Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 3

Thomas' Calculus, Sections 1.5, 1.6, 2.1, 2.2 and 2.4

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examples: finding formulas for composites

$$\begin{aligned} f(x) &= \sqrt{x} \quad \text{with} \quad D(f) = [0, \infty) \\ g(x) &= x + 1 \quad \text{with} \quad D(g) = (-\infty, \infty) \end{aligned}$$

composite	**domain**
$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
$(g \circ g)(x) = g(g(x)) = g(x) + 1 = x + 2$	$(-\infty, \infty)$

$$\begin{aligned} f(x) &= \sqrt{x} \quad \text{with} \quad D(f) = [0, \infty) \\ g(x) &= x^2 \quad \text{with} \quad D(g) = (-\infty, \infty) \end{aligned}$$

composite	**domain**
$(f \circ g)(x) = \lvert x \rvert$	$(-\infty, \infty)$
$(g \circ f)(x) = x$	$[0, \infty)$

Shifting a graph of a function:

Shift Formulas

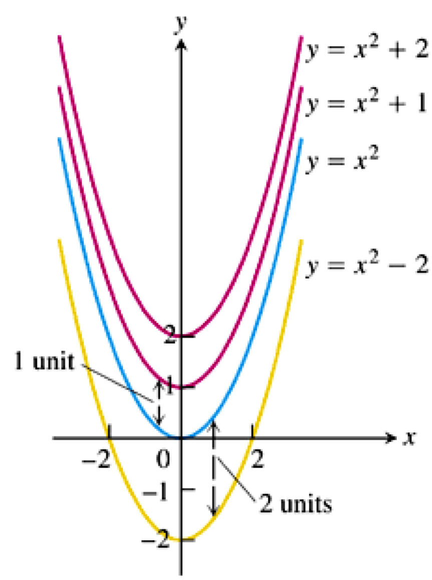
Vertical Shifts

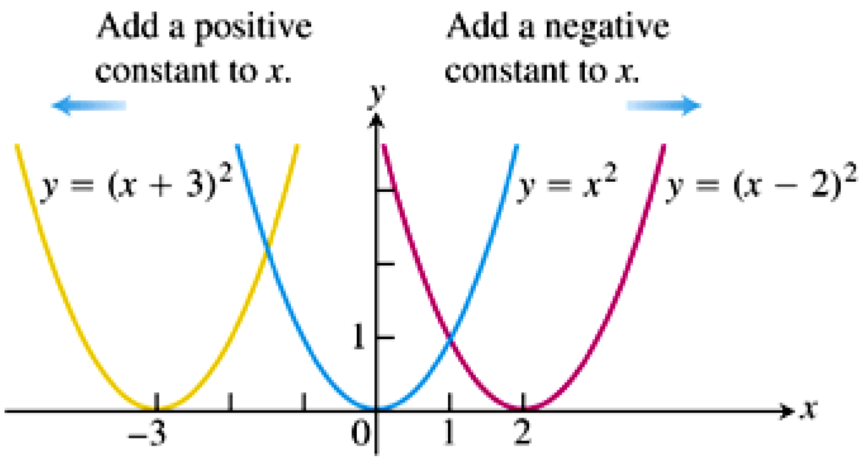
$y = f(x) + k$ — Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ — Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

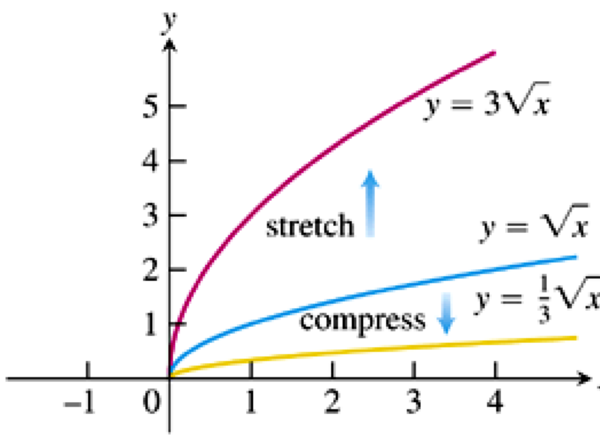
examples:



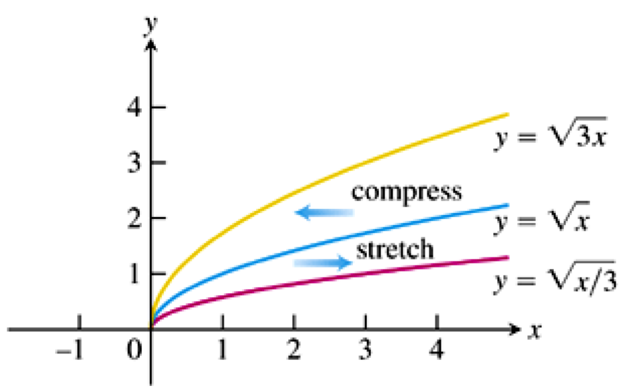


Scaling and reflecting a graph of a function:
For $c > 1$,

$y = cf(x)$	*stretches* the graph of f *along the y-axis* by a factor of c
$y = \frac{1}{c}f(x)$	*compresses* the graph of f *along the y-axis* by a factor of c

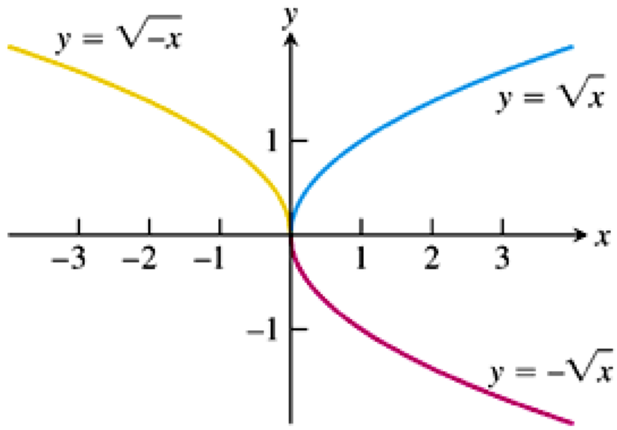


$y = f(cx)$	*compresses* the graph of f *along the x-axis* by a factor of c
$y = f(x/c)$	*stretches* the graph of f *along the x-axis* by a factor of c



For $c = -1$,

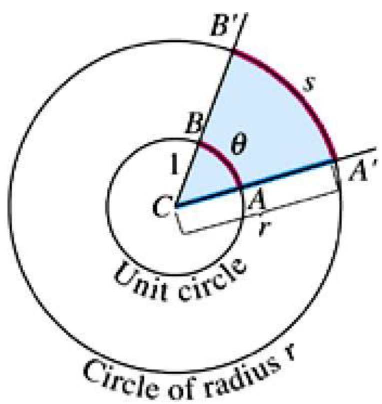
$y = -f(x)$ *reflects* the graph of f *across the x-axis*



$y = f(-x)$ *reflects* the graph of f *across the y-axis*

Combining scalings and reflections: *see next exercise sheet for examples!*

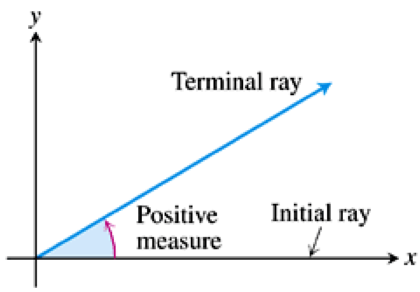
Trigonometric functions

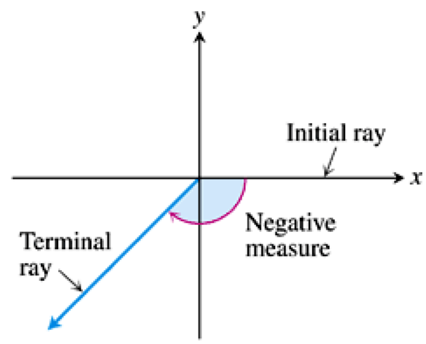


The **radian measure** of the angle ACB is the length θ of arc AB on the unit circle.
$s = r\theta$ is the *length of arc* on a circle of radius r when θ is measured in radians.

conversion formula degrees $\leftrightarrow$ radians:

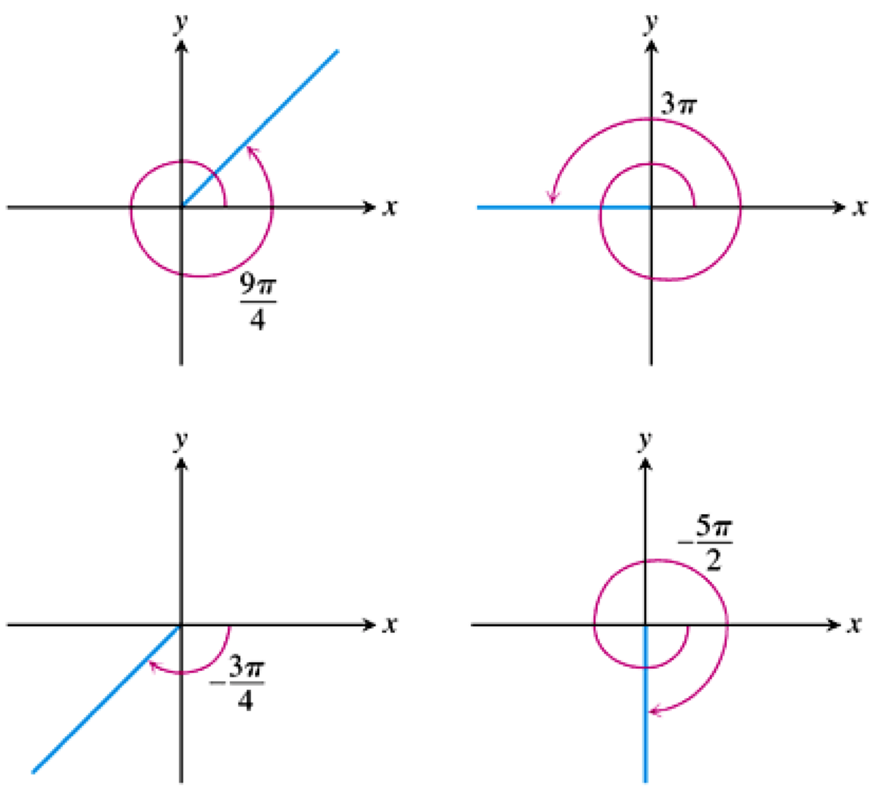
$$360^\circ \text{ corresponds to } 2\pi \Rightarrow \boxed{\frac{\text{angle in radians}}{\text{angle in degrees}} = \frac{\pi}{180}}$$



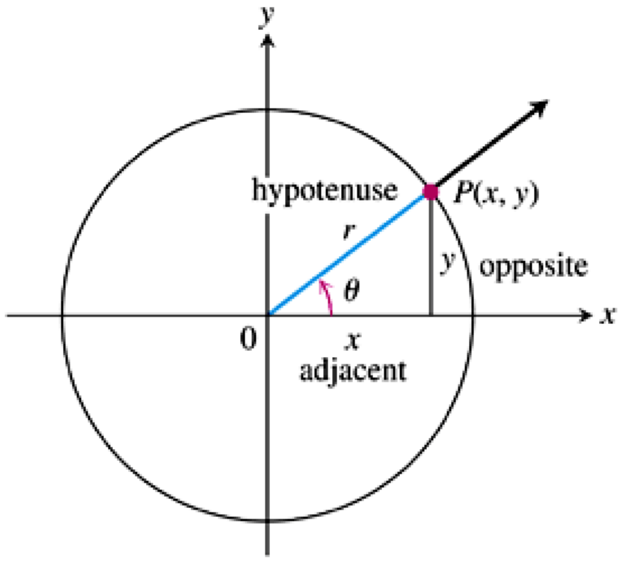


- angles are **oriented**
- *positive angle*: counter-clockwise
- *negative angle*: clockwise

angles can be *larger* (counter-clockwise) *smaller* (clockwise) than 2π:

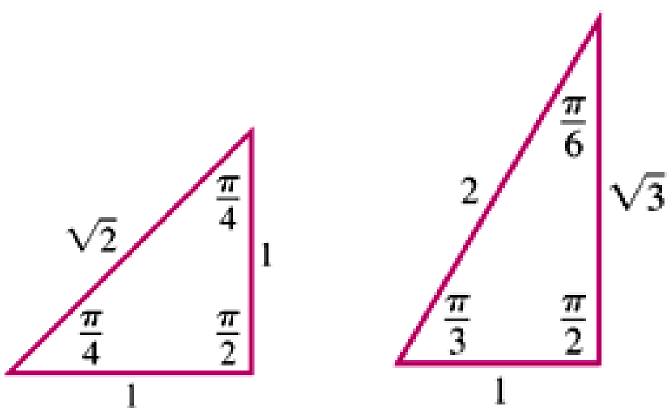


reminder: the six **basic trigonometric functions**



sine:	$\sin\theta = \frac{y}{r}$	**cosecant:**	$\csc\theta = \frac{r}{y}$
cosine:	$\cos\theta = \frac{x}{r}$	**secant:**	$\sec\theta = \frac{r}{x}$
tangent:	$\tan\theta = \frac{y}{x}$	**cotangent:**	$\cot\theta = \frac{x}{y}$

note: These definitions hold not only for $0 \leq \theta \leq \pi/2$ but also for $\theta < 0$ and $\theta > \pi/2$. recommended to memorize the following two triangles:

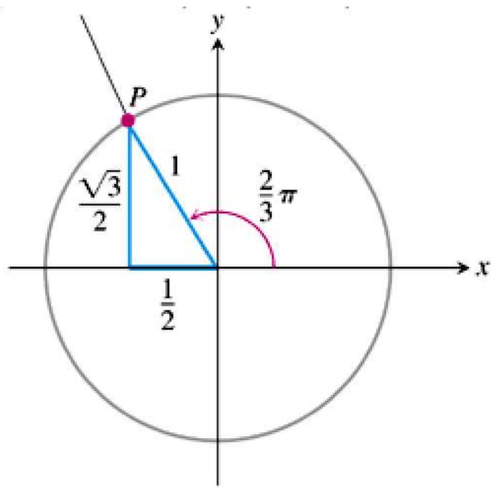


because *exact values* of trigonometric ratios in the *surds form* can be read from them

example:

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ; \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

a more non-trivial **example:**



From the above triangle and with $r = 1$, $x = -1/2$ and $y = \sqrt{3}/2$ we can read off the values of all trigonometric functions:

$$\sin\left(\frac{2}{3}\pi\right) = \frac{y}{r} = \frac{\sqrt{3}}{2} \qquad \csc\left(\frac{2}{3}\pi\right) = \frac{r}{y} = \frac{2}{\sqrt{3}}$$
$$\cos\left(\frac{2}{3}\pi\right) = \frac{x}{r} = -\frac{1}{2} \qquad \sec\left(\tfrac{2}{3}\pi\right) = \tfrac{r}{x} = -2$$
$$\tan\left(\frac{2}{3}\pi\right) = \frac{y}{x} = -\sqrt{3} \qquad \cot\left(\frac{2}{3}\pi\right) = \frac{x}{y} = -\frac{1}{\sqrt{3}}$$

note: For an angle of measure θ and an angle of measure $\theta + 2\pi$ we have the *very same* trigonometric function values (why?)

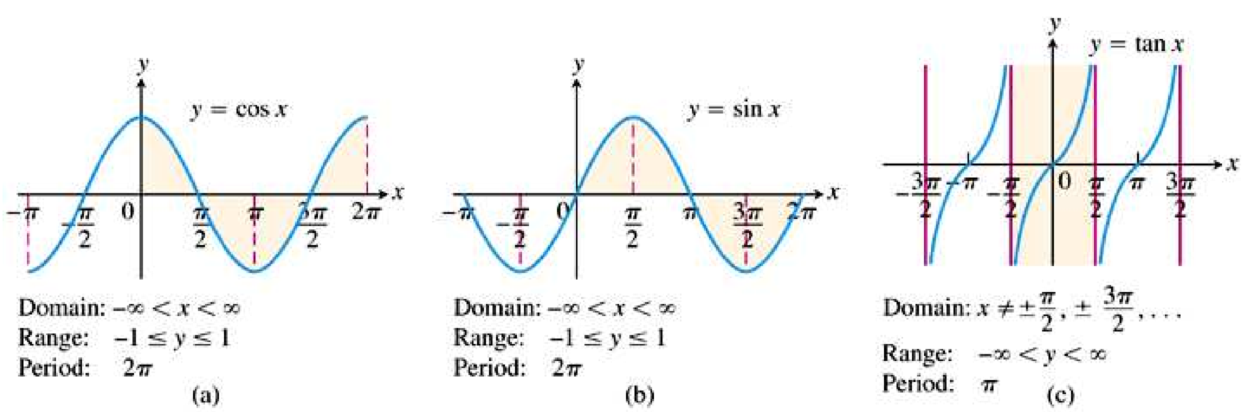
$$\sin(\theta + 2\pi) = \sin\theta \quad ; \quad \cos(\theta + 2\pi) = \cos\theta \quad ; \quad \tan(\theta + 2\pi) = \tan\theta$$

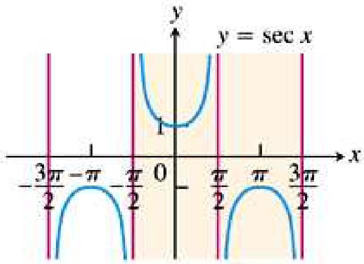
and so on.

DEFINITION Periodic Function

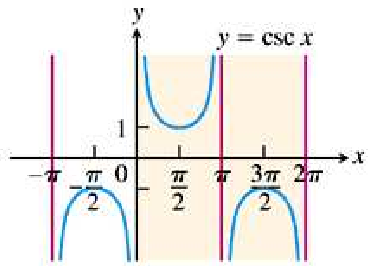
A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Graphs of trigonometric functions:

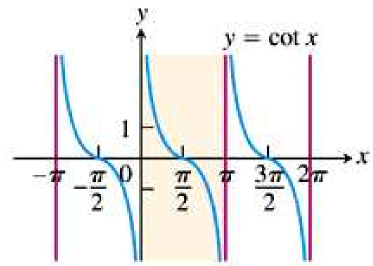




Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \leq -1$ and $y \geq 1$
Period: 2π
(d)



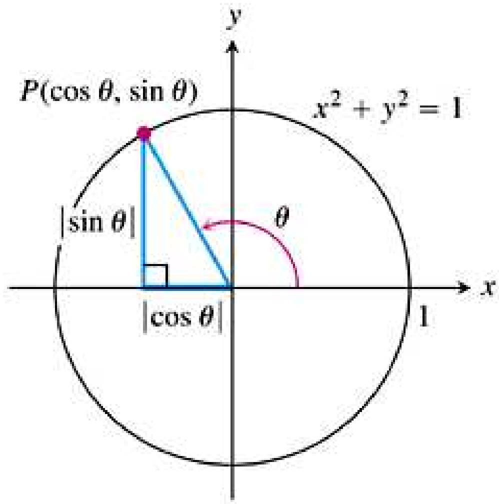
Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \leq -1$ and $y \geq 1$
Period: 2π
(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π
(f)

An important **trigonometric identity**: Since $x = r\cos\theta$ and $y = r\sin\theta$ by definition, for a triangle with $r = 1$ we immediately have

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$



This is an example of an **identity**, i.e., an equation that remains true *regardless of the values of any variables that appear within it.*

counterexample:

$$\cos\theta = 1$$

This is *not* an identity, because it is only true for *some* values of θ, not all.

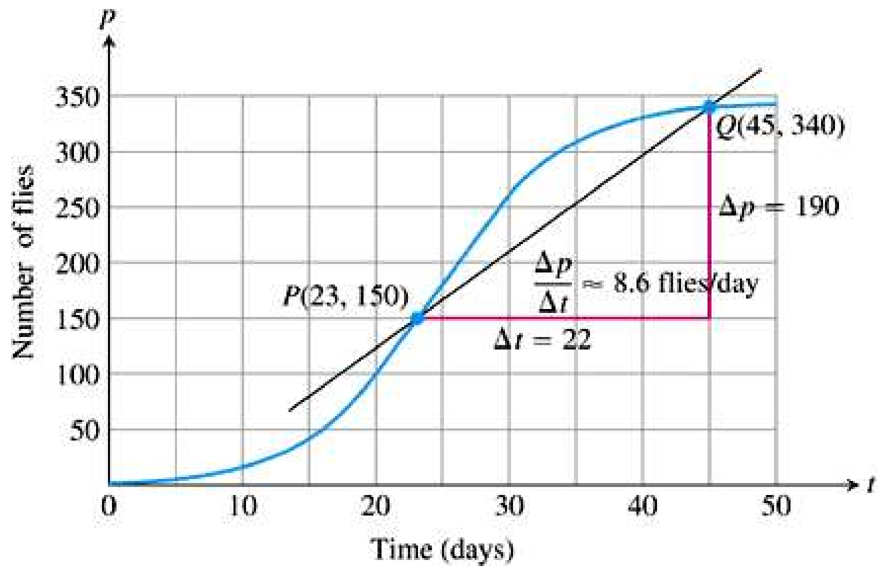
Reading Assignment: **Read Thomas' Calculus**

- short **paragraph** about ellipses, p.44/45
- **Section 1.6**, p.53-55 about trigonometric function symmetries and identities

You will need this for Coursework 2!

Rates of change and limits

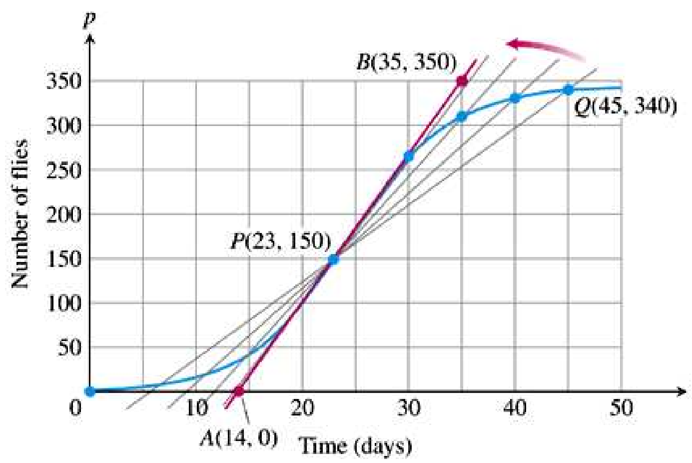
example: growth of a fruit fly population measured experimentally



Average rate of change from day 23 to day 45?

For **growth rate on a specific day**, e.g., day 23, study the average rates of change over *increasingly short time intervals* starting at day 23:

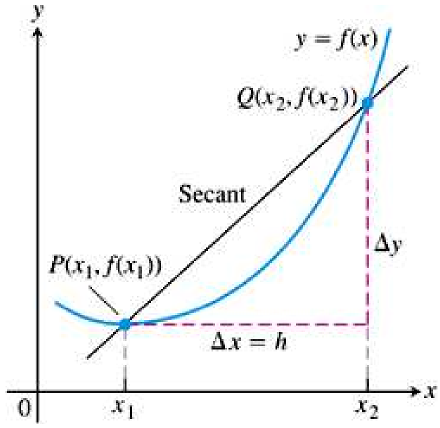
Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340-150}{45-23} \approx 8.6$
(40, 330)	$\frac{330-150}{40-23} \approx 10.6$
(35, 310)	$\frac{310-150}{35-23} \approx 13.3$
(30, 265)	$\frac{265-150}{30-23} \approx 16.4$



Lines approach the red *tangent* at point P with slope

$$\frac{350-0}{35-14} \simeq 16.7 \text{ flies/day}$$

Summary: average rate of change and limit



$x_2 = x_1 + h$

DEFINITION **Average Rate of Change over an Interval**

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Show animation!

To move from **average rates of change** to **instantaneous rates of change** we need to consider **limits**!

Definition 1 (informal) *Let $f(x)$ be defined on an open interval about x_0* except possibly at x_0 itself*. If $f(x)$ gets* **arbitrarily close** *to the number L (as close to L as we like) for all x* **sufficiently close** *to x_0, we say that f approaches the* limit *L as x approaches x_0, and we write*

$$\lim_{x \to x_0} f(x) = L\,,$$

which is read "the limit of $f(x)$ as x approaches x_0."

This is an *informal* definition, because: What do "arbitrarily close" and "sufficiently close" mean? This will be made mathematically precise in *Convergence and Continuity, MTH5104*; see also Thomas' Calculus, Section 2.3, if you're curious.

example: How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

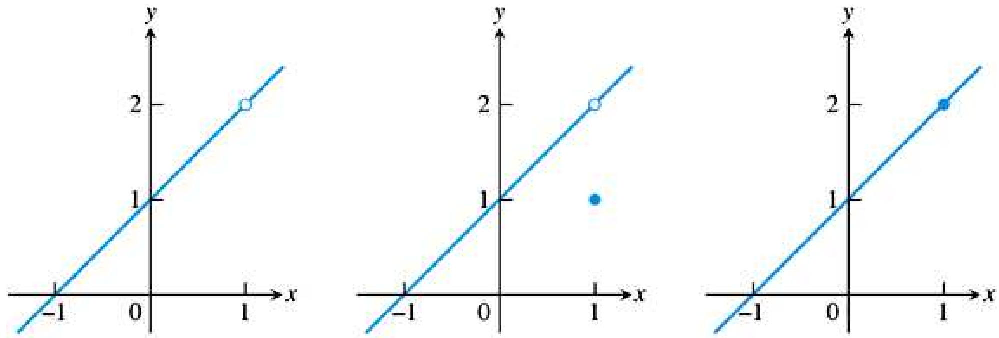
behave near $x_0 = 1$?
Problem: $f(x)$ is not defined for $x_0 = 1$.
But: we can *simplify*:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \text{ for } x \neq 1$$

This *suggests* that

$$\lim_{x \to 1} f(x) = 1 + 1 = 2$$

Graphs of these two functions, see (a) and (c):



(a) $f(x) = \frac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

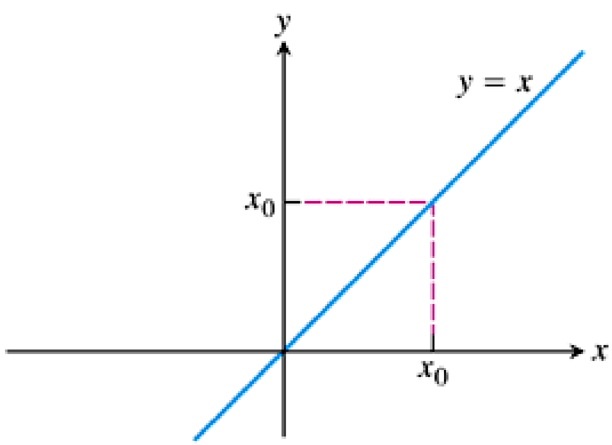
We say that $f(x)$ approaches the **limit** 2 as x approaches 1 and write

$$\lim_{x \to 1} f(x) = 2 \,.$$

note: The limit value does not depend on how the function is defined at x_0. All the above 3 functions have limit 2 as $x \to 1$! However, only for h we have equality of limit and function value:

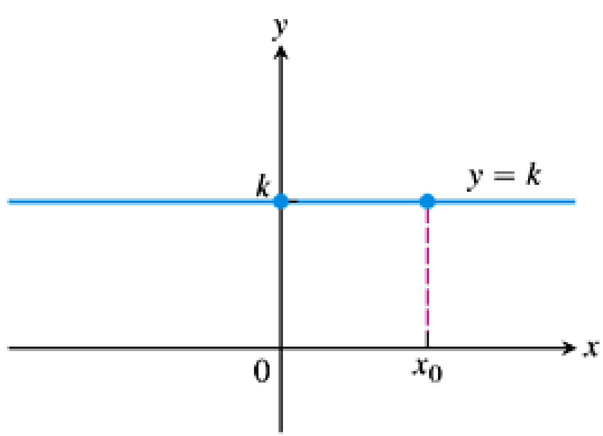
$$\lim_{x \to 1} h(x) = h(1)$$

Limits at every point:

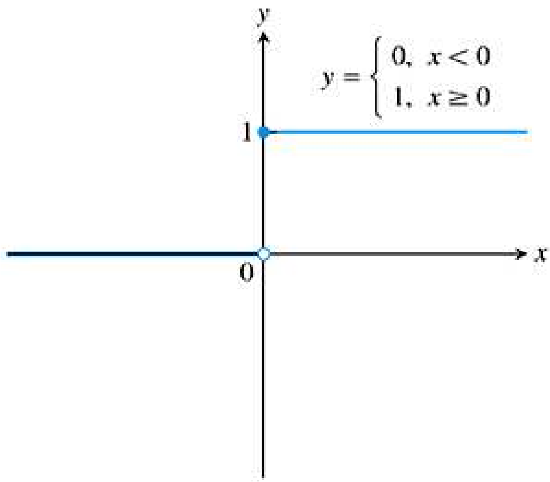


For any value of x_0 we have $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} x = x_0$.

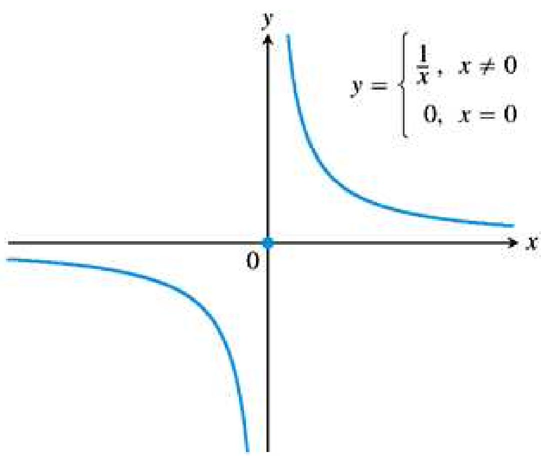
example: $\lim\limits_{x \to 3} x = 3$



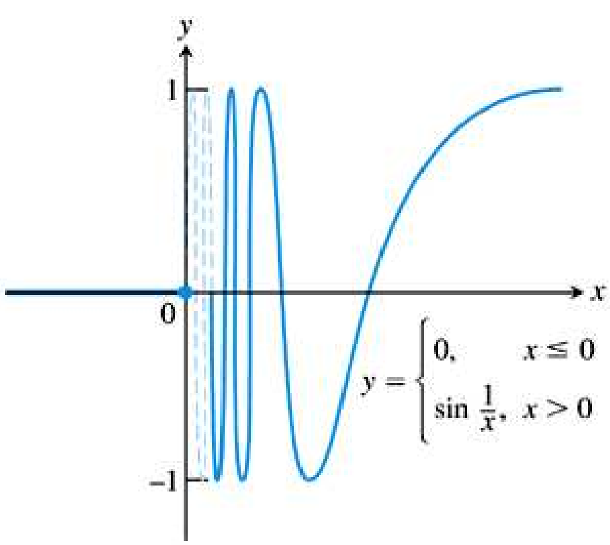
For any value of x_0 we have $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} k = k$.
example: For $k=5$ we have $\lim_{x\to -12} 5 = \lim_{x\to 7} 5 = 5$.
Limits can fail to exist! **No limit** at $x=0$ — three different **examples:**



values that *jump*



values that *grow too large*



values that *oscillate too much*

We have just "convinced ourselves" that for real constants k and c

$$\lim_{x\to c} x = c$$

and

$$\lim_{x\to c} k = k \quad .$$

The following theorem provides the basis to calculate **limits of functions that are arithmetic combinations** of the above two functions (like polynomials, rational functions, powers):

Theorem 1 (Limit laws) *If L, M, c and k are real numbers and* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$*, then*

1. Sum Rule: $\lim_{x\to c}(f(x) + g(x)) = L + M$
   The limit of the sum of two functions is the sum of their limits.
2. Difference Rule: $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. Product Rule: $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
4. Constant Multiple Rule: $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
5. Quotient Rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. Power Rule: *If s and r are integers with no common factor and $s \neq 0$, then*
   $$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$
   provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

For a proof of this theorem see Thomas' Calculus Section 2.3 and Appendix 2, or MTH5104.

examples:

- $\lim_{x\to c}(x^3 - 4x + 2) =$ *(rules 1,2)*
  $= \lim_{x\to c} x^3 - \lim_{x\to c} 4x + \lim_{x\to c} 2 =$ *(rules 3 or 6,4)*
  $= c^3 - 4c + 2$
- $\lim_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{4(-2)^2 - 3} = \sqrt{13}$ *(rules 6,2,3)*

So "sometimes" you can just *substitute the value of x.*

THEOREM 2 **Limits of Polynomials Can Be Found by Substitution**

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 **Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero**

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

example: Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

- substitution of $x = 1$? *No!*
- *but* algebraic simplification is possible:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x+2)(x-1)}{x(x-1)} = \frac{x+2}{x}\ ,\ x \neq 1$$

- therefore,

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x+2}{x} = 3$$

example: Evaluate

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$$

- substitution of $x = 0$?
- **trick**: algebraic simplification

$$\begin{aligned} \frac{\sqrt{x^2+100} - 10}{x^2} &= \frac{\sqrt{x^2+100} - 10}{x^2} \frac{\sqrt{x^2+100} + 10}{\sqrt{x^2+100} + 10} \\ &= \frac{(x^2+100) - 100}{x^2(\sqrt{x^2+100} + 10)} \\ &= \frac{1}{\sqrt{x^2+100} + 10} \end{aligned}$$

- therefore

$$\lim_{x \to 0} \frac{\sqrt{x^2+100} - 10}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+100} + 10} = \frac{1}{20}$$