}$$

example: Differentiate $f(x) = \sqrt{x}$ by using the alternative formula for derivatives.

$$\begin{aligned}
f'(x) &= \lim_{z\to x}\frac{f(z)-f(x)}{z-x}\\
&= \lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{z-x}\\
&= \lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{(\sqrt{z}-\sqrt{x})(\sqrt{z}+\sqrt{x})}\\
&= \lim_{z\to x}\frac{1}{\sqrt{z}+\sqrt{x}}\\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

note: For $f'(x)$ at $x = 4$, one sometimes writes

$$f'(4) = \frac{d}{dx}\sqrt{x}\bigg|_{x=4} = \frac{1}{2\sqrt{x}}\bigg|_{x=4}.$$

One-sided derivatives

In analogy to one-sided limits, we define **one-sided derivatives**:

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h} \quad \textbf{right-hand derivative} \text{ at } x$$

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h} \quad \textbf{left-hand derivative} \text{ at } x$$

f is differentiable at x if and only if these two limits exist and are equal.

example: Show that $f(x) = |x|$ is not differentiable at $x = 0$. [2009 exam question]

- right-hand derivative at $x = 0$:

$$\lim_{h\to 0^+}\frac{|0+h|-|0|}{h} = \lim_{h\to 0^+}\frac{|h|}{h} = 1$$

- left-hand derivative at $x = 0$:

$$\lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1 ,$$

so the right-hand and left-hand derivatives *differ*.

Theorem 1 *If f has a derivative at $x = c$, then f is continuous at $x = c$.*

Proof: *Trick:* For $h \neq 0$, write

$$f(c+h) = f(c) + \frac{f(c+h) - f(c)}{h} h .$$

By assumption, $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$. Therefore,

$$\lim_{h\to 0} f(c+h) = f(c) + f'(c) \cdot 0 = f(c) .$$

According to definition of continuity, f is continuous at $x = c$. q.e.d.

caution: The converse of the theorem is *false*!
note: The theorem implies that if a function is *discontinuous* at $x = c$, then it is *not differentiable* there.

Differentiation rules ('machinery')

Proof of one rule see ff; proof of other rules see book, Section 3.2.

Rule 1 (Derivative of a Constant Function) *If f has the constant value $f(x) = c$, then*

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0 .$$

Rule 2 (Power Rule for Positive Integers) *If n is a positive integer, then*

$$\frac{d}{dx} x^n = n x^{n-1} .$$

Rule 3 (Constant Multiple Rule) *If u is a differentiable function of x, and c is a constant, then*

$$\frac{d}{dx}(cu) = c\frac{du}{dx} .$$

Proof:

$$\begin{aligned} \frac{d}{dx} cu &= \\ \textit{(def. of derivative)} &= \lim_{h\to 0} \frac{cu(x+h) - cu(x)}{h} \\ \textit{(limit laws)} &= c \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\ \textit{(u is differentiable)} &= c\frac{du}{dx} \end{aligned}$$

q.e.d.

Rule 4 (Derivative Sum Rule) *If u and v are differentiable functions of x, then*

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \ .$$

example: Differentiate $y = 3x^4 + 2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(3x^4+2) \\ \textit{(rule 4)} \qquad &= \frac{d}{dx}(3x^4) + \frac{d}{dx}(2) \\ \textit{(rule 3)} \qquad &= 3\frac{d}{dx}(x^4) + \frac{d}{dx}(2) \\ \textit{(rule 2)} \qquad &= 3\cdot 4x^3 + \frac{d}{dx}(2) \\ \textit{(rule 1)} \qquad &= 12x^3 + 0 = 12x^3 \end{aligned}$$

Rule 5 (Derivative Product Rule) *If u and v are differentiable functions of x, then*

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} \ .$$

example: Differentiate $y = (x^2+1)(x^3+3)$.

$$\text{here: } u = x^2+1 \ , \quad v = x^3+3$$

$$u' = 2x \ , \qquad v' = 3x^2$$

$$y' = 2x(x^3+3) + (x^2+1)3x^2 = 5x^4 + 3x^2 + 6x$$

Rule 6 (Derivative Quotient Rule) *If u and v are differentiable functions of x and $v(x) \neq 0$, then*

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \ .$$

example: Differentiate $y = \dfrac{t-2}{t^2+1}$.

$$\text{here: } u = t-2 \ , \quad v = t^2+1$$

$$u' = 1 \ , \qquad v' = 2t$$

$$y' = \frac{1(t^2+1) - (t-2)2t}{(t^2+1)^2} = \frac{-t^2+4t+1}{(t^2+1)^2}$$

Common mistakes: $(uv)' = u'v'$ **and** $(u/v)' = u'/v'$ are generally **WRONG!**

Rule 7 (Power Rule for Negative Integers) *If n is a negative integer and $x \neq 0$, then*

$$\frac{d}{dx}x^n = nx^{n-1} \ .$$

(Proof: define $n = -m$ and use the quotient rule.)

example: $\frac{d}{dx}\left(\frac{1}{x^{11}}\right) = \frac{d}{dx}(x^{-11}) = -11x^{-12}$.

Higher-order derivatives

If f' is differentiable, we call $f'' = (f')'$ the **second derivative** of f.
Notation: $f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''$.
Similarly, we write $f''' = (f'')'$ for the third derivative, and generally for the n**-th derivative**, $n \in \mathbb{N}_0$: $f^{(n)} = (f^{(n-1)})'$ with $f^{(0)} = f$.

example: Differentiate repeatedly $f(x) = x^3$ and $g(x) = x^{-2}$.

$$\begin{aligned} f'(x) &= 3x^2 & g'(x) &= -2x^{-3} \\ f''(x) &= 6x & g''(x) &= 6x^{-4} \\ f'''(x) &= 6 & g'''(x) &= -24x^{-5} \\ f^{(4)}(x) &= 0 & g^{(4)}(x) &= \dots \end{aligned}$$