MTH4100 Calculus I

Lecture notes for Week 4

Thomas' Calculus, Sections 2.4 to 2.6

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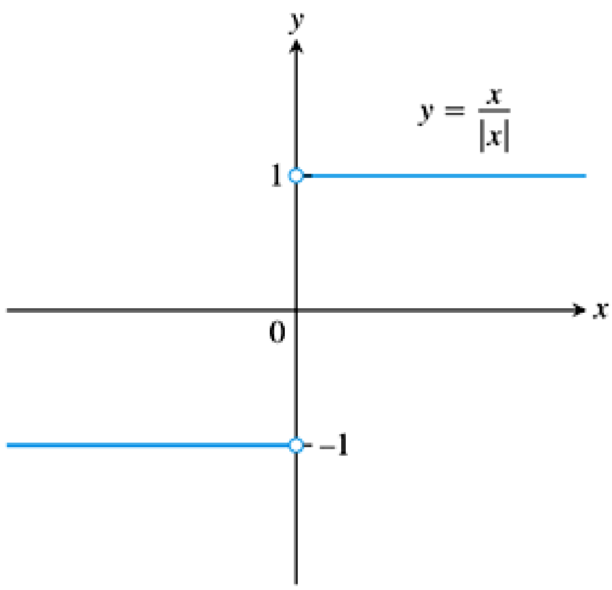
One-sided limits and limits at infinity

To have a *limit* L as $x \to c$, a function f must be defined on both sides of c (**two-sided limit**). If f fails to have a limit as $x \to c$, it may still have a **one-sided limit** if the approach is only from the right (*right-hand limit*) or from the left (*left-hand limit*).

We write $\boxed{\lim_{x\to c^+} f(x) = L}$ or $\boxed{\lim_{x\to c^-} f(x) = M}$.
The symbol $x \to c^+$ means that we only consider values of x *greater than* c. The symbol $x \to c^-$ means that we only consider values of x *less than* c.

example:



- $\lim_{x\to 0^+} f(x) = 1$
- $\lim_{x\to 0^-} f(x) = -1$
- $\lim_{x\to 0} f(x)$ **does not exist**

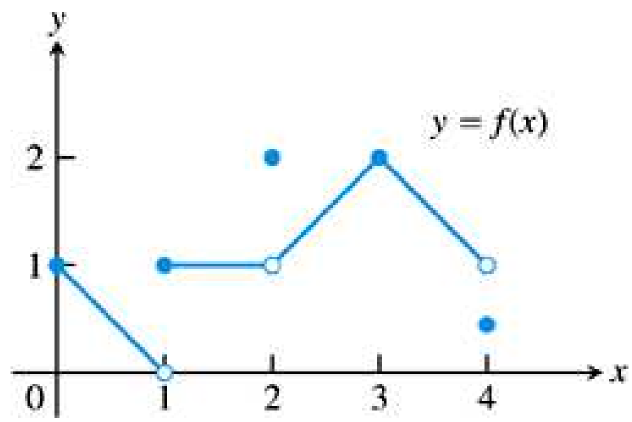
THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

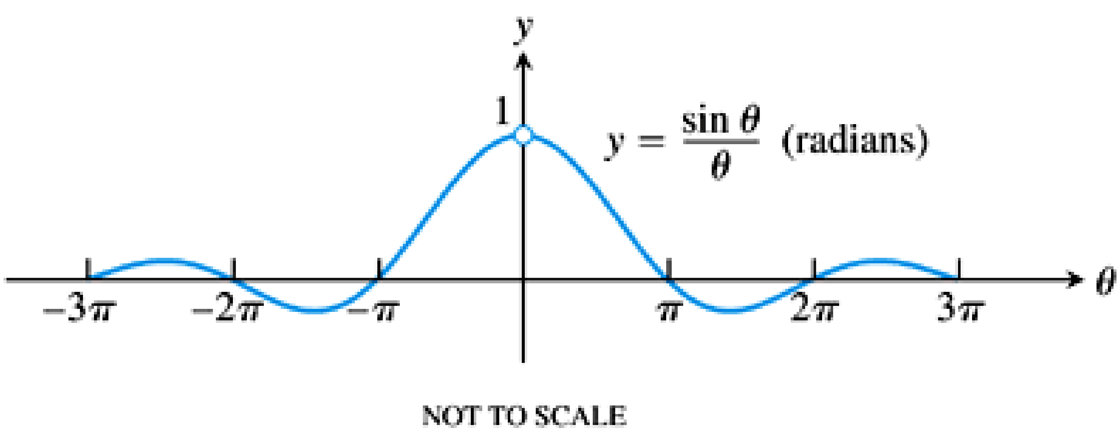
Limit laws and theorems for limits of polynomials and rational functions all hold for one-sided limits.

example:



c	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	$\lim_{x\to c} f(x)$
0	n.a.	1	n.a.
1	0	1	n.a.
2	1	1	1
3	2	2	2
4	1	n.a.	n.a.

Limits involving $\dfrac{\sin\theta}{\theta}$:



Theorem 1

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \quad (\theta \textit{ in radians})$$

Proof: Show equality of left-hand and right-hand limits at $x = 0$ by using the 'Sandwich Theorem' (Thomas' Calculus p.105ff).

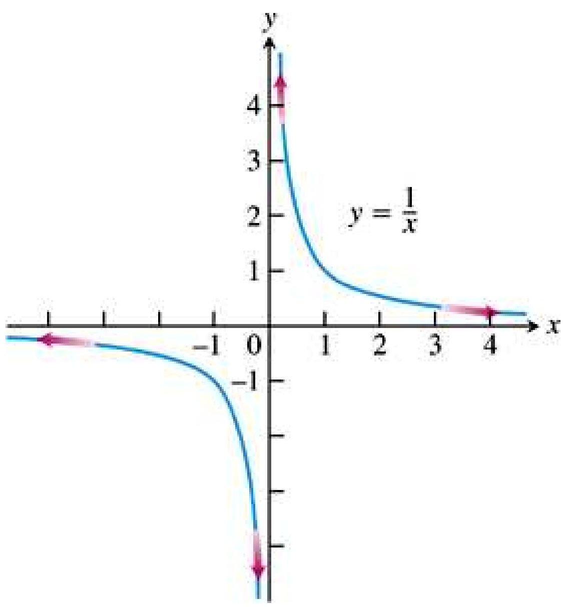
example:
Compute

$$\begin{aligned} & \lim_{h\to 0}\frac{\cos h - 1}{h} = \qquad [\sin^2(h/2) = (1-\cos h)/2] \\ = & \lim_{h\to 0}\frac{-2\sin^2(h/2)}{h} \\ = & \lim_{h\to 0} -\frac{\sin(h/2)}{h/2}\sin(h/2) \qquad [\theta = h/2] \\ = & \lim_{\theta\to 0} -\frac{\sin\theta}{\theta}\sin\theta \qquad [\text{limit laws}] \\ = & -1\cdot 0 = 0 \end{aligned}$$

Special case of a limit:

x approaching positive/negative infinity

example:



similar to *one-sided limit*

Definition 1 (informal) *1. We say that $f(x)$ has the* **limit** L **as** x **approaches infinity** *and write*

$$\lim_{x\to\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L.

2. We say that $f(x)$ has the **limit** L **as** x **approaches minus infinity** *and write*

$$\lim_{x\to -\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

examples:

$$\lim_{x\to\pm\infty} k = k \quad \text{and} \quad \lim_{x\to\pm\infty} \frac{1}{x} = 0$$

Simply replace $x \to c$ by $x \to \pm\infty$ in the previous limit laws theorem:

Theorem 2 (Limit laws as x approaches infinity) *If L, M and k are real numbers and $\lim_{x\to\pm\infty} f(x) = L$ and $\lim_{x\to\pm\infty} g(x) = M$, then*

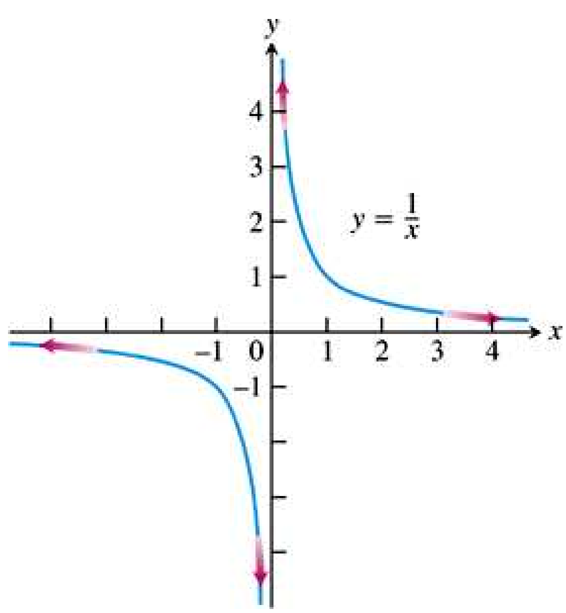
1. Sum Rule: $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$
2. Difference Rule: $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$
3. Product Rule: $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$
4. Constant Multiple Rule: $\lim_{x\to\pm\infty}(k \cdot f(x)) = k \cdot L$
5. Quotient Rule: $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. Power Rule: *If s and r are integers with no common factor and $s \neq 0$, then*
$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$
provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

example:

$$\begin{aligned} \lim_{x\to\infty}\left(5 + \frac{1}{x}\right) &= && \text{[sum rule]} \\ &= \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x} && \text{[known results]} \\ &= 5 \end{aligned}$$

This leads us to **horizontal asymptotes**.

example:



$$\lim_{x\to\infty}\frac{1}{x}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x}=0$$

The graph approaches the line $y = 0$ **asymptotically**: The line is an **asymptote** of the graph.

DEFINITION Horizontal Asymptote

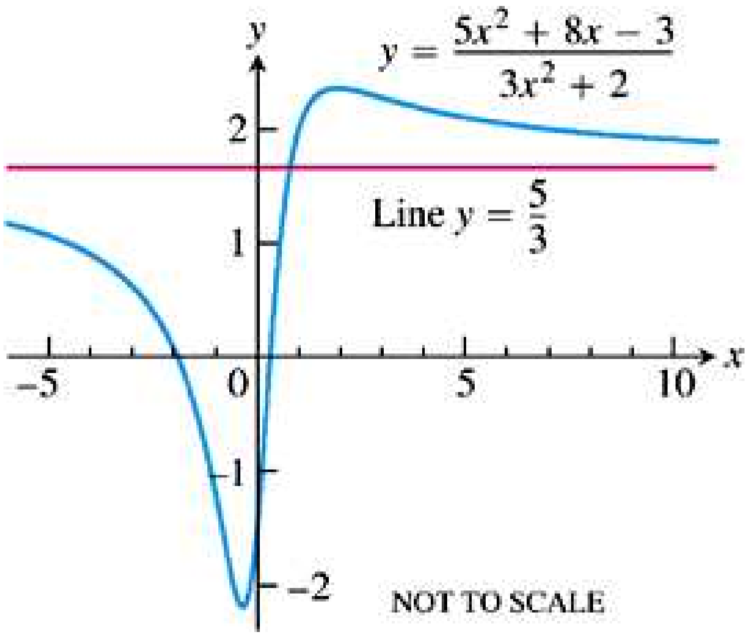
A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

example:

Calculate the horizontal asymptote for rationals: *pull out the highest power of x.*

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{x^2(5+8/x-3/x^2)}{x^2(3+2/x^2)}=\frac{5}{3}$$



The graph has the line $y = 5/3$ as a **horizontal asymptote** on *both the left and the right*, because

$$\lim_{x\to\pm\infty} f(x) = \frac{5}{3} \quad .$$

What happens if the degree of the polynomial in the numerator is one greater than that in the denominator? Do polynomial division:

example:

$$f(x)=\frac{2x^2-3}{7x+4}=\frac{2}{7}x-\frac{8}{49}-\frac{115}{49(7x+4)}$$

with

$$\lim_{x\to\pm\infty} \frac{-115}{49(7x+4)} = 0$$

If for a rational function $f(x) = p(x)/q(x)$ the degree of $p(x)$ is *one greater* than the degree of $q(x)$, polynomial division gives

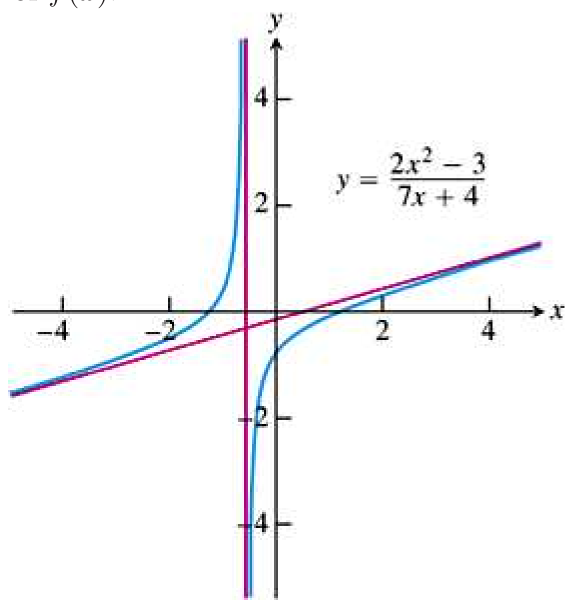
$$f(x) = ax + b + r(x) \qquad \text{with} \lim_{x\to\pm\infty} r(x) = 0$$

$y = ax + b$ is called an **oblique (slanted) asymptote**.

For the above example

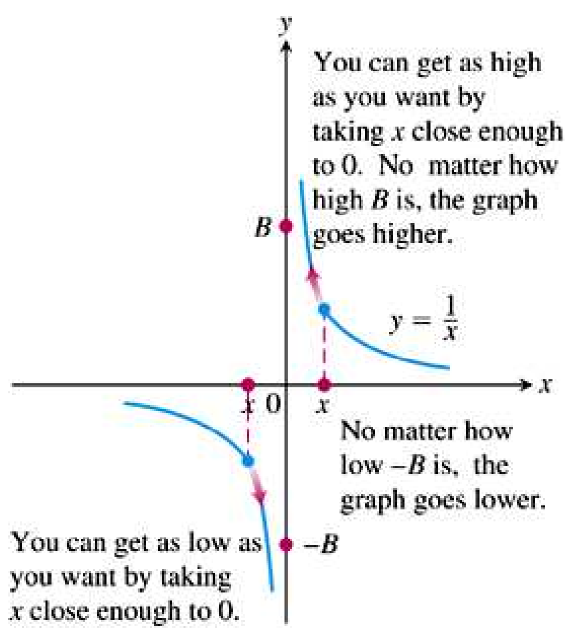
$$y = \frac{2}{7}x - \frac{8}{49}$$

is the oblique asymptote of $f(x)$.



One-sided infinite limits

example:



$f(x) = \dfrac{1}{x}$ has *no limit* as $x \to 0^+$. However, it is convenient to still say that $f(x)$ *approaches* ∞ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

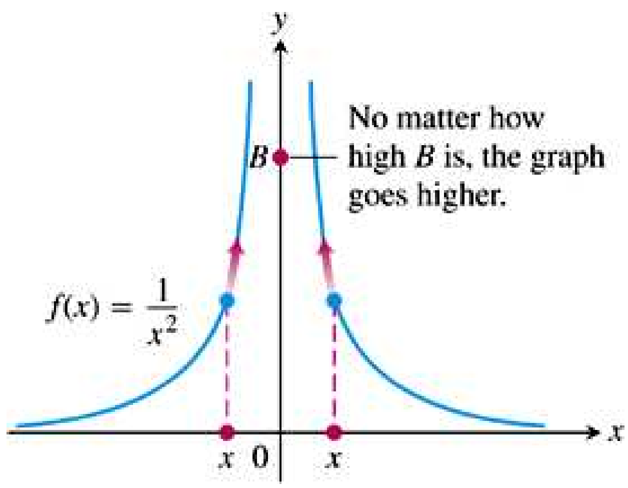
Similarly,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

note: $\lim_{x \to 0^+} \dfrac{1}{x} = \infty$ really means that **the limit does not exist, because** $1/x$ **becomes arbitrarily large and positive as** $x \to 0^+$**!**

Two-sided infinite limits

example: What is the behaviour of $f(x) = \dfrac{1}{x^2}$ near $x = 0$?



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

as the values of $1/x^2$ are positive and become arbitrarily large as $x \to 0$.

Definition 2 (informal) *1. We say that* $f(x)$ **approaches infinity as** x **approaches** x_0 *and write*

$$\lim_{x \to x_0} f(x) = \infty$$

if, as $x \to x_0$*, the values of* f *grow without bound, eventually reaching and surpassing every positive real number.*

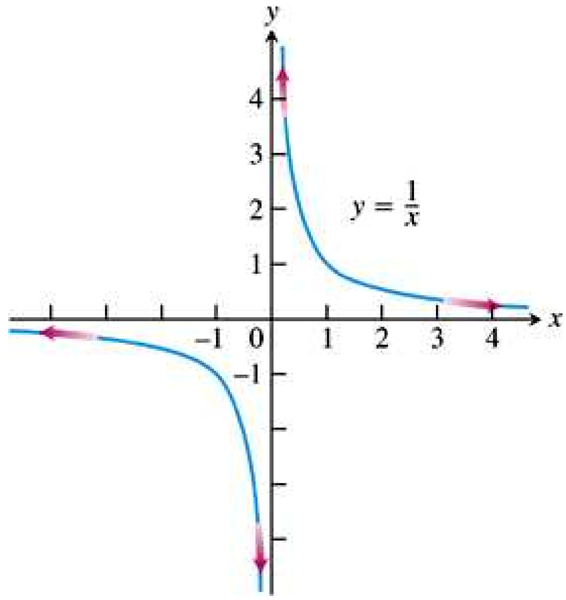
2. We say that $f(x)$ **approaches negative infinity as** x **approaches** x_0 *and write*

$$\lim_{x \to x_0} f(x) = -\infty$$

if, as $x \to x_0$*, the values of* f *become arbitrarily large and negative.*

Vertical asymptotes

example:



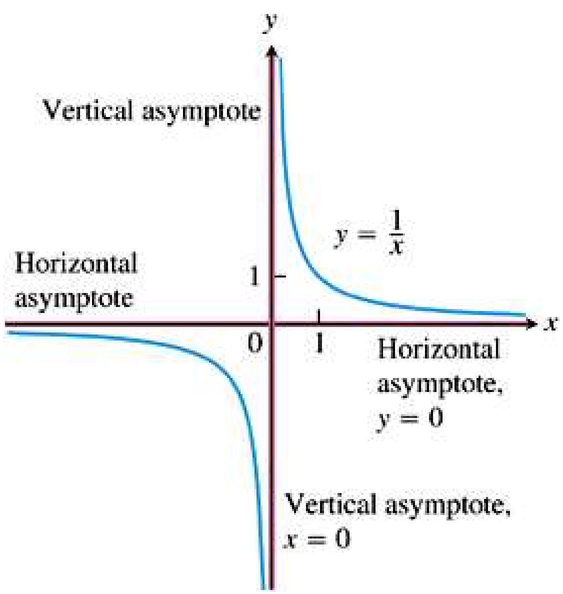
Recall that $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. This means that the graph approaches the line $x = 0$ **asymptotically**: The line is an **asymptote** of the graph.

DEFINITION Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty.$$

Summary: asymptotes for $y = 1/x$



Further asymptotic behavior

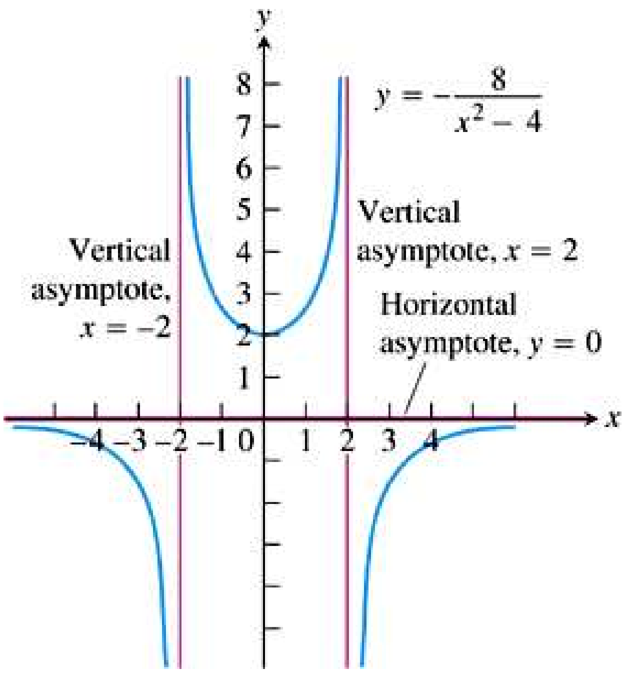
example: Find the horizontal and vertical asymptotes of

$$f(x) = -\frac{8}{x^2 - 4}$$

Check for the behaviour as $x \to \pm\infty$ and as $x \to \pm 2$ (why?):

- $\lim_{x\to\pm\infty} f(x) = 0$, approached from *below*.
- $\lim_{x\to -2^-} f(x) = -\infty$, $\lim_{x\to -2^+} f(x) = \infty$
- $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = -\infty$ (because $f(x)$ is *even*)

Asymptotes are (why?) $\boxed{y = 0\ ,\ x = -2\ ,\ x = 2}$.



The graph approaches the x-axis from **only one side**: Asymptotes do not have to be two-sided!

example: Find the asymptotes of

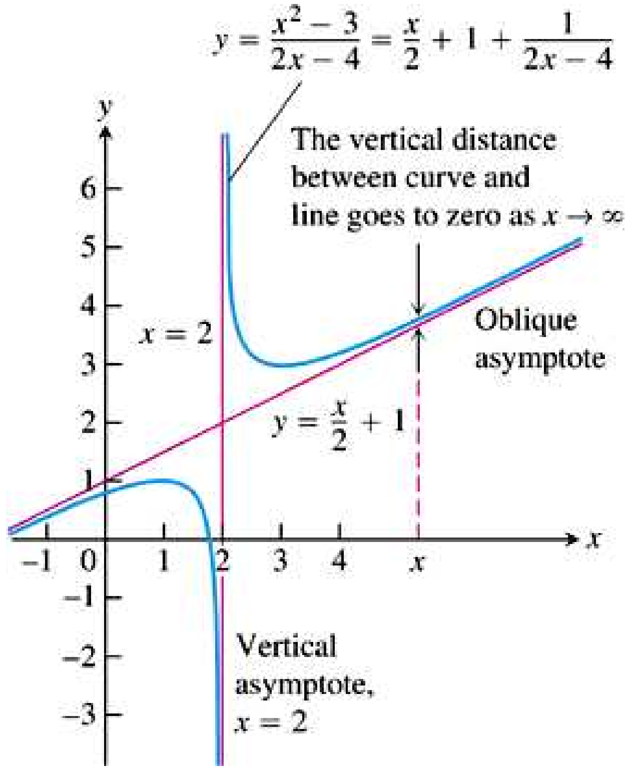
$$f(x) = \frac{x^2 - 3}{2x - 4}$$

- Rewrite by **polynomial division:**

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

.

- Asymptotes are $\boxed{y = \frac{x}{2} + 1\,, \quad x = 2}$.

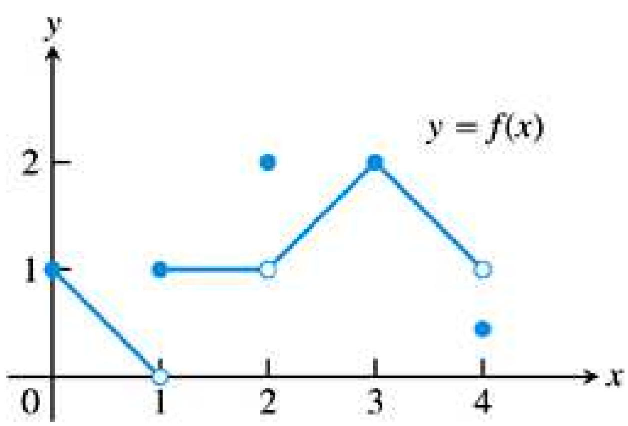


We say that $x/2 + 1$ **dominates** when x is large and that $1/(2x - 4)$ **dominates** when x is near 2.

Continuity

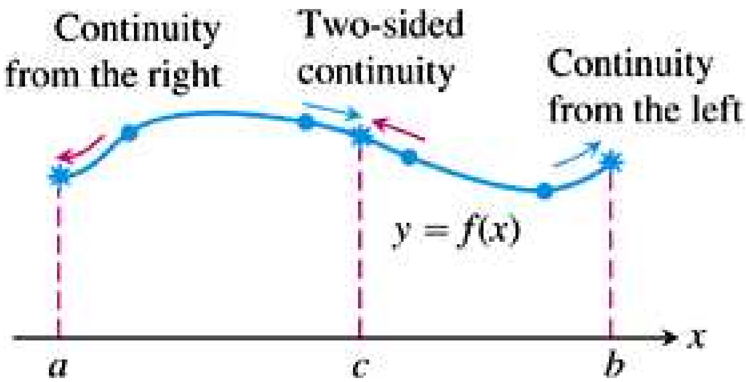
Definition 3 (informal) *Any function whose graph can be sketched* over its domain *in one continuous motion, i.e.* without lifting the pen, *is an example of a* **continuous function**.

example:



This function is continuous on $[0, 4]$ *except at* $x = 1, x = 2$ and $x = 4$.
More precisely, we need to define continuity at *interior* and at *end points*.

example:



DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b), \quad \text{respectively}.$$

For any $x = c$ in the domain of f one defines:

- **right-continuous**: $\lim_{x\to c^+} f(x) = f(c)$
- **left-continuous**: $\lim_{x\to c^-} f(x) = f(c)$

A function f is **continuous at an interior point** $x = c$ if and only if it is **both right-continuous and left-continuous at** c.

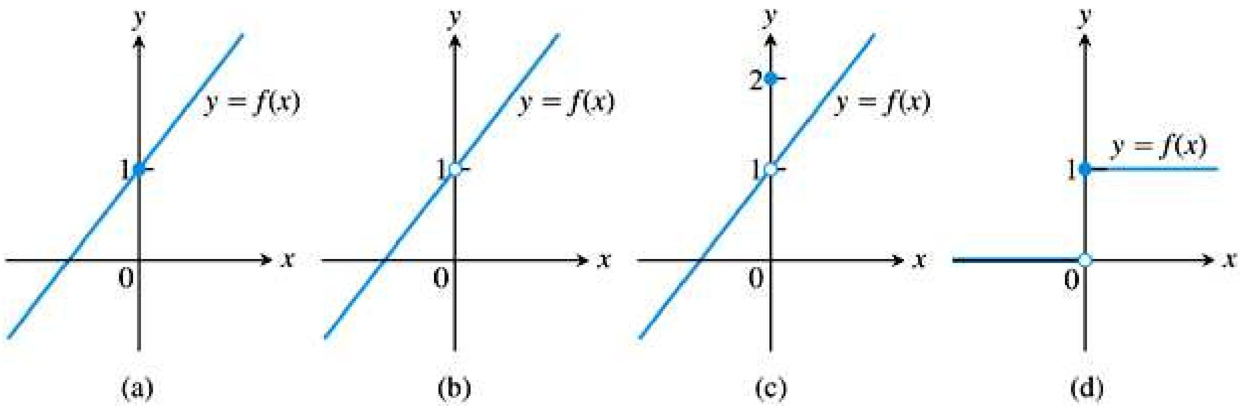
Remark 1 (Continuity Test) A function $f(x)$ is continuous at an interior point of its domain $x = c$ if and only if it meets the following three conditions:

1. $f(c)$ exists.
2. f has a limit as x approaches c.
3. The limit equals the function value.

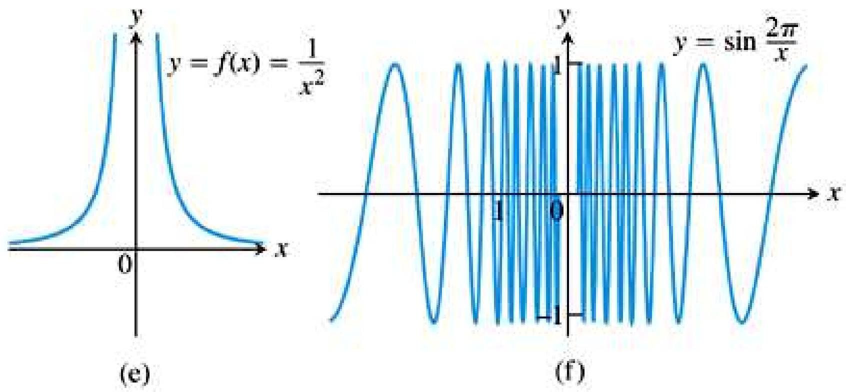
Note the difference to a function merely having a limit!

If a function f is not continuous *at a point* c, we say that f is **discontinuous** at c. Note that c need not be in the domain of f!

examples: Continuity and discontinuity at $c = 0$.



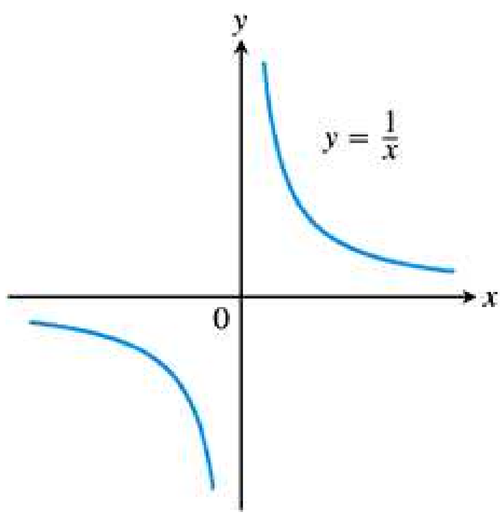
continuous   not continuous   jump discontinuity



infinite discontinuity   oscillating discontinuity

- A function is **continuous on an interval** if and only if it is continuous at every point of the interval.
- A **continuous function** is a function that is continuous at every point of its domain.

example:



- $y = \frac{1}{x}$ is a continuous function: It is continuous at every point of its domain.
- It has nevertheless a **discontinuity** at $x = 0$: No contradiction, because it is not defined there.

Previous limit laws straightforwardly imply:

THEOREM 9 Properties of Continuous Functions

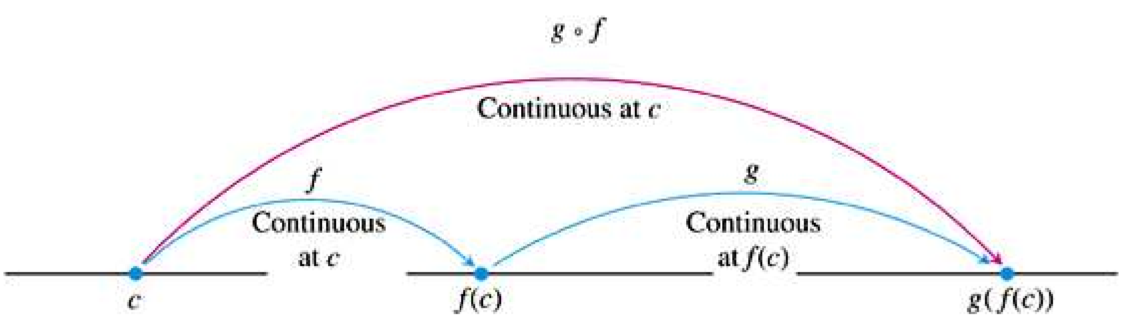
If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

example: $f(x) = x$ and constant functions are continuous $\Rightarrow$ polynomials and rational functions are also continuous

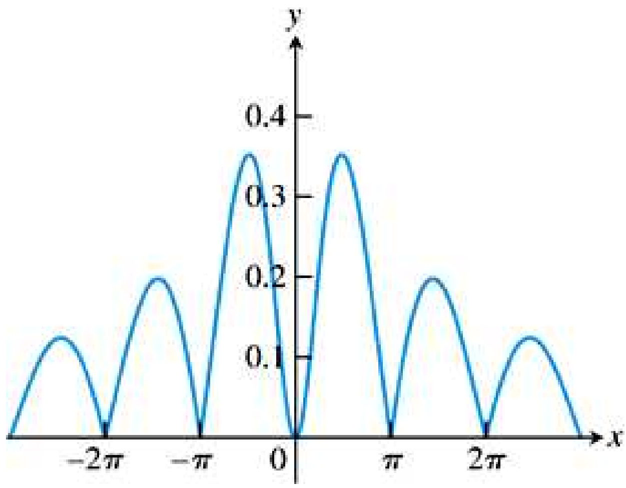
THEOREM 10 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



example: Show that $y = \left|\frac{x \sin x}{x^2+2}\right|$ is everywhere continuous.

- Note that $y = \sin x$ (and $y = \cos x$) are everywhere continuous.
- $f(x) = \frac{x \sin x}{x^2+2}$ is continuous (why?).
- $g(x) = |x|$ is continuous (why?).
- Therefore $y = g \circ f$ is continuous.


y
0.4
0.3
0.2
0.1
−2π
−π
0
π
2π
x