Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 2

Thomas' Calculus, Sections 1.3 to 1.5

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Reading Assignment: read

Thomas' Calculus, Chapter 1.2: Lines, Circles, and Parabolas

What is a function?

examples:
height of the floor of the lecture hall depending on distance; stock market index depending on time; volume of a sphere depending on radius

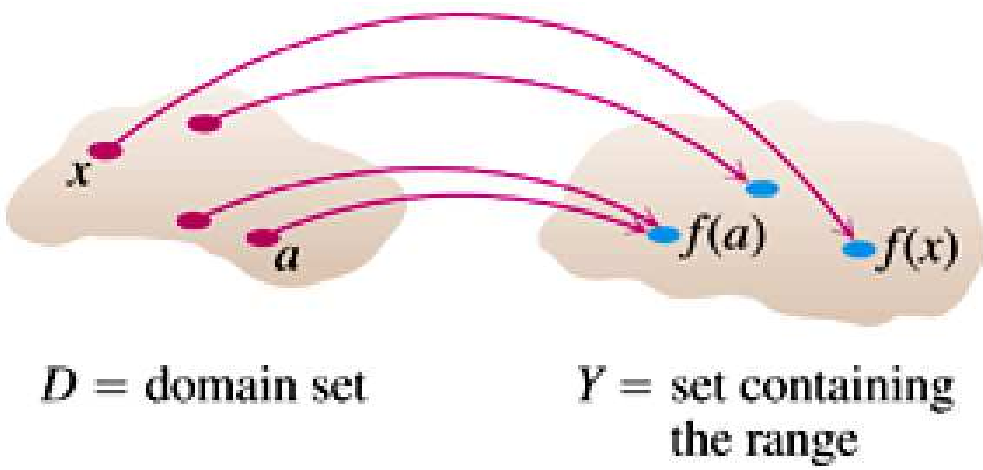
What do we mean when we say *y is a function of x?* Symbolically, we write $y = f(x)$, where

- x is the *independent variable* (input value of f)
- y is the *dependent variable* (output value of f at x)
- f is a *function* ("rule that assigns x to y" – further specify!)

A function acts like a "little machine":

Important: There is *uniqueness*, i.e., we have only *one value* $f(x)$ for every x!

Definition 1 *A* **function** *from a set D to a set Y is a rule that assigns a* unique *(single) element $f(x) \in Y$ to each element $x \in D$.*



- The set D of all possible *input values* is called the *domain* of f.
- The set R of all possible *output values* of $f(x)$ as x varies throughout D is called the *range* of f.

note: $R \subseteq Y$!

- We write f *maps D to Y* symbolically as

$$f : D \to Y$$

- We write f *maps x to $y = f(x)$* symbolically as

$$f : x \mapsto y = f(x)$$

Note the different arrow symbols used!
The **natural domain** is the largest set of real x which the rule f can be applied to.

examples:

Function	**Domain** $x \in D$	**Range** $y \in R$
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

note: A function is specified by the rule f *and* the domain D:

$$f : x \mapsto y = x^2\ , \quad D(f) = [0, \infty)$$

and

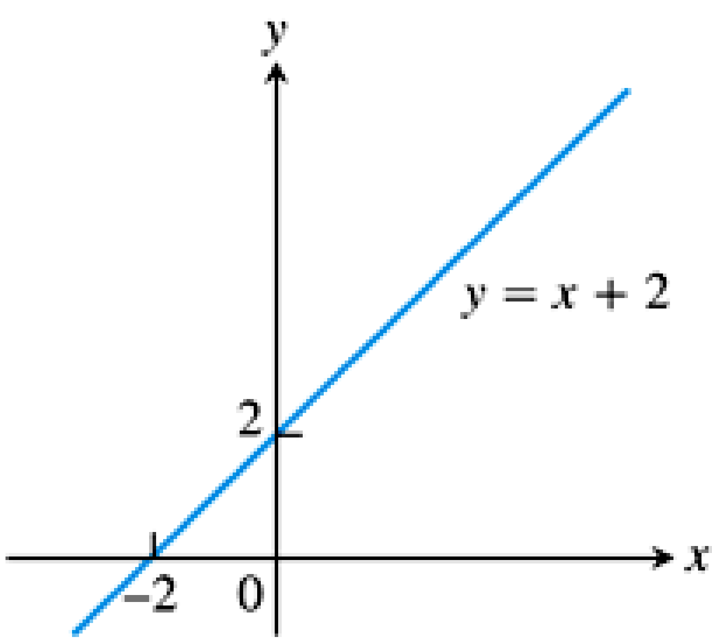
$$f : x \mapsto y = x^2\ , \quad D(f) = (-\infty, \infty)$$

are *different* functions!

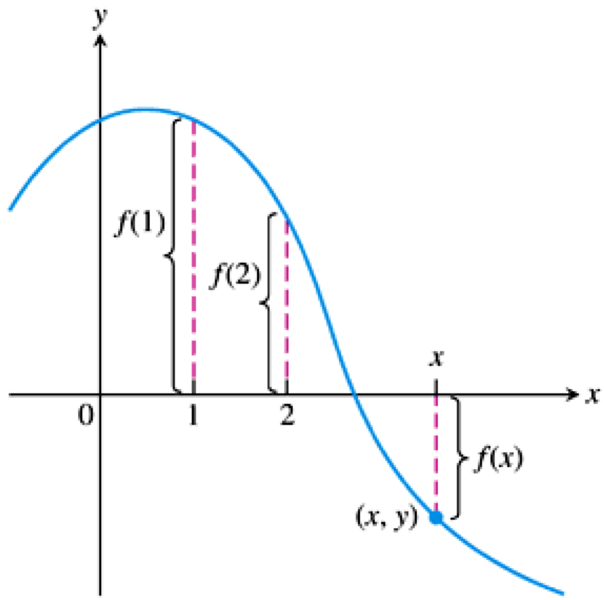
Definition 2 *If f is a function with domain D, its* **graph** *consists of the points (x, y) whose coordinates are the input-output pairs for f:*

$$\{(x, f(x))|x \in D\}$$

examples:

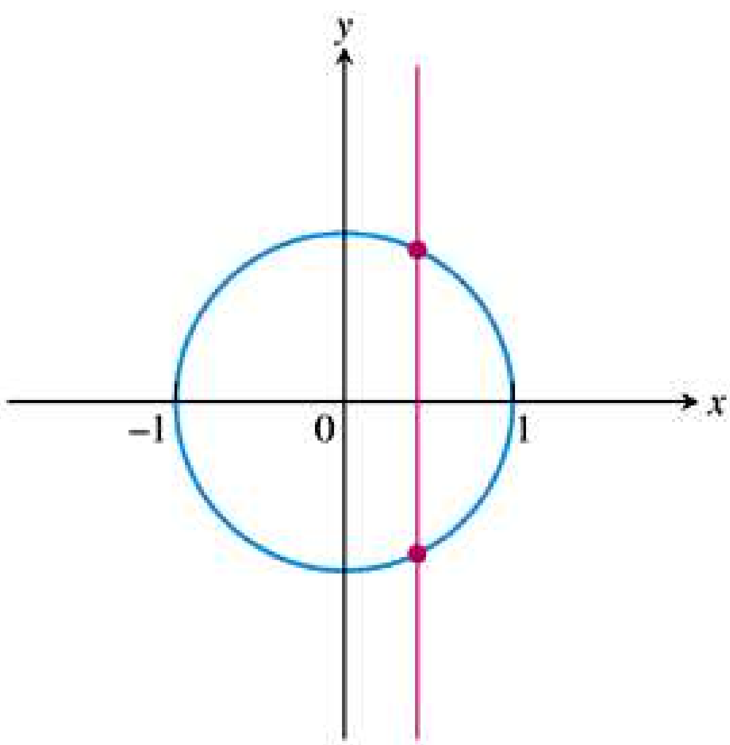


Given the function, one can *sketch* the graph.

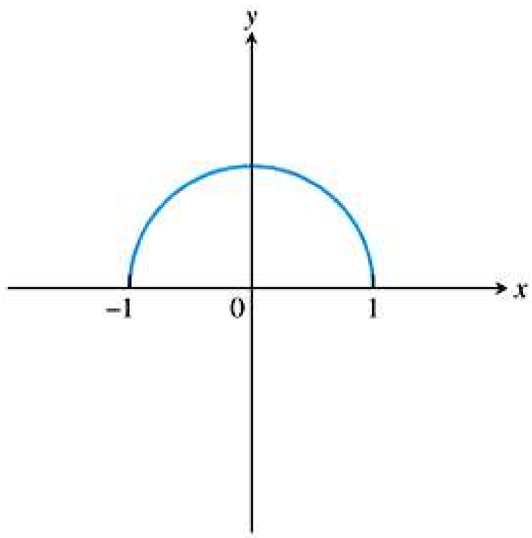


$y = f(x)$ is the *height* of the graph above/below x.
recall: A function f can have only *one value* $f(x)$ *for each* x in its domain! This leads to *the vertical line test*:

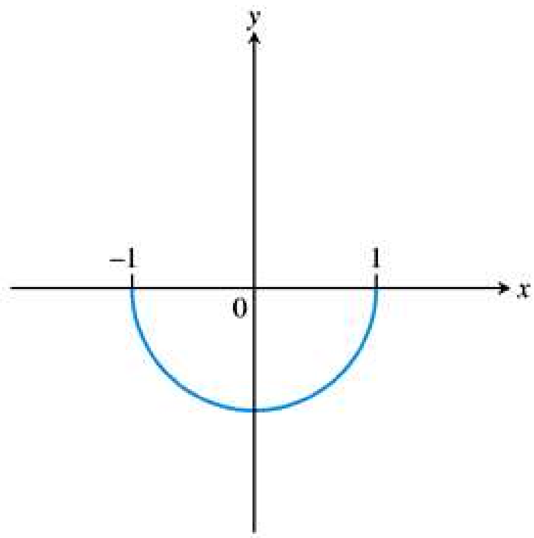
No vertical line can intersect the graph of a function *more than once*.



(a) $x^2 + y^2 = 1$



(b) $y = \sqrt{1 - x^2}$



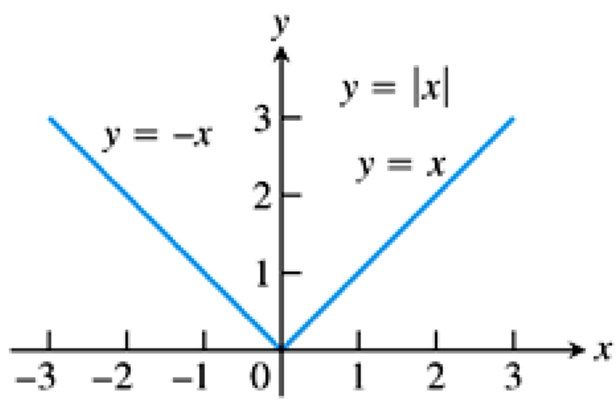
(c) $y = -\sqrt{1 - x^2}$

A **piecewise defined function** is a function that is is described by using *different formulas on different parts of its domain.*

examples:

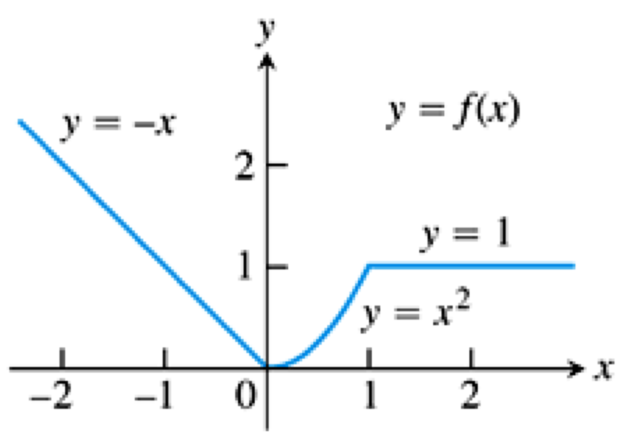
- the *absolute value function*

$$f(x) = |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



- some other function

$$f(x) = \begin{cases} -x & , x < 0 \\ x^2 & , 0 \leq x \leq 1 \\ 1 & , x > 1 \end{cases}$$

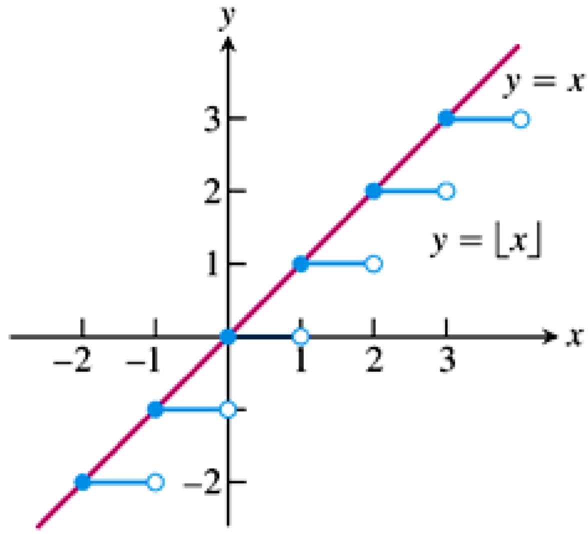


- the *floor function*

$$f(x) = \lfloor x \rfloor$$

is given by the greatest integer less than or equal to x:

$$\lfloor 1.3 \rfloor = 1,\ \lfloor -2.7 \rfloor = -3$$

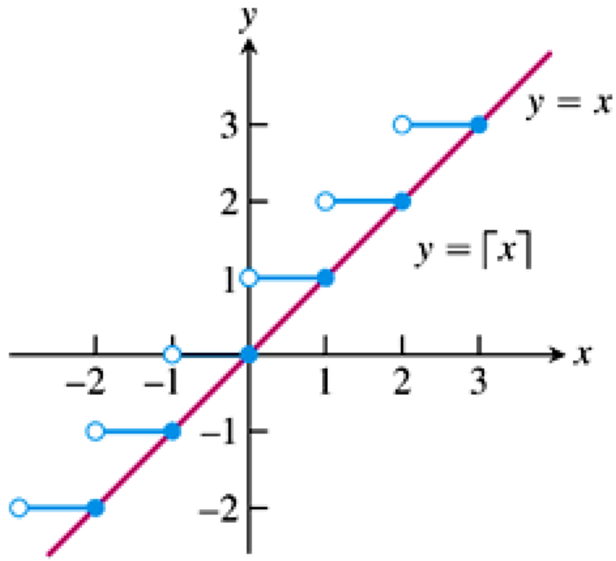


- the *ceiling function*

$$f(x) = \lceil x \rceil$$

is given by the smallest integer greater than or equal to x:

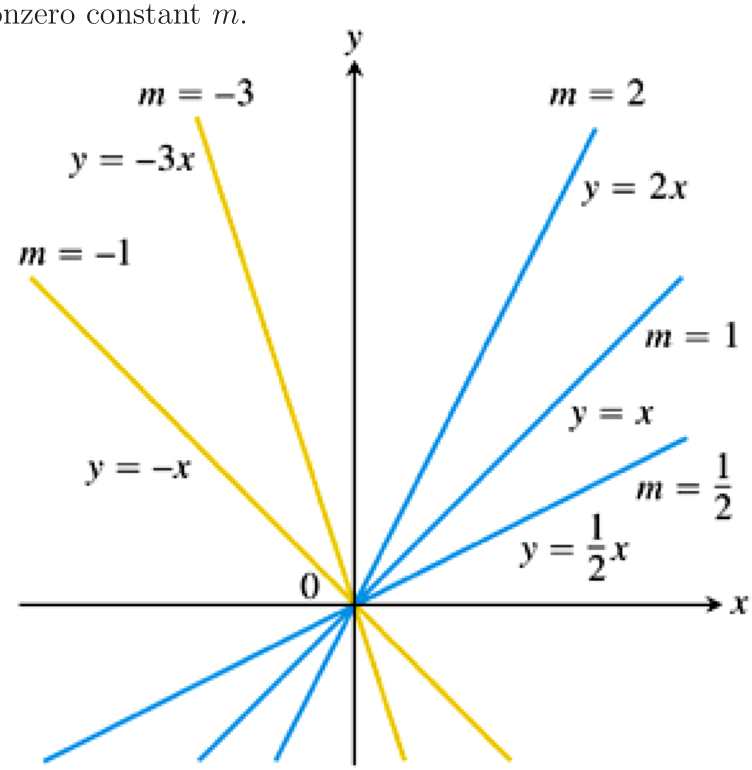
$$\lceil 3.5 \rceil = 4,\ \lceil -1.8 \rceil = -1$$



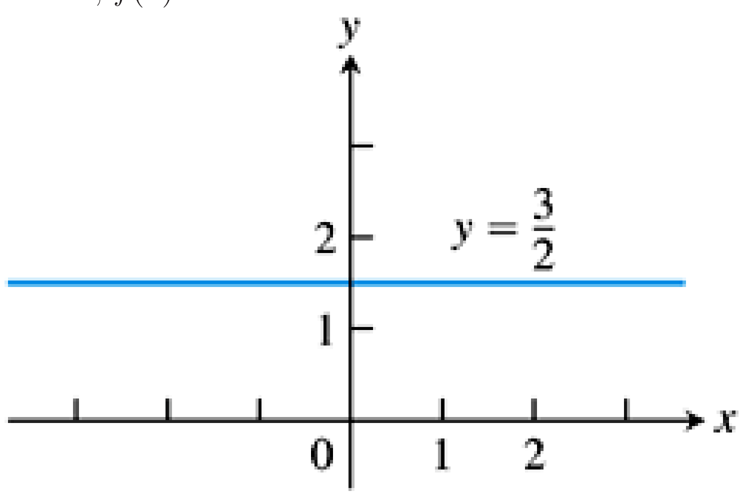
Some fundamental types of functions

• **linear function:** $\boxed{f(x) = mx + b}$

$b = 0$: all lines pass through the origin, $f(x) = mx$. One also says "$y = f(x)$ is proportional to x" for some nonzero constant m.

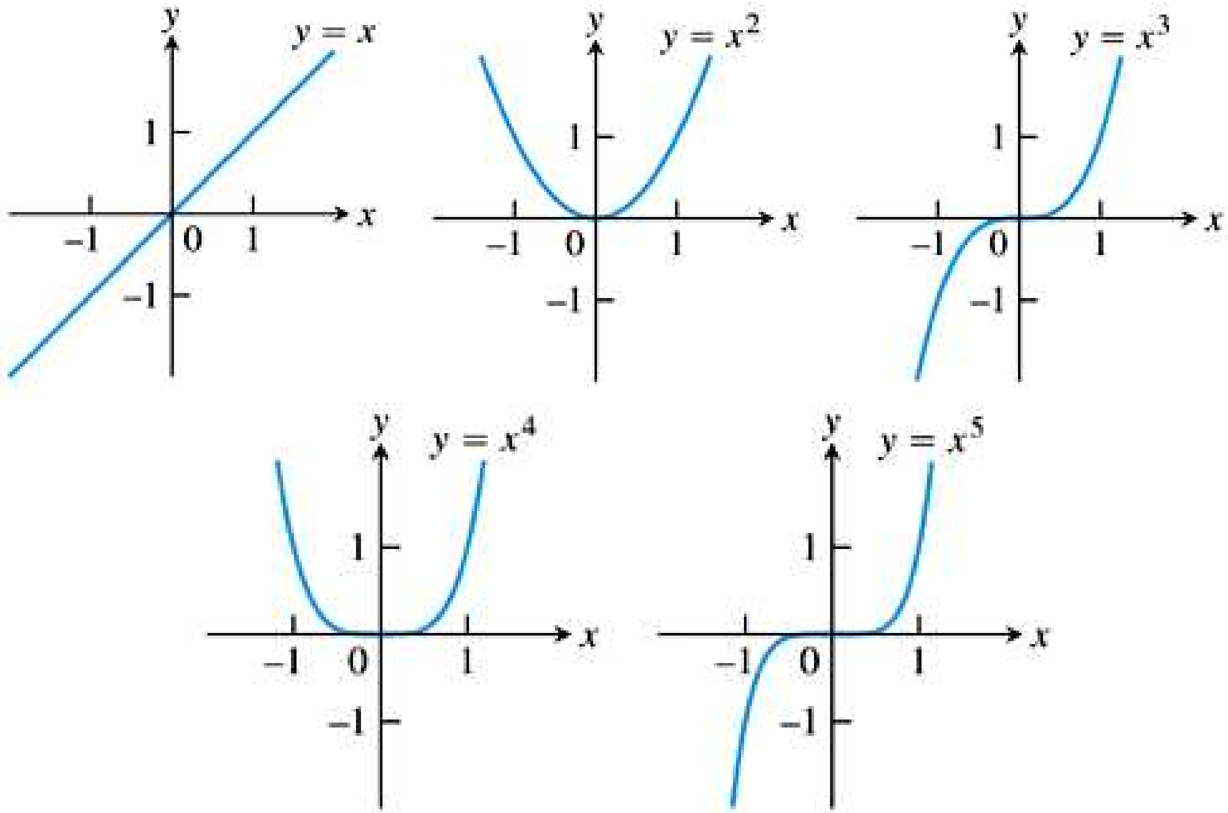


$m = 0$: *constant function*, $f(x) = b$

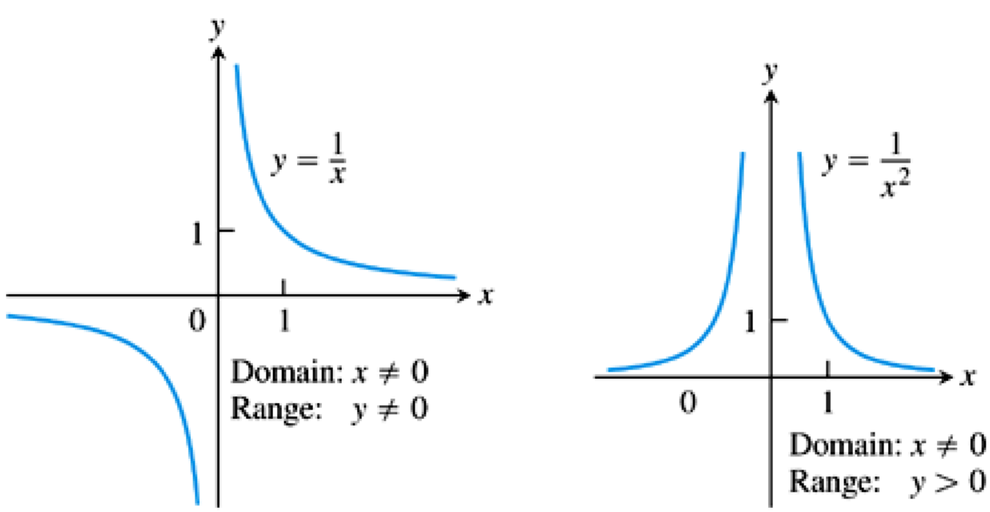


• **power function:** $\boxed{f(x) = x^a}$

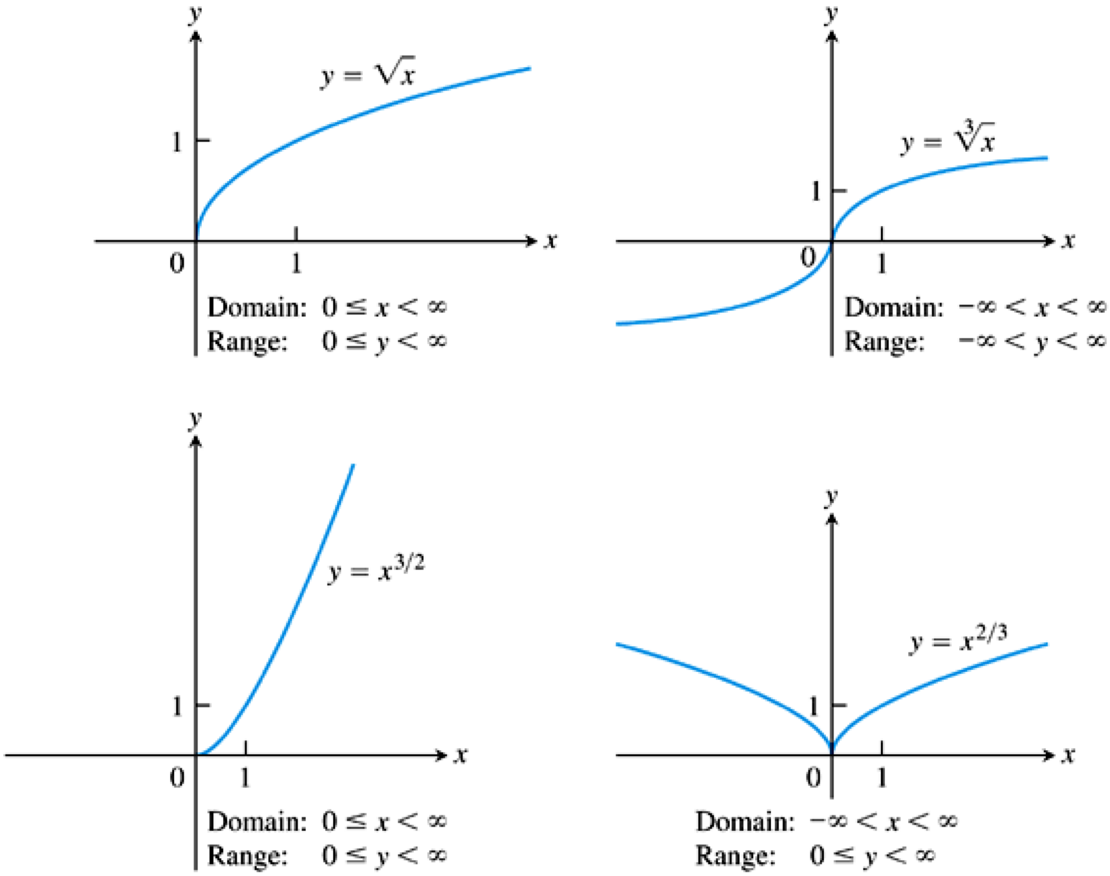
$a = n \in \mathbb{N}$: graphs of $f(x)$ for $n = 1, 2, 3, 4, 5$



$a = -n$, $n \in \mathbb{N}$: graphs of $f(x)$ for $n = -1, -2$

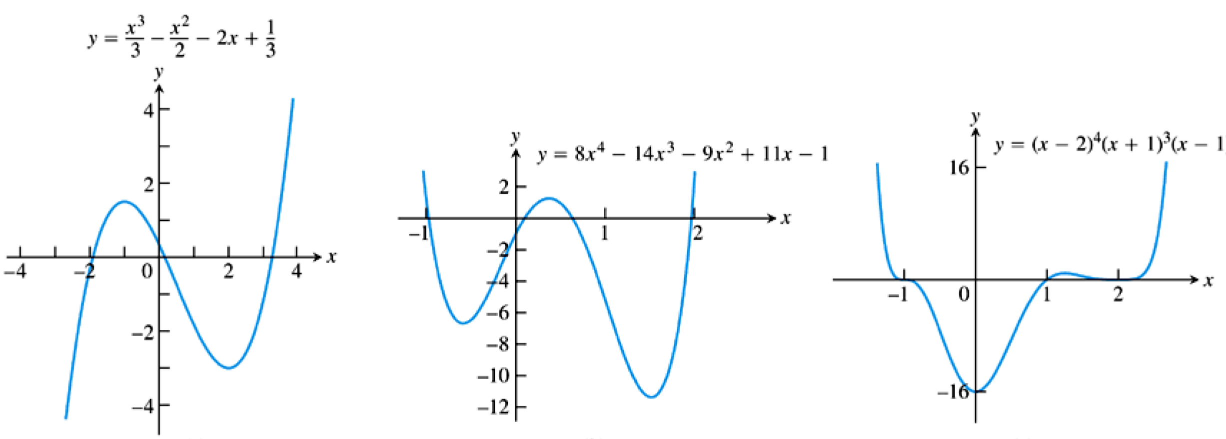


$a \in \mathbb{Q}$: graphs of $f(x)$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$



• **polynomials:** $\boxed{p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\ ,\ n \in \mathbb{N}_0}$
with *coefficients* $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ and domain $\mathbb{R}$
If the leading coefficient $a_n \neq 0$, $n > 0$, n is called the *degree* of the polynomial.

examples: Linear functions with $m \neq 0$ are polynomials of degree 1.
Three polynomial functions and their graphs:

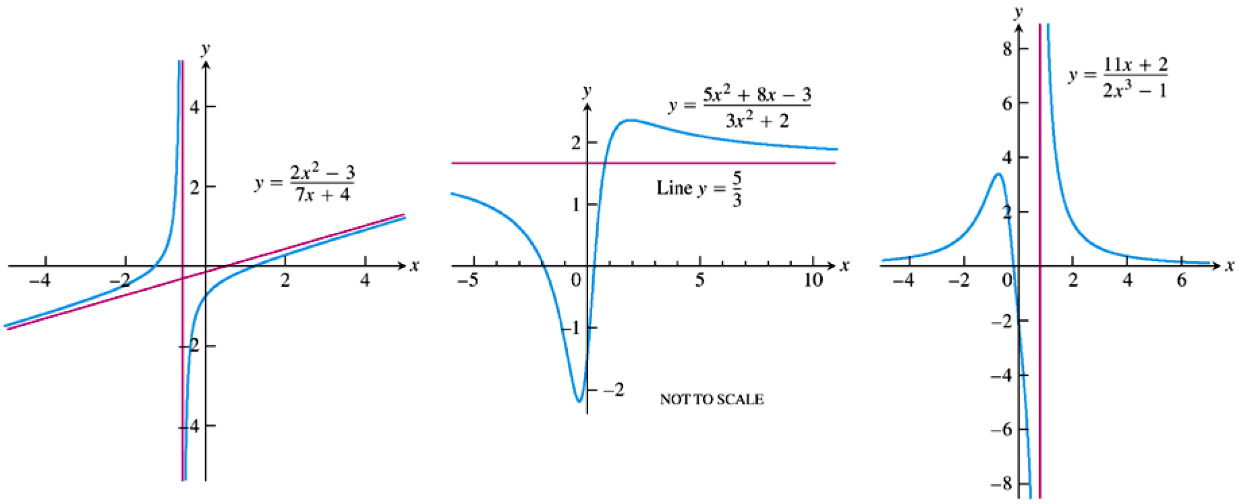


(a) (b) (c)

• **rational functions:** $\boxed{f(x) = \dfrac{p(x)}{q(x)}}$

with $p(x)$ and $q(x)$ polynomials and domain $\mathbb{R} \setminus \{x | q(x) = 0\}$ (never divide by zero!)

examples: three rational functions and their graphs

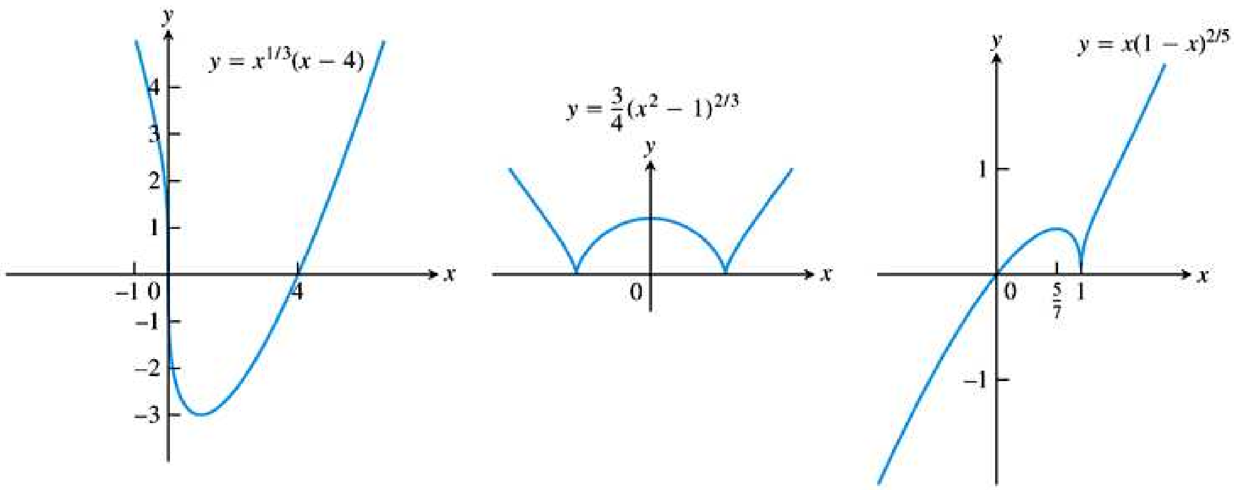


(a) (b) (c)

• **other classes of functions** (to come later):

algebraic functions: any function constructed from polynomials using algebraic operations (including taking roots)

examples:



(a) (b) (c)

trigonometric functions

exponential and *logarithmic functions*

transcendental functions: any function that is not algebraic

examples: trigonometric or exponential functions

...

Informally,

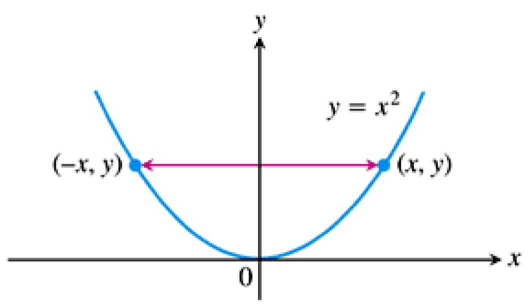
- a function is called **increasing** if the graph of the function “climbs” or “rises” as you move *from left to right.*
- a function is called **decreasing** if the graph of the function “descends” or “falls” as you move *from left to right.*

examples:

function	where increasing	where decreasing
$y = x^2$	$0 \le x < \infty$	$-\infty < x \le 0$
$y = 1/x$	nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = x^{2/3}$	$0 \le x < \infty$	$-\infty < x \le 0$

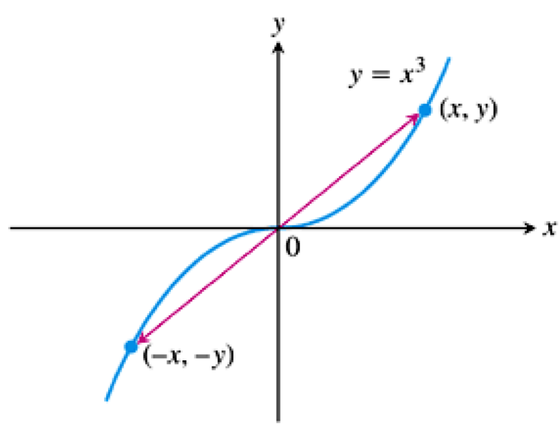
Definition 3 *A function* $y = f(x)$ *is an*
even function of x *if* $f(-x) = f(x)$*,*
odd function of x *if* $f(-x) = -f(x)$*,*
for every x *in the function's domain.*

examples:



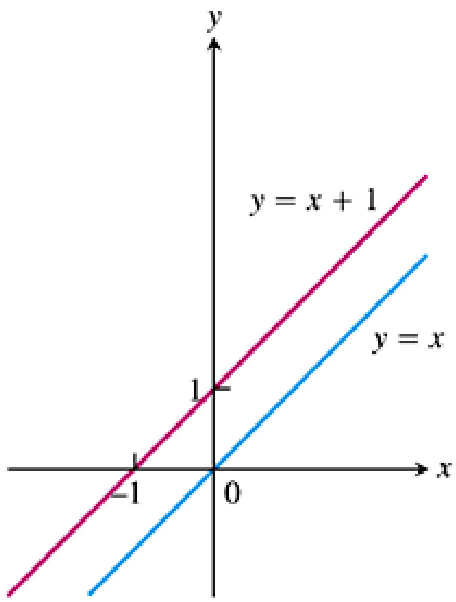
(a)

$f(-x) = (-x)^2 = x^2 = f(x)$: even function; graph is *symmetric about the* y*-axis*



(b)

$f(-x) = (-x)^3 = -x^3 = -f(x)$: odd function; graph is *symmetric about the origin*



1. $f(-x) = -x = -f(x)$: odd function

2. $f(-x) = -x + 1 \neq f(x)$ and $-f(x) = -x - 1 \neq f(-x)$: *neither even nor odd!*

Combining functions

If f and g are functions, then for every $x \in D(f) \cap D(g)$ (that is, for every x that belongs to the domains of *both* f and g) we define *sums, differences, products and quotients:*

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\(f-g)(x) &= f(x) - g(x)\\(fg)(x) &= f(x)g(x)\\(f/g)(x) &= f(x)/g(x) \quad \text{if } g(x) \neq 0\end{aligned}$$

algebraic operation on *functions* = algebraic operation on function *values*

Special case - multiplication by a constant $c \in \mathbb{R}$: $(cf)(x) = c\,f(x)$ (take $g(x) = c$ constant function)

examples: combining functions algebraically

$$f(x) = \sqrt{x} \quad , \quad g(x) = \sqrt{1-x}$$

(natural) domains:

$$D(f) = [0, \infty) \qquad D(g) = (-\infty, 1]$$

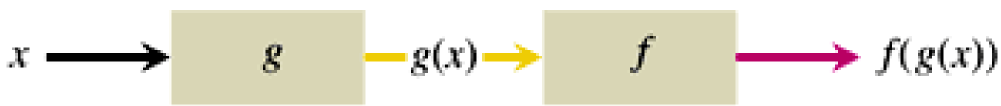
intersection of both domains:

$$D(f) \cap D(g) = [0, \infty) \cap (-\infty, 1] = [0, 1]$$

function	formula	domain
$f+g$	$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$	$[0,1]=D(f)\cap D(g)$
$f-g$	$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x)=\sqrt{1-x}-\sqrt{x}$	$[0,1]$
$f\cdot g$	$(f\cdot g)(x)=f(x)g(x)=\sqrt{x(1-x)}$	$[0,1]$
f/g	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x=1$ excluded)
g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x=0$ excluded)

Definition 4 (Composition of functions) *If f and g are functions, the* **composite** *function $f\circ g$ ("f composed with g") is defined by*

$$(f\circ g)(x)=f(g(x))$$



The domain *of $f\circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f, i.e.*

$$D(f\circ g)=\{x|x\in D(g)\ \text{and}\ g(x)\in D(f)\}$$