Mathematical Tripos 2004, Part IB : Electromagnetism, Lent. 2004
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1 Introduction

1.1 Electric Charge

The existence of electric charge was well-known already to the ancient Greeks, from the rubbing of amber with fur.

Experiments show that there are charges of two kinds, positive and negative. All stable charged matter owes its charge to a preponderance of electrons, if negative, and of protons, if positive. In fact, each electron and each proton carry a charge $\mp e$, where

$$e = 1.6 \times 10^{-19}\, C, \quad (C = \text{Coulomb}), \tag{1}$$

a magnitude so small that total charge can be regarded as a continuous variable. Thus we can refer to the charge density $\rho(\mathbf{r})$ as the charge per unit volume at a point $\mathbf{r}$ of a spatial distribution of charge.

Experiment shows also that, when we consider stationary particles P_1 and P_2 situated at $\mathbf{r}_1$ and $\mathbf{r}_2$ with charges q_1 and q_2, then P_1 experiences a force

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1\, q_2}{{r_{12}}^2}\,\frac{\mathbf{r}_{12}}{r_{12}} = \frac{1}{4\pi\epsilon_0}\frac{q_1\, q_2}{{r_{12}}^2}\hat{\mathbf{r}}_{12}, \tag{2}$$

due to P_2. This expresses the inverse-square or Coulomb law. Here

$$\mathbf{r}_{12} = -\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2, \quad r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|, \quad \hat{\mathbf{r}}_{12} = \mathbf{r}_{12}/r_{12}, \tag{3}$$

with $\hat{\mathbf{r}}_{12}$ a unit vector pointing from P_2 to P_1.

If $q_1\, q_2$ is positive (same sign charges) then $\mathbf{F}_{12}$ is an repulsive force; if negative (opposite sign charges), then it is attractive.

The factor $\dfrac{1}{4\pi\epsilon_0}$ is a dimensional quantity arising because of our use of SI or Système Internationale units (=MKS, metre, kilogram, second units).

Next we consider the force on charge q_1 at $\mathbf{r}_1$ due to a set of charges q_j at $\mathbf{r}_j$. This is given by

$$\mathbf{F}_1 = \frac{q_1}{4\pi\epsilon_0}\sum_{j\neq 1}\frac{q_j \mathbf{r}_{1j}}{{r_{1j}}^3}. \tag{4}$$

Hence, for the force on a charge q at $\mathbf{r}$ due to charge of density $\rho(\mathbf{r}')$ continuously distributed over a spatial volume V, we have

$$\mathbf{F}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|^3} \quad . \tag{5}$$

Now we define the electric field $\mathbf{E}(\mathbf{r})$ of such a distribution of charge to be the force it would exert on a unit charge if one were to be placed at $\mathbf{r}$, *i.e.*

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{6}$$

Similarly for a system of point charge q_j at $\mathbf{r}_j$, and to charge of density $\sigma(\mathbf{r}')$ distributed over a surface S, we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_j \frac{q_j(\mathbf{r}-\mathbf{r}_j)}{|\mathbf{r}-\mathbf{r}_j|^3} \tag{7}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{(\mathbf{r}-\mathbf{r}')\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} dS'. \tag{8}$$

Ex. q at the origin O gives rise to the electric field $\mathbf{E}(\mathbf{r})$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r},\ r = |\mathbf{r}|,\ |\hat{\mathbf{r}}| = 1. \tag{9}$$

and q', if placed at $\mathbf{r}$, would experience a force (due to this field),

$$\mathbf{F}(\mathbf{r}) = q'\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\,q'}{r^2}\hat{\mathbf{r}}. \tag{10}$$

1.2 Electric current

The ancient Greeks were well-aware too of magnetic material like lodestone, and of its effects. However a modern view is that the magnetic field $\mathbf{B}(\mathbf{r})$ and related forces are due to charges in motion, *i.e.* to electric currents. So we look next at the idea of electric current.

There are very many types of electric current flow. Here we confine ourselves to getting an intuitive picture of current flow in a copper wire.

First we recall that an atom is an electrically neutral system with a central nuclei containing Z protons with Z electrons moving around it 'in orbits' governed by the laws of quantum mechanics.

We use a battery to apply an electric field to a length of copper wire (or similarly to some suitable piece of crystalline material capable of conducting an electric current). Then some of the electrons of the copper atoms of the wire are detached from the atoms, leaving them as positively charged ions. These ions are held in position by the mechanical forces that describe the constitution of the material, and the detached electrons are moved like a gas, by the applied electric field, through the essentially fixed ionic background. In other words the detached electrons (called conduction electrons) constitute an electric current flowing in the wire (material).

Suppose we have a distribution of charge carriers, here electrons of charge q, N per unit volume, whose average motion is a drift velocity $\mathbf{v}$.

This distribution has charge density $\rho = Nq$, and constitutes electric current flow of current density $\mathbf{J} = Nq\mathbf{v} = \rho\mathbf{v}$. To see that (or *how*) $\mathbf{J}$ describes the rate of flow of electric charge, let $\delta\mathbf{S}$ be a small plane element of area, and let $\mathcal{C}$ be a cylinder of current flow of cross-section $\delta\mathbf{S}$ with generators parallel to $\mathbf{v}$ of magnitude $|\mathbf{v}|$. Then $\mathcal{C}$ has volume $\mathbf{v}.\delta\mathbf{S}$, contains charge $Nq\mathbf{v}.\delta\mathbf{S} = \mathbf{J}.\delta\mathbf{S}$, and all of this charge flows across $\delta\mathbf{S}$ in unit time. Thus, writing $\delta\mathbf{S} = \delta S\mathbf{n}$, we see that in the current flow of current density $\mathbf{J}$, an amount of charge $\mathbf{J}.\mathbf{n}$ crosses unit area perpendicular to $\mathbf{n}$ in unit time.

The total charge per unit time passing through a surface S is called the electric current I through S

$$I = \int_S \mathbf{J}\cdot\mathbf{n}dS = \int_S \mathbf{J}\cdot\mathbf{dS}, \quad \mathbf{dS} = \mathbf{n}dS. \tag{11}$$

We comment here on the generic term *flux*: The flux f of a vector field $\mathbf{v}$ through a surface S is defined by

$$f = \int_S \mathbf{v} \cdot \mathbf{dS}. \tag{12}$$

Here S can either be closed bounding a spatial volume V, so that f is the flux of $\mathbf{v}$ out of $S = \partial V$, as in the Gauss theorem context of sec. 1.5 below, or else open and bounded by a curve $C = \partial S$, as in the definition just given, (11), of current I as the flux of current density through S, or through C. Physically what we have seen is that I measures the rate of flow of charge through S.

1.3 Magnetism

Magnetic fields $\mathbf{B}(\mathbf{r})$ arise from bar magnets, or from electric currents in wires, coils, etc. If a particle of charge q has position vector $\mathbf{r}$ and velocity $\mathbf{v} = \dot{\mathbf{r}}$, and moves in the presence of electric and magnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, it is an experimental fact that it experiences a force (the Lorentz force)

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}), \tag{13}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{r})$ and $\mathbf{B} = \mathbf{B}(\mathbf{r})$.

Consider the effect of the field $\mathbf{B}$ of the bar magnet on the wire. The current in the wire involves particles of charge q moving along the wire with velocity $\mathbf{v}$. Each one feels a (magnetic) force $q\mathbf{v}\wedge\mathbf{B}$ which, for positive qv, tends to push them downwards. One can see such a wire move downwards in experiment.

In a related experiment, one employs a fixed current carrying circuit connected to a battery instead of the bar magnet. In this case the second circuit experiences a force of attraction towards the first one due to the magnetic field of the first circuit. See Sec. 3.7.

One can also give the (magnetic) force per unit volume on a medium carrying N charges q per unit volume each moving with velocity $\mathbf{v}$

$$\mathbf{f} = Nq\mathbf{v}\wedge\mathbf{B} = \mathbf{J}\wedge\mathbf{B}. \tag{14}$$

1.4 Maxwell's Equations

It was the great achievement of Maxwell to unify the separate subjects electricity and magnetism into a single consistent formalism involving a set of equations (Maxwell's equations) capable of describing all classical electromagnetic phenomena. For charges and currents in a non-polarisable and non-magnetisable medium, such as the vacuum, these are

$$\nabla\wedge\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \tag{15}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{16}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{17}$$

$$\nabla\wedge\mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}) \tag{18}$$

where ρ and $\mathbf{J}$ are the charge and current densities.

These equations involve two constants ϵ_0 and μ_0 that are not themselves of much physical significance (but see (46) below). The last term of (18) features the displacement current postulated by Maxwell in order to achieve a formalism that *consistently* unified previous theories of electricity and magnetism.

Certain more general media can be described by means of a suitable generalisation of the set (15–18) of Maxwell's equations, but this lies beyond the present course syllabus.

First we observe the consistency of Maxwell's equations. Since $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\wedge\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$, (15-16) imply

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{B}) = \boldsymbol{\nabla}\cdot(-\boldsymbol{\nabla}\wedge\mathbf{E}) = 0. \tag{19}$$

So $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ is preserved in time.

Similarly $\boldsymbol{\nabla}\cdot(...)$ of (18) implies

$$\begin{aligned} 0 &= \boldsymbol{\nabla}\cdot\mathbf{J} + \epsilon_0\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{E}) \\ 0 &= \boldsymbol{\nabla}\cdot\mathbf{J} + \frac{\partial\rho}{\partial t}. \end{aligned} \tag{20}$$

Here (17) has been used. Eq. (20) expresses the conservation of charge. Integrating (20) over a fixed volume V containing total charge Q

$$Q = \int_V \rho d\tau, \tag{21}$$

we derive

$$\frac{dQ}{dt} = \int_V \frac{\partial\rho}{\partial t}d\tau = -\int_V \boldsymbol{\nabla}\cdot\mathbf{J}d\tau = -\int_{\partial V}\mathbf{J}\cdot\mathbf{dS}, \tag{22}$$

which states that the rate of decrease of the charge contained in V is equal to the flux of $\mathbf{J}$ out of V (through the surface $S = \partial V$). It is noted that the presence of the displacement term in (18) is essential in this demonstration of consistency.

1.5 Integral forms of Maxwell's equations

Maxwell's equations involve divs and curls. We can therefore convert them into useful integral forms by integrating over fixed volumes using the divergence theorem, or over fixed surfaces using Stokes's theorem.

$$\frac{\rho}{\epsilon_0} = \boldsymbol{\nabla}\cdot\mathbf{E} \Rightarrow \frac{1}{\epsilon_0}\int_V \rho d\tau = \int_V \boldsymbol{\nabla}\cdot\mathbf{E}d\tau \tag{23}$$

Hence

$$\frac{1}{\epsilon_0}Q = \int_{S=\partial V}\mathbf{E}\cdot\mathbf{dS}. \tag{24}$$

The right-hand side is the flux of $\mathbf{E}$ out of V. The statement (24) is Gauss's Law. It is of practical use.

Ex. Consider a point charge q at rest at O, and let V be the sphere of radius r centred at O. By symmetry the electric field must be of the form

$$\mathbf{E}(\mathbf{r}) = E(r)\mathbf{e}_r = E(r)\mathbf{n}, \tag{25}$$

so that

$$\int_{\partial V}\mathbf{E}\cdot\mathbf{dS} = \int_{\partial V}\mathbf{E}\cdot\mathbf{n}dS = E(r)\int_{\partial V}dS, \tag{26}$$

and hence

$$\begin{aligned} \frac{1}{\epsilon_0}q &= E(r)\,4\pi r^2 \\ \mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\mathbf{e}_r. \end{aligned} \tag{27}$$

Similarly (16) implies that

$$\int_{\partial V} \mathbf{B} \cdot \mathbf{dS} = 0, \tag{28}$$

for any closed surface $S = \partial V$. This can be interpreted as the statement that there are no magnetic 'charges' or magnetic monopoles.

Next (17) yields

$$\int_S \boldsymbol{\nabla}\wedge \mathbf{B} \cdot \mathbf{dS} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS} + \mu_0 \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS}. \tag{29}$$

Hence, in the case of steady current (no time dependence), Stokes's theorem implies

$$\begin{aligned} \int_C \mathbf{B} \cdot \mathbf{dr} &= \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS} \\ &= \mu_0 \text{ (flux of } \mathbf{J} \text{ through open } S \text{ bounded by } C) \\ &= \mu_0 I, \end{aligned} \tag{30}$$

where $I = \int_S \mathbf{J} \cdot \mathbf{dS}$ is the total current through S (or C). This is Ampère's Law. It too is useful in practice.

Ex. Consider an infinite straight wire lying along the z-axis and carrying a current I in the positive direction.

By symmetry, expect $\mathbf{B}$ of the form $\mathbf{B} = B(s)\mathbf{e}_\phi$ using cylindrical polars $(s,\ \phi,\ z)$. Then apply Ampère for C any circle centred on the z-axis and lying in a horizontal plane. On C we have

$$\mathbf{r} = s\mathbf{e}_s(\phi) \quad \text{so that, at constant } s, \quad \mathbf{dr} = s\mathbf{de}_s = s\frac{\partial \mathbf{e}_s}{\partial \phi} d\phi = s\mathbf{e}_\phi d\phi. \tag{31}$$

Then Ampère's law implies

$$B(s)s \int_0^{2\pi} d\phi = \mu_0 I \tag{32}$$

and hence

$$B(s) = \frac{\mu_0 I}{2\pi s}. \tag{33}$$

Finally (15) implies

$$\int_C \mathbf{E} \cdot \mathbf{dr} = \int_S \boldsymbol{\nabla}\wedge \mathbf{E} \cdot \mathbf{dS} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{34}$$

by applying Stokes's theorem to a fixed curve $C = \partial S$ bounding a fixed open surface S. If we define the electromotive force (or electromotance) acting in C by

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dr}, \tag{35}$$

and the flux of $\mathbf{B}$ through (the open surface) S by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}, \tag{36}$$

then we get Faraday's Law of induction

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{37}$$

This will be studied later.

1.6 Electromagnetic waves

Here we consider Maxwell's equations in the absence of charges and of currents, *e.g.* in the vacuum.

$$\begin{aligned} \boldsymbol{\nabla}\wedge\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0 &(38)\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 &(39)\\ \boldsymbol{\nabla}\cdot\mathbf{E} &= 0 &(40)\\ \boldsymbol{\nabla}\wedge\mathbf{B} &= \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}. &(41)\end{aligned}$$

Take $\boldsymbol{\nabla}\wedge(...)$ of (38) and use

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E}, \tag{42}$$

where the first term is zero by (40), and $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$. Then we have

$$\nabla^2\mathbf{E} = \boldsymbol{\nabla}\wedge\frac{\partial\mathbf{B}}{\partial t} = \frac{\partial}{\partial t}(\boldsymbol{\nabla}\wedge\mathbf{B}) = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{43}$$

Thus each (Cartesian) component of $\mathbf{E}$ satisfies a wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{E} = 0, \tag{44}$$

where the wave speed c is given by

$$c^2 = \frac{1}{\epsilon_0\mu_0}. \tag{45}$$

Check that (39) and (41) can be used similarly to show that each component of $\mathbf{B}$ satisfies the same wave equation. In other words, each of $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are propagated as waves of speed c.

The values of the quantities ϵ_{00} and μ_0 appropriate to SI units are fixed by experiment, and these values indicate that

$$c = 3\times 10^8 m/s = \text{the speed of light}. \tag{46}$$

Maxwell's equations with the crucial displacement current term, necessary for consistency, can describe electromagnetic wave phenomena across its entire frequency spectrum: see the Table. For waves of frequency ν, measured in hertz, and wavelength λ, measured in metres, $c = \lambda\nu$. Also, in quantum theory, the energy of a quantum of given frequency ν is $E = h\nu$, where h is Planck's constant. (One hertz equals one cycle per second).

Frequency spectrum					
radiation	ν	λ	radiation	ν	λ
γ	10^{19}	10^{-11}	infra-red	10^{14}	10^{-6}
X-rays	10^{18}	10^{-10}	μ-wave	10^{13}	10^{-5}
ultra-violet	10^{16}	10^{-8}	mm	10^{11}	10^{-3}
visible light	10^{15}	10^{-7}	radio	10^{6}	10^{2}

1.7 Discontinuity formulas

Here we collect, for easy reference but *without discussion at this stage* a class of formulas that logically belong together but whose occurrences are scattered throughout several sections of the course material.

Let S be a surface with unit normal $\mathbf{n}$ which separates regions $V_\pm$ of space, with $\mathbf{n}$ pointing from S into V_+.

a). Let S carry charge density σ per unit area. Let $\mathbf{E}_\pm$ denote the electric fields just inside the $V_\pm$ sides of S. Then

$$\mathbf{n}\cdot\mathbf{E}|_-^+ = \frac{1}{\epsilon_0}\sigma \tag{47}$$

$$\mathbf{n}\wedge\mathbf{E}|_-^+ = 0. \tag{48}$$

Eq. (47) is proved on the basis of Gauss's theorem in Sec. 2.2. Note eqs. (47) and (48) respectively involve the components of $\mathbf{E}$ normal and tangential ($\mathbf{n}\cdot\mathbf{n}\wedge\mathbf{E}=0$)to the surface S.

b). Let S carry current density $\mathbf{s}$ per unit length (charge crossing unit length in S in unit time). Let $\mathbf{B}_\pm$ denote the magnetic fields just inside the $V_\pm$ sides of S.

$$\mathbf{n}\cdot\mathbf{B}|_-^+ = 0 \tag{49}$$

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0\mathbf{s}. \tag{50}$$

Eq. (49) is proved in the same way as used for (47). Eq. (50) is a consequence of Stokes's theorem, as is (48). A special case of (50) occurs in Sec. 3.3

The correspondence between Maxwell's equations and the discontinuity formulas is clear: drop $\dfrac{\partial}{\partial t}$ terms, and replace $\boldsymbol{\nabla}(...)$ by $\mathbf{n}(...)|_-^+$. Thus, from (20), we expect that $\mathbf{n}\cdot\mathbf{J}|_-^+=0$ at a surface of discontinuity, one that may carry surface density of charge.

Force per unit area on S

In case (a), consider only the special case when $\mathbf{E}_\pm$ only have normal components $\mathbf{n}\cdot\mathbf{E}_\pm = E_\pm$. Then the force per unit area on a surface S (carrying surface charge σ) has magnitude

$$\frac{1}{2}\sigma(E_+ + E_-). \tag{51}$$

In case (b), consider only the special case in which $\mathbf{B}_\pm$ only have tangential components $B_\pm$. Then the force per unit area on a surface S (carrying surface current $\mathbf{s}$) is normal to S, and has magnitude

$$\frac{1}{2}s(B_+ + B_-). \tag{52}$$

We do not prove the results (51) and (52); the most convenient method of proof lies outside the scope of this course.

2 Electrostatics

2.1 Electrostatic potential

Electrostatics is the study of time independent electromagnetic phenomena in the absence of currents and magnetic fields. Then Maxwell's equations are

$$\boldsymbol{\nabla}\wedge\mathbf{E} = 0 \tag{53}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho. \tag{54}$$

Eq. (53) can be satisfied by defining the (electrostatic) potential ϕ by means of

$$\mathbf{E} = -\boldsymbol{\nabla}\phi, \tag{55}$$

so that (54) yields Poisson's equation

$$\boldsymbol{\nabla}^2\phi = -\frac{1}{\epsilon_0}\rho. \tag{56}$$

In this way the study of electrostatics is reduced to the study of a single equation – Poisson's equation. In regions of space where there is no electric charge $\rho = 0$, this reduces to Laplace's equation

$$\boldsymbol{\nabla}^2\phi = 0. \tag{57}$$

ϕ is defined by (55) only to within an additive constant. Usually one choses this constant in such a way that $\phi(\mathbf{r}) \longrightarrow 0$ as $r = |\mathbf{r}| \longrightarrow \infty$. For the point charge q at O, the electric field given by (9) reads

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2}\mathbf{e}_r = -\frac{\partial\phi}{\partial r}\mathbf{e}_r, \quad \mathbf{e}_r = \frac{\mathbf{r}}{r}, \tag{58}$$

and an integration (with constant zero) gives

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r}. \tag{59}$$

Since Poisson's equation is a linear equation for ϕ, the 'superposition principle' applies and tells us that any linear combination (superposition) of solutions is again a solution. A first example of this is

The electric dipole

Consider a system of two point charges $\pm q$, $-q$ at O and $+q$ at $\mathbf{d}$. The superposition principle implies that

$$4\pi\epsilon_0\phi(\mathbf{r}) = q(-\frac{1}{r} + \frac{1}{|\mathbf{r}-\mathbf{d}|}). \tag{60}$$

For all such examples the easiest method of expansion involves the vector statement of Taylor's theorem:

$$f(\mathbf{r}+\mathbf{h}) = f(\mathbf{r}) + \mathbf{h}\cdot\boldsymbol{\nabla}f(\mathbf{r}) + \frac{1}{2}(\mathbf{h}\cdot\boldsymbol{\nabla})^2 f(\mathbf{r}) + \ldots. \tag{61}$$

Here

$$\frac{1}{|\mathbf{r}-\mathbf{d}|} = \frac{1}{r} - \mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\boldsymbol{\nabla})^2\frac{1}{r} + \ldots. \tag{62}$$

So for $d = |\mathbf{d}|$ small we have

$$4\pi\epsilon_0\phi = -q\mathbf{d}\cdot\boldsymbol{\nabla}\frac{1}{r}. \tag{63}$$

The electric dipole arises by taking the limits $q \to \infty, d \to 0$ in such a way that qd remains constant, at a finite value $qd = p$. Then $\mathbf{p} = q\mathbf{d}$ defines the dipole moment of the electrical dipole, and its potential is given by

$$4\pi\epsilon_0\phi = -\mathbf{p}\cdot\boldsymbol{\nabla}\frac{1}{r} = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{\mathbf{p}\cdot\mathbf{e}_r}{r^2}. \tag{64}$$

Taking $\mathbf{d} = d\mathbf{k} = (0,0,1)$ in the z-direction, then, working initially in Cartesians so that $\mathbf{p}\cdot\boldsymbol{\nabla} = p\frac{\partial}{\partial z}$, we find that (64) gives us

$$4\pi\epsilon_0\phi = -p\frac{\partial}{\partial z}\frac{1}{r} = -p(-\frac{1}{r^2}\frac{z}{r}) = \frac{pz}{r^3} \quad = \frac{p\cos\theta}{r^2}. \tag{65}$$

In the last step, we used spherical polars with $z = r\cos\theta$.

The electric quadrupole: not lectured

We can easily go further to the linear quadrupole with charges $-q$ at $\pm\mathbf{d}$ and $2q$ at the origin, so that the system has zero total charge and also zero dipole moment. (It looks like a pair of dipoles pointing in oppposite directions.)

$$\begin{aligned}\frac{4\pi\epsilon_0}{q}\phi &= \frac{2}{r} - \frac{1}{|\mathbf{r}+\mathbf{d}|} - \frac{1}{|\mathbf{r}-\mathbf{d}|} \\ &= \frac{2}{r} - [\frac{1}{r} + \mathbf{d}\cdot\nabla\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\nabla)^2\frac{1}{r}] - [\frac{1}{r} - \mathbf{d}\cdot\nabla\frac{1}{r} + \frac{1}{2}(\mathbf{d}\cdot\nabla)^2\frac{1}{r}] \\ &= -(\mathbf{d}\cdot\nabla)^2\frac{1}{r}. \end{aligned} \tag{66}$$

Note that this approach gets the cancellation of unwanted terms to happen ahead of their evaluation. Hence

$$4\pi\epsilon_0\phi = -q(\mathbf{d}\cdot\nabla)^2\frac{1}{r} = -qd^2\frac{\partial^2}{\partial z^2}\frac{1}{r} = -qd^2\frac{\partial}{\partial z}(-\frac{z}{r^3}) = qd^2(\frac{1}{r^3} - \frac{3z^2}{r^5}). \tag{67}$$

In spherical polars the quadrupole potential is

$$4\pi\epsilon_0\phi = qd^2\frac{1-3\cos^2\theta}{r^3}. \tag{68}$$

We note that the point charge, electric dipole and quadrupole potentials go to zero as r goes to infinity respectively like $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$.

The general charge distribution $\mathcal{D}$

Suppose $\mathcal{D}$ has electric charge density ρ non-zero only throughout some finite subset $\hat{V} \subset V$ =all space. To find the potential due to $\mathcal{D}$, we view it as linear superposition of contributions due to 'elementary charges' $\rho(\mathbf{r}')\,d\tau'$ throughout $\hat{V}$. Then the superposition principle gives

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|}. \tag{69}$$

Since

$$\nabla\frac{1}{|\mathbf{r}-\mathbf{a}|} = -\frac{\mathbf{r}-\mathbf{a}}{|\mathbf{r}-\mathbf{a}|^3}, \quad \mathbf{r} \neq \mathbf{a} \tag{70}$$

we get

$$-\nabla\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V (-\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|})\rho(\mathbf{r}')d\tau' = \frac{1}{4\pi\epsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{E}(\mathbf{r}), \tag{71}$$

consistently with (6) of chapter 1, at least for $\mathbf{r} \notin \hat{V}$. We do not have time to provide the proof, by standard methods in vector calculus, that (69) satisfies Poisson's equation for all $\mathbf{r} \in V$.

Large distance behaviour of (69)

Taking an origin near to or within $\hat{V}$, we want to find how the potential due to $\mathcal{D}$ behaves at large distances r. We will follow the same procedure as above, using Taylor's theorem (61). We find

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \left(\frac{1}{r} - \mathbf{r}'\cdot\nabla\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\cdot\nabla)^2\frac{1}{r}\dots\right)\rho(\mathbf{r}')d\tau'. \tag{72}$$

The leading term of $4\pi\epsilon_0\phi$ (going like $\frac{1}{r}$) is the total charge term, namely

$$\frac{Q}{r}, \quad Q = \int_V \rho(\mathbf{r}')d\tau', \tag{73}$$

unless $Q = 0$. In the latter case the leading term (going like $\frac{1}{r^2}$) is the dipole term

$$-\left(\int_V \mathbf{r}'\rho(\mathbf{r}')d\tau'\right)\cdot\boldsymbol{\nabla}\frac{1}{r} = -\mathbf{P}\cdot\boldsymbol{\nabla}\frac{1}{r}, \tag{74}$$

where the dipole moment of the distribution is

$$\mathbf{P} = \int_V \mathbf{r}'\rho(\mathbf{r}')d\tau', \tag{75}$$

unless of course $\mathbf{P} = 0$, in which case the leading term is a quadrupole type term which goes like $\frac{1}{r^3}$ ....

Uniqueness

Suppose we are given a charge distribution $\rho(\mathbf{r})$ throughout a fixed spatial volume V, then Poisson's equation in V has a unique solution provided that, on $S = \partial V$, either

(i) (Dirichlet boundary conditions) $\phi(\mathbf{r})$ is specified for all $\mathbf{r} \in S$,

or

(ii) (Neumann boundary conditions) $\dfrac{\partial\phi}{\partial n} = \mathbf{n}\cdot\boldsymbol{\nabla}\phi(\mathbf{r}) = -\mathbf{n}\cdot\mathbf{E}(\mathbf{r})$ is specified for all $\mathbf{r} \in S$.

We will see soon that the latter option corresponds to specifying the surface density of charge on S.

For the case of $\mathcal{D}$ above, with a choice of origin near $\hat{V}$, it follows that (69) is the unique solution of Poisson's equation which satisfies the (Dirichlet) boundary condition that $\phi \to 0$ as $r \to \infty$.

Field lines and equipotentials

We mention a way of gaining some insight into the nature of the electric field surrounding a system of charges.

One draws the field lines of $\mathbf{E}$ for the system. A field line here is a line at each of whose points $\mathbf{E}$ is tangent to the line.

Also one draws on the same diagram the equipotentials of the system. These are surfaces ϕ =constant. As $\mathbf{E} = -\boldsymbol{\nabla}\phi$, and $\boldsymbol{\nabla}\phi$ is everywhere normal to such surfaces, it follows that the field lines cut the equipotentials at right angles.

2.2 Gauss's theorem and the calculation of electric fields

In Sec. 1.5 we proved Gauss's theorem

$$\frac{1}{\epsilon_0}Q = \int_S \mathbf{E}\cdot\mathbf{dS}, \tag{76}$$

where

$$Q = \int_V \rho d\tau, \tag{77}$$

is the total charge contained in the spatial volume V, $\partial V = S$. We now apply it to the calculation of the electric fields of simple systems of charge.

a) The point charge q at the origin has been treated in Sec 1.1.

b) Line charge lying along the z-axis with uniform (line) density of charge η (Coulombs) per unit length. Let S be the closed surface of a right circular cylinder of unit length coaxial with the line charge. By symmetry, it is clear that $\mathbf{E}$ is radial, so $\mathbf{E} \cdot \mathbf{n} = 0$ on the ends of S. In fact $\mathbf{E}(\mathbf{r}) = E(s)\mathbf{e}_s$ where s and $\mathbf{e}_s$ are the radial coordinate of cylindrical polars and its associated unit vector. Thus Gauss gives

$$E\, 2\pi s = \frac{1}{\epsilon_0}\eta, \quad \mathbf{E}(s) = \frac{\eta}{2\pi\epsilon_0}\frac{1}{s}\mathbf{e}_s. \tag{78}$$

This corresponds to a potential given by

$$2\pi\epsilon_0\phi = -\eta \log \frac{s}{s_0}. \tag{79}$$

In this example, $\phi(s)$ does not go to zero as $s \to \infty$, so we were forced to demand that $\phi = 0$ for some fixed but arbitrary value s_0 of s.

To check that (79) is correct, use

$$-\boldsymbol{\nabla}\phi = -\mathbf{e}_s \frac{\partial \phi}{\partial s} \tag{80}$$

c) Plane sheet P occupying the plane $z = 0$, carrying uniform charge density σ per unit area.

Here we use the 'Gaussian pillbox': a cylinder of cross-sectional area A, with axis $\mathbf{k} = (0, 0, 1)$, with plane ends at $z = h$ and $z = -h$. By symmetry $\mathbf{E}$ is perpendicular to P. Above P we have $\mathbf{E} = E\mathbf{k}$ and below $\mathbf{E} = -E\mathbf{k}$ for some $E = E(h)$. This time $\mathbf{E} \cdot \mathbf{dS}$ is zero on the curved sides of the pill-box, and Gauss gives

$$EA - (-E)A = \frac{\sigma\, A}{\epsilon_0}, \quad E = \frac{1}{2\epsilon_0}\sigma, \quad \text{independent of } h. \tag{81}$$

d) Parallel plane sheets in the planes $z = 0$ and $z = a$, carrying uniform distributions of charge respectively of charge with surface densities $\pm\sigma$ (Coulombs) per unit area. Using the result of c) twice and the principle of superposition, we find that

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\mathbf{k}, \quad \mathbf{k} = (0, 0, 1), \tag{82}$$

in the spatial region between the plates and zero outside.

Writing $\mathbf{E} = -\frac{\partial\phi}{\partial z}\mathbf{k}$, we get $\phi = \phi_0 - \frac{\sigma}{\epsilon_0}z$, where ϕ_0 is the potential of the $z = 0$ sheet. If the $z = a$ sheet has potential ϕ_a, then the potential difference between the sheets is $\phi_0 - \phi_a = \frac{\sigma}{\epsilon_0}a$.

e) Spherical shell, centre at O, radius r', uniform charge density σ per unit area, and thus total charge $Q = 4\pi r'^2 \sigma$. By symmetry, as for a point charge at O, we have $\mathbf{E} = E(r)\mathbf{e}_r$.

To apply Gauss's theorem, take spheres of radius r, concentric with the shell. Let these have surfaces S_1 and S_2, in the cases (i) $r > r'$ and (ii) $r < r'$

In case (i):

$$\begin{aligned} \int_{S_1} \mathbf{E} \cdot \mathbf{dS} = \int_{S_1} E(r)\mathbf{e}_r \cdot \mathbf{e}_r dS &= \frac{1}{\epsilon_0} Q \\ 4\pi r^2 E(r) &= \frac{1}{\epsilon_0} Q, \quad E(r) = \frac{\sigma}{\epsilon_0} \frac{r'^2}{r^2}. \end{aligned} \tag{83}$$

For case (ii), we have $E(r) = 0$, since there is no charge in the volume V_2.

It is to be noted that the result (78) is the same (for $r > r'$) as applies to a point charge Q situated at the origin.

Check that $E = \mathbf{E} \cdot \mathbf{e}_r$, the normal component of $\mathbf{E}$, has discontinuity $\frac{1}{\epsilon_0}\sigma$ at $r = r'$.

f) Sphere of radius R carrying uniform charge of density ρ (Coulombs) per unit volume, and thus total charge $Q = \frac{4\pi}{3} R^3 \rho$.

For $r > R$ by superposition of shells and the result of e), we learn that the potential is the same as it would be if we had a point charge Q at the origin.

$$\mathbf{E}(\mathbf{r}) = E_1(r)\mathbf{e}_r, \quad E_1(r) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}. \tag{84}$$

For $r < R$, applying Gauss to a sphere S_2 centre the origin of radius r, only the charge inside S_2 is relevant, and we have

$$\mathbf{E}(\mathbf{r}) = E_2(r)\mathbf{e}_r, \quad E_2(r)\, 4\pi r^2 = \frac{1}{\epsilon_0} \rho \frac{4\pi}{3} r^3, \tag{85}$$

so that inside the charge distribution

$$E_2(r) = \frac{Qr}{4\pi\epsilon_0 R^3}. \tag{86}$$

We have obtained (86) by direct application of Gauss, but we could otherwise have found it from e) by a suitable application of the superposition principle.

Note that $E(r)$, the normal (and here only) component of $\mathbf{E}$, is continuous at $r = R$.

We can use $\mathbf{E} = -\boldsymbol{\nabla}\phi = -\mathbf{e}_r \dfrac{\partial \phi}{\partial r}$ to determine the potentials ϕ_1 outside, and ϕ_2 inside, the charge distribution.

$$\begin{aligned} -\frac{\partial \phi_1}{\partial r} &= \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \quad \Rightarrow \quad \phi_1 = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} + A \\ -\frac{\partial \phi_2}{\partial r} &= \frac{Qr}{4\pi\epsilon_0 R^3} \quad \Rightarrow \quad \phi_2 = -\frac{Qr^2}{8\pi\epsilon_0 R^3} + B. \end{aligned} \tag{87}$$

Here A and B are constants of integration. Demanding that $\phi \to 0$ as $r \to \infty$, we look at ϕ_1 and require $A = 0$. To find B, we use the fact that ϕ is continuous at $r = R$. This leads to

$$\phi_2 = \frac{Q}{8\pi\epsilon_0 R^3}(3R^2 - r^2). \tag{88}$$

g) The discontinuity law at a surface carrying surface charge.

Suppose a surface S with normal $\mathbf{n}$ carrying charge of uniform charge density σ per unit area, separates regions 1 and 2 of empty space, with $\mathbf{n}$ pointing into 2. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the electric fields in regions 1 and 2.

Use Gauss with a Gaussian pillbox of very small height, and cross sectional area A, with the end with normal $\mathbf{n}$ just inside 2 and the other end with normal $-\mathbf{n}$, just inside 1. In fact we take the height so small that the curved sides of the box contribute negligibly to the surface integral of the theorem. Then

$$[\mathbf{n} \cdot \mathbf{E}_2 + (-\mathbf{n}) \cdot \mathbf{E}_1]A = \frac{\sigma A}{\epsilon_0}, \quad \mathbf{n} \cdot \mathbf{E}|_{-}^{+} = \frac{1}{\epsilon_0}\sigma, \tag{89}$$

as stated in Sec. 1.7.

See that examples c), d), e), and f) conform to this, there being no surface charge present in f).

Solutions of Laplace's equations

In spherical polars (r, θ, ϕ), the general solution of Laplace's equations with spherical symmetry (with no dependence on θ and ϕ) is

$$\phi = a + \frac{b}{r}. \tag{90}$$

Next we have solutions, like the dipole potential, $\propto \cos\theta$,

$$\phi = -Er\cos\theta + \frac{c}{r^2}\cos\theta = \phi_1 + \phi_2. \tag{91}$$

Here the first term gives an electric field $\mathbf{E} = -\boldsymbol{\nabla}\phi_1 = -\frac{\partial\phi_1}{\partial z}\mathbf{k} = E\mathbf{k}$ of constant magnitude in the z-direction.

In cylindrical polars (s, ϕ, z), the general solution of Laplace's equations with cylindrical symmetry is

$$\phi = a + b\ln s. \tag{92}$$

2.3 Perfect conductors

In a perfect conductor any movable charges present are free to move within it under an applied electric field without resistance. In electrostatics, we deal with situations in which all such movable charges have reached positions of equilibrium. In particular there is no current flow, the atoms of the material keep their conduction electrons and are neutral.

Consider then a perfect conductor $\mathcal{C}$ with surface S, with perfectly non-conducting empty space (the vacuum) outside, to which some non-zero total charge has been supplied.

We shall see in Sec. 3.1 that inside $\mathcal{C}$ we must have $\mathbf{E} = 0$, and hence $\rho = 0$. Thus it follows that the charge supplied must reside on the surface S of $\mathcal{C}$. Further $\mathbf{E} = E\mathbf{n}$ on S, else charges would be able to move along S. Thus S is an equipotential of constant ϕ, since $\mathbf{E} = -\boldsymbol{\nabla}\phi$ is normal to it. Also, because $\mathbf{E} = 0$ inside S, ϕ is constant throughout the interior of $\mathcal{C}$, with a value equal to the surface equipotential value. Finally the charge σ per unit area on S follows from g) of Sec. 2.2. This gives

$$\frac{1}{\epsilon_0}\sigma = \mathbf{n} \cdot \mathbf{E}|_{-}^{+} = \mathbf{n} \cdot \mathbf{E} = E \tag{93}$$

This uses the fact that $\mathbf{E} = 0$ inside $\mathcal{C}$, (*i.e.* on the minus side of the surface S of $\mathcal{C}$).

The Force on a charged conductor

We do not have time to give a proof, but note the force per unit area exerted on the surface of a perfect conductor carrying charge per unit area σ is given by

$$F = \frac{1}{2\epsilon_0}\sigma^2. \tag{94}$$

This is a special case of the result (51) of Sec. 1.7.

2.4 Electrostatic energy

The potential energy (PE) of a point charge q at $\mathbf{r}$ in an electric field of potential $\phi(\mathbf{r})$ is the work that must be done on q to bring it from infinity (where $\phi = 0$) to $\mathbf{r}$

$$PE = q\phi(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r}, \quad \mathbf{F} = q\mathbf{E}. \tag{95}$$

Consider a system of point charges $q_i, \quad i = 1, 2, \ldots, n$, bringing them from infinity to their final positions in order, doing work

$$\begin{aligned} \text{on} \quad q_1; \quad W_1 &= 0 \\ \text{on} \quad q_2; \quad W_2 &= \frac{q_2}{4\pi\epsilon_0}\frac{q_1}{r_{12}} \\ \text{on} \quad q_3; \quad W_3 &= \frac{q_3}{4\pi\epsilon_0}(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}) \\ \text{on} \quad q_i; \quad W_i &= \frac{q_i}{4\pi\epsilon_0}\sum_{j<i}\frac{q_j}{r_{ji}} \\ W = \sum_{i=1}^{n} W_i &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{4\pi\epsilon_0}\frac{q_i\, q_j}{r_{ij}}. \end{aligned} \tag{96}$$

Here $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, \quad r_{ij} = |\mathbf{r}_{ij}|$, and $\sum_{i=1}^{n}\sum_{j<i} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}$. Thus W by construction gives the electrostatic energy of the system.

But the potential at q_i due to all the other charges is

$$\phi_i = \frac{1}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{r_{ij}}, \tag{97}$$

so that

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\phi_i. \tag{98}$$

The corresponding result for a continuous distribution of charge of charge density $\rho(\mathbf{r})$ in volume V then is

$$\begin{aligned} W &= \frac{1}{2}\int_V \rho(\mathbf{r})\phi(\mathbf{r})d\tau \\ &= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int_V\int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau\, d\tau'. \end{aligned} \tag{99}$$

If there are conductors $\mathcal{C}_i$ with charges Q_i at potentials ϕ_i, then the contribution which they make to W is given by

$$\frac{1}{2}\sum_i \int_{S_i}\sigma_i\phi_i dS_i = \frac{1}{2}\sum_i \phi_i\int_{S_i}\sigma_i dS_i = \frac{1}{2}\sum_i \phi_i Q_i. \tag{100}$$

(Recall that the potential is constant on a conductor).

Field energy in electrostatics

Given a charge distribution $\rho(\mathbf{r}')$ distributed over a finite volume $\hat{V}$ and a set of conductors all in some finite region of space in which an origin is taken. Let V be all space bounded by a sphere S at infinity, but excluding the interiors of the conductors.

Then

$$W = \frac{1}{2}\int_V \rho\phi d\tau + \frac{1}{2}\sum_i Q_i\phi_i. \tag{101}$$

Use

$$\begin{aligned} \rho\phi &= \epsilon_0 \phi \boldsymbol{\nabla} \cdot \mathbf{E} \\ &= \epsilon_0 [\boldsymbol{\nabla} \cdot (\phi \mathbf{E}) - \mathbf{E} \cdot \boldsymbol{\nabla} \phi] \\ &= \epsilon_0 \boldsymbol{\nabla} \cdot (\phi \mathbf{E}) + \epsilon_0 \mathbf{E}^2. \end{aligned} \tag{102}$$

Then W is given by

$$\frac{1}{2}\epsilon_0 [\int_V \mathbf{E}^2 d\tau + \int_S \phi \mathbf{E} \cdot \mathbf{dS} + \sum_i \int_{\mathcal{C}_i} \phi \mathbf{E} \cdot \mathbf{dS}_i] + \frac{1}{2} \sum_i Q_i \phi_i. \tag{103}$$

As $\phi = 0$ on S (at infinity) the second term of (103) is zero. In the third term of (103), the divergence theorem dictates that $\mathbf{dS}_i = -\mathbf{n} dS_i$ points into $\mathcal{C}_i$, and so, in this term, we have

$$-\epsilon_0 \int_{\mathcal{C}_i} \phi \mathbf{n} \cdot \mathbf{E} dS_i = -\epsilon_0 \phi_i \int_{\mathcal{C}_i} \mathbf{n} \cdot \mathbf{E} dS_i = -\phi_i \int_{\mathcal{C}_i} \sigma_i dS_i = -\phi_i Q_i. \tag{104}$$

It follows that the third and the fourth terms of (103) cancel. And so, for the energy of the electrostatic field, we have the important result

$$W = \frac{1}{2}\epsilon_0 \int_V \mathbf{E}^2 d\tau. \tag{105}$$

We note this involves an integral over all of V, including the regions unoccupied by charge, whereas the first term of (101) is really an integral over the region $\hat{V} \subset V$ occupied by charge.

2.5 Capacitors and capacitance

A pair of conductors carrying charges $\pm Q$ constitute a capacitor (or a condenser). Since their potentials are proportional to Q, the same applies to their potential difference $V = \phi_1 - \phi_2$.

Therefore we define the capacitance C of the capacitor by

$$V = \frac{1}{C} Q. \tag{106}$$

It turns out always to be a constant that depends on the configuration of the two conductors.

a) Parallel-plate capacitor.

The field lines are mainly straight lines perepndicular to the plates. We assume the distance a between the plates is small on a scale set by the area A of the plates. Thus we may neglect ' edge effects', so called because the electric field lines near to the edges of the plates bulge out from between the plates.

¿From d) of Sec. 2.2, we know that $\mathbf{E} = E\mathbf{k}$, $\quad E = \dfrac{\sigma}{\epsilon_0}$ between the plates, with $E = 0$ elsewhere. Here $\mathbf{k} = (0, 0, 1)$. Hence

$$-\frac{d\phi}{dz} = E \Rightarrow \phi = -Ez + c. \tag{107}$$

If $\phi = \phi_1$ at $z = 0$, then $c = \phi_1$, and then $\phi = \phi_2$ at $z = a$ gives

$$\phi_2 = -Ea + \phi_1, \quad \text{and} \quad V = \phi_1 - \phi_2 = aE = \frac{a\sigma}{\epsilon_0} = \frac{aQ}{\epsilon_0 A}. \tag{108}$$

So

$$C = \frac{A\epsilon_0}{a}. \tag{109}$$

The energy of the capacitor is given now by (98), so that

$$W = \frac{1}{2}\sum_i q_i\phi_i = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C}. \tag{110}$$

But the energy can also be calculated from the field energy expression (105), which gives

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{A\epsilon_0}{2}\int_0^a (\frac{\sigma}{\epsilon_0})^2 dz = \frac{\sigma^2 Aa}{2\epsilon_0} = \frac{1}{2}\frac{Q^2}{C}. \tag{111}$$

b) Concentric spheres S_1 and S_2 of radii a and $b > a$, carrying charges Q and $-Q$. Take $\phi = 0$ at $r = b$ and $\phi = V$ at $r = a$. ¿From previous studies we know that for $r \in \{a \leq r \leq b\}$ (outside S_1 and inside S_2) we have

$$4\pi\epsilon_0 E = -4\pi\epsilon_0\frac{\partial\phi}{\partial r} = \frac{Q}{r^2}, \tag{112}$$

and

$$4\pi\epsilon_0\phi = \frac{Q}{r} - \frac{Q}{b}. \tag{113}$$

Hence

$$4\pi\epsilon_0 V = Q(\frac{1}{a} - \frac{1}{b}) \tag{114}$$

and

$$C = \frac{4\pi\epsilon_0 a\, b}{(b-a)}. \tag{115}$$

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3 Steady electric currents and magnetism

3.1 Steady current flow

Here we study steady current flow in conducting material. This is governed by Maxwell's equations without $\frac{\partial}{\partial t}$ terms, so that we have

$$\nabla\wedge\mathbf{B} = \mu_0\mathbf{J}, \quad \nabla\wedge\mathbf{E} = 0, \tag{1}$$

together with the experimental law, valid for simple conductors, but not, for example, for non-isotropic materials such as crystalline material,

$$\mathbf{J} = \sigma\mathbf{E}, \tag{2}$$

where σ is the conductivity of the material.

(Both conductivity and surface charge are normally denoted by the same symbol σ. We seldom have contexts in which both arise.)

Note that (1) implies

$$\nabla\cdot\mathbf{J} = 0. \tag{3}$$

This agrees the continuity equation, eq. (20) of chapter one, as $\frac{\partial\rho}{\partial t} = 0$ applies here. Eq. (2) also implies

$$\nabla\cdot\mathbf{E} = 0, \tag{4}$$

and hence also $\rho = 0$ within the material. This makes sense in contexts such as current flowing in copper wires in which electrons flow through a background of positively charge ions, so that it is reasonable to suppose that $\rho = 0$ for the total charge density of the material, electrons plus ions.

We have a remark here promised in Sec. 2.3 which talks about perfect conductors for which the conductivity σ goes to infinity. In order for finite currents ($|\mathbf{J}|$ finite) to flow in such material, it is necessary that $|\mathbf{E}|$ goes to zero, so that also ρ goes to zero.

In this section, we are concerned only with current flow. In later sections of this chapter, we study the magnetic fields that arise from the (time-independent) flow of electric currents.

Consider steady current flow in regions of conducting material, outside of batteries.

This is governed by the equations

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\wedge\mathbf{E} = 0, \tag{5}$$

together with the experimental law (2).

If we set $\mathbf{E} = -\nabla\phi$ then the flow is governed by the single equation, Laplace's equation, plus (2), so that we can solve problems of steady current flow by finding $\phi, \mathbf{E}, \mathbf{J}$ in turn.

We might ask: can we obtain an understanding of the elementary form

$$V = IR \tag{6}$$

of Ohm's law, relating the potential difference across the ends of a conductor to the current that flows within it?

We do this here for a simple example; there are two others in Problem Set 2.

Uniform current enters the plate of uniform thickness δ shown in the diagram. In cylindrical polars, (with polar angle called θ since ϕ is reserved here for the potential), we have the solution

$$\phi = d - c\theta, \quad c,\ d \quad \text{constants}, \tag{7}$$

of Laplace's equation, so that the potential difference (PD) between AB and CD is $V = c\alpha$. Hence

$$\mathbf{E} = -\frac{1}{s}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta = \frac{c}{s}\mathbf{e}_\theta, \tag{8}$$

and the lines of $\mathbf{E}$ and of $\mathbf{J}$ are arcs of circles centred on O, as shown. Also

$$\mathbf{J} = \sigma\mathbf{E} = \frac{\sigma c}{s}\mathbf{e}_\theta = \frac{\sigma V}{\alpha s}\mathbf{e}_\theta \tag{9}$$

so that the total current entering at AB (which of course equals the current leaving at CD) is

$$I = \int_{AB} \mathbf{J}\cdot\mathbf{dS} \ = \ \frac{\sigma V\delta}{\alpha}\int_{s_1}^{s_2}\frac{1}{s}ds = \frac{\sigma V\delta}{\alpha}\ln\frac{s_2}{s_1}, \tag{10}$$

where we used $\mathbf{dS} = \mathbf{e}_\theta ds\delta$, and (9). This is indeed of the form (6) of Ohm's law, with

$$R = \frac{\alpha}{\sigma\delta\ln(s_2/s_1)}. \tag{11}$$

So resistance is inversely proportional to conductivity σ, and, like capacitance, depends on the geometry of the current flow set-up.

Generation of heat by steady current flow

Consider the tube of flow shown, *i.e.* the cylinder whose sides are lines of $\mathbf{E}$ and $\mathbf{J}$ and whose ends are equipotentials. Current of density $\mathbf{J}$ enters at the end A where the potential is ϕ_A and leaves at B where the potential is $\phi_B < \phi_A$. The potential difference is

$$V = \phi_A - \phi_B = -\delta\mathbf{r}\cdot\boldsymbol{\nabla}\phi = \delta r\, E. \tag{12}$$

In unit time charge $J\delta S$ enters the tube at A in unit time and leaves at B. The work done on this charge moving it through the potential difference V in unit time is

$$(J\delta S)V = (J\delta S)(E\delta r) = JE(\delta S\ \delta r) = (\mathbf{J}\cdot\mathbf{E})\delta\tau. \tag{13}$$

This work done corresponds to the conversion of electrical energy into heat, *i.e.* to the loss of electrical energy. The energy loss per unit time in volume τ, with surface S is

$$W = \int_\tau \mathbf{J}\cdot\mathbf{E}d\tau = -\int_\tau \mathbf{J}\cdot\boldsymbol{\nabla}\phi d\tau. \tag{14}$$

We use

$$\mathbf{J} \cdot \boldsymbol{\nabla}\phi = \boldsymbol{\nabla} \cdot (\phi\mathbf{J}) - \phi(\boldsymbol{\nabla} \cdot \mathbf{J}), \tag{15}$$

where the second term is zero owing to (3), and the first term allows us to apply the divergence theorem to (14). We obtain

$$W = -\int_S \phi\mathbf{n} \cdot \mathbf{J}dS, \tag{16}$$

where $\mathbf{n}$ is the unit normal on S pointing out of τ.

Consider a conductor with current entering it and leaving it at ends S_1 and S_2, which are equipotentials of potentials ϕ_1 and ϕ_2. Then, remembering that the $\mathbf{n}$ of (16) for S_1 is the negative of $\mathbf{n}_1$ in the diagram, we have from (16)

$$\begin{aligned} W &= (\phi_1 - \phi_2)I, \quad I = \int_{S_1} \mathbf{n}_1 \cdot \mathbf{J}dS = \int_{S_2} \mathbf{n}_2 \cdot \mathbf{J}dS \\ &= VI, \end{aligned} \tag{17}$$

where V is the potential difference between the ends. Using the elementary form (6) of Ohm's law, we have shown that the energy generation per unit time in a conductor of resistance R through which flows a current I is

$$W = RI^2. \tag{18}$$

This is a formula familiar from elementary studies for the energy dissipated in unit time as heat.

3.2 Magnetostatics

This deals with steady currents and the associated (time independent) magnetic fields. It is governed by the equations

$$\begin{aligned} \boldsymbol{\nabla}\wedge\mathbf{B} &= \mu_0\,\mathbf{J}, \quad (\Rightarrow \boldsymbol{\nabla} \cdot \mathbf{J} = 0) & (19) \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0. & (20) \end{aligned}$$

Eq. (20) is automatically satisfied when the vector potential $\mathbf{A}$ is introduced via

$$\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}, \tag{21}$$

since

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\wedge\mathbf{A}) = \partial_i\epsilon_{ijk}\partial_j A_k = \boldsymbol{\nabla}\wedge\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{22}$$

For given $\mathbf{B}$ however (21) does not determine $\mathbf{A}$ uniquely, because we can transform the vector potential according to

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{23}$$

where χ is an arbitrary scalar field. Since

$$\boldsymbol{\nabla}\wedge\mathbf{A}' = \boldsymbol{\nabla}\wedge\mathbf{A} + \boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\chi = \boldsymbol{\nabla}\wedge\mathbf{A} = \mathbf{B}, \tag{24}$$

the transformed vector potential serves our needs just as well as does $\mathbf{A}$.

In fact we can make use of (23) to impose a simplifying condition on the vector potentials we use in practice. Suppose we have found some $\mathbf{A}$ which yields the required $\mathbf{B}$ via (21), and is such that $\boldsymbol{\nabla} \cdot \mathbf{A} = \psi$, where ψ is a scalar field, calculable, as is obvious, from $\mathbf{A}$. We shall pass by means of (23) to a vector potential $\mathbf{A}'$ such that

$$\boldsymbol{\nabla} \cdot \mathbf{A}' = 0. \tag{25}$$

This can always be done, since (25) implies

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \cdot \mathbf{A} + \boldsymbol{\nabla}^2\chi \\ &= \psi + \boldsymbol{\nabla}^2\chi, \end{aligned} \tag{26}$$

which is an equation of Poisson type for which a (particular integral) solution for χ in terms of ψ can always be found.

In what follows, we therefore assume that we can deal with vector potentials $\mathbf{A}$ which obey

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{27}$$

[**Some language**: Eq. (23) is called a gauge transformation, the condition (27) is called a gauge condition, and the physical theory is said to be gauge-invariant, because it depends only on $\mathbf{B}$, and not on the gauge condition that has been used]

Return now to (19). Since

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A}, \tag{28}$$

(19) reduces, with the aid crucially of our gauge condition (27), to

$$\boldsymbol{\nabla}^2\mathbf{A} = -\mu_0\,\mathbf{J}. \tag{29}$$

In Cartesian coordinates this reads as

$$\boldsymbol{\nabla}^2 A_k = -\mu_0 J_k \qquad (k = 1, 2, 3), \tag{30}$$

which, for each k, is of Poisson type, so that as in electrostatics, we can write down the solution

$$\begin{aligned} A_k(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \frac{J_k(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\ \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'. \end{aligned} \tag{31}$$

Since it is not obvious that the expression (31) for $\mathbf{A}$ satisfies (27), we ought to prove that it does. When this is done, it follows that

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\wedge\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')\wedge(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\tau', \tag{32}$$

satisfies (19). In calculating $\mathbf{B}$, note that $\boldsymbol{\nabla}\wedge$ acts only on the $\mathbf{r}$ variable, found only in the denominator factor of expression (31) for $\mathbf{A}$, so that we may use eq. (64) of Sec. 2.1 to finish the verification.

Consider a current of density $\mathbf{J}$ flowing in an element $\delta\mathbf{r}$ of a very thin wire of cross-sectional area A. Then $\mathbf{J}\delta V = J\,(A\delta\mathbf{r}) = (JA)\,\delta\mathbf{r} = I\delta\mathbf{r}$. Neglecting the thickness of the wire, we can write, for the vector-potential and the magnetic field due to a wire which carries a current I and takes the form of a simple curve C, the expressions

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_C \frac{\mathbf{dr}'}{|\mathbf{r}-\mathbf{r}'|} \tag{33}$$

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\int_C \frac{(\mathbf{r}-\mathbf{r}')\wedge\mathbf{dr}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{34}$$

The results (32) and (34) for $\mathbf{B}$ are each often referred to the Biot-Savart law.

Proof that (31) satisfies (27).

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)d\tau' \quad (\boldsymbol{\nabla}\text{ acts on }\mathbf{r}\text{ and not on }\mathbf{r}')$$

$$
\begin{aligned}
&= \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_V \left[\boldsymbol{\nabla}'\cdot\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{J}(\mathbf{r}')\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{r}')\right]d\tau' \\
&= -\frac{\mu_0}{4\pi}\int_S \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{n}'\cdot\mathbf{J}(\mathbf{r}')dS'. \qquad (35)
\end{aligned}
$$

Here V is all space, but if we suppose that a physical current distribution occupies a finite volume $\hat{V}\subset V$ near the origin, then $\mathbf{J}(\mathbf{r}')=0$ on S and the proof is complete.

Note the use of a now well-known identity for $\boldsymbol{\nabla}\cdot(\phi\mathbf{F})$ in the third line, $\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{r}')=0$ in the fourth line, and finally the ubiquitous divergence theorem.

[**Care with $\boldsymbol{\nabla}^2\mathbf{F}$ for a vector field $\mathbf{F}$** may be needed. There is no problem in Cartesians, and hence probably not in the material of this course:

$$(\boldsymbol{\nabla}^2\mathbf{F})_k = (\partial_j\partial_j)F_k \qquad (36)$$

where $\nabla^2=\partial_j\partial_j$ is the usual expression used in Laplace's equation. In other coordinate systems, where the unit basis vectors are themselves coordinate dependent, $(\boldsymbol{\nabla}^2\mathbf{F})_\alpha$, the component of the vector $\boldsymbol{\nabla}^2\mathbf{F}$ along the unit vector $\mathbf{e}_\alpha$, is no longer given by $(\nabla^2)\mathbf{F}_\alpha$. The correct result however follows from use of $\boldsymbol{\nabla}^2\mathbf{F}=-\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{F})+\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F})$ where each of the two terms on the right is calculable by two well-defined steps in any system of orthogonal curvilinear co-ordinates.
]

3.3 Magnetic fields of simple current distributions

To calculate these one may use Ampère's law, the Biot-Savart law or perhaps first calculate $\mathbf{A}$ from (31) or (33).

a) **Infinite straight wire carrying current I**

Take the z-axis along the wire, take O in the xy-plane throught the point P, and calculate $\mathbf{B}$ at P, $\mathbf{r}=\vec{OP}$ using Biot-Savart. Using cylindrical polars, (s,ϕ,z), we have

$$\mathbf{r}=s\mathbf{e}_s, \quad \mathbf{r}'=z'\mathbf{k}, \quad \mathbf{dr}'=dz'\mathbf{k}, \quad |\mathbf{r}-\mathbf{r}'|=(s^2+z'^2)^{1/2}. \qquad (37)$$

Now $-(\mathbf{r}-\mathbf{r}')\wedge\mathbf{dr}'=s\,dz'\mathbf{e}_\phi$ so that we have proved that $\mathbf{B}$ is everywhere in the direction of $\mathbf{e}_\phi$. Hence, from (34)

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{s\,dz'}{(s^2+z'^2)^{3/2}}\mathbf{e}_\phi$$

$$= \frac{\mu_0 I}{4\pi s}\int_{-\pi/2}^{\pi/2} \cos\alpha \, d\alpha \, \mathbf{e}_\phi, \quad (z' = s\tan\alpha)$$
$$= \frac{\mu_0 I}{2\pi s}\mathbf{e}_\phi. \tag{38}$$

We got the same answer in Sec. 1.4, arguing there that $\mathbf{B} = B(s)\mathbf{e}_\phi$ by 'symmetry considerations'.

b) Long solenoid

This is a continuous wire carrying current I wound round a very long right circular cylinder, so long that end effects can be ignored. Assume there are N turns of wire per unit length, with N large, wound in a spiral of very small pitch, so that we can regard the cylindrical surface as carrying a surface current. Use cylindrical polars (r, ϕ, z), with z-axis at the axis of the cylinder. Then $\mathbf{s} = NI\mathbf{e}_\phi$ gives the current density, *i.e.* the current per unit length, measuring the charge crossing unit length in unit time. Note that **we called the radial coordinate of cylindrical polars** r here because the symbol s denotes the magnitude of the surface current.

Since $|\mathbf{B}|$ is clearly independent of both z and ϕ, we take $\mathbf{B}$ of the form

$$\mathbf{B} = B_z(r)\mathbf{k}, \quad \mathbf{k} = (0, 0, 1). \tag{39}$$

This satisfies $\boldsymbol{\nabla}.\mathbf{B} = 0$. Also $\boldsymbol{\nabla}\wedge\mathbf{B} = 0$, which holds where there is no (volume) density of current, implies

$$\frac{\partial B_z}{\partial r} = 0, \quad \text{so that} \quad B_z = \text{constant}. \tag{40}$$

(The cylindrical polar coordinate detail of each of these statements should be checked.)

Outside the cylinder this constant is zero, because $|\mathbf{B}| = 0$ for infinite r. To find $|\mathbf{B}|$ inside the cylinder use the rectangular contour C shown in the diagram on P21. Only the vertical line inside the solenoid contributes to $\oint \mathbf{dr}\cdot\mathbf{B}$, so that Ampère leads to

$$B_z\, z = \mu_0 NIz, \quad B_z = \mu_0 NI, \quad \mathbf{B} = \mu_0 NI\mathbf{k}. \tag{41}$$

The answer obtained here illustrates the general discontinuity law given as eq. (50) of Sec. 1.7, and proved in Sec. 3.8

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0\,\mathbf{s}, \tag{42}$$

at a surface of discontinuity carrying a surface current density $\mathbf{s}$ per unit length. We have $\mathbf{n}\wedge\mathbf{B}|^+ = 0$, and

$$\mathbf{n}\wedge\mathbf{B}|_- = (-)\mathbf{e}_r\wedge(\mu_0\, NI\mathbf{k}) = \mu_0\, NI\mathbf{e}_\phi = \mu_0\mathbf{s}. \tag{43}$$

c) Long cylindrical conductor

Consider current, flowing in a long right circular cylinder and distributed uniformly over its circular cross-section, of area $A = \pi a^2$, so that

$$\mathbf{J} = J\mathbf{k}, \quad \pi a^2 J = I, \quad \mathbf{k} = (0, 0, 1). \tag{44}$$

Assume that magnetic fields can be calculated within the conducting material by the same formulas as apply in the vacuum or free-space. This is a good approximation for good conductors, which do have similar magnetic properties to free-space.

Use cylindrical polars (s, ϕ, z) with z-axis along the axis of the conductor. By symmetry $\mathbf{B} = B(s)\mathbf{e}_\phi$, and we apply Ampère to horizontal circles centred on the z-axis for (i) $s > a$ and $s < a$.

$$\text{outside} \quad 2\pi s B = \mu_0 I, \quad B = \frac{\mu_0 I}{2\pi s} \tag{45}$$

$$\text{inside} \quad 2\pi s B = \mu_0 \pi s^2 J, \quad B = \frac{\mu_0 I s}{2\pi a^2} = \frac{\mu_0 I s}{2}. \tag{46}$$

Note that outside the conductor the magnetic field is the same as for a very thin wire, as in example **a**).

Note also that here there is no surface current, and hence we expect

$$\mathbf{n}\wedge\mathbf{B}|_{-}^{+} = 0. \tag{47}$$

Here $\mathbf{n}\wedge\mathbf{B} = \mathbf{e}_s\wedge B(s)\mathbf{e}_\phi = B(s)\mathbf{k}$ and continuity of the tangential component of $\mathbf{B}$ at $s = a$ follows (45) and (46).

3.4 The large distance expansion of the vector potential

Consider the formula (31) for the vector potential in the situation in which the distribution of current density is confined to a subset $\hat{V} \subset V$ =allspace. Chose the origin near to it. Then a long calculation based on Taylor's theorem yields the following leading large r approximation to $\mathbf{A}(\mathbf{r})$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\mathbf{m}\wedge\mathbf{r}, \tag{48}$$

where the magnetic dipole moment of the current distribution is defined by

$$\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\wedge\mathbf{J}(\mathbf{r})d\tau. \tag{49}$$

We consider in detail only the following case.

3.5 The current loop

Here we look at the vector potential $\mathbf{A}$ (33) of a current loop, *i.e.* a wire of negligible cross-section shaped in the form of a closed contour C, carrying a current I. For simplicity, let C define a plane loop of area $\mathbf{S} = S\mathbf{n}$ of unit normal $\mathbf{n}$.

Chose an origin near the loop and seek the vector potential of its magnetic field, at distances large on a scale set by the physical dimensions of the loop. (Or, consider $\mathbf{A}(\mathbf{r})$ due to a small loop.)

Let S be a surface such that $\partial S = C$. Let $\mathbf{c}$ be an arbitrary constant vector, and work on $\mathbf{c}\cdot\mathbf{A}$

$$\begin{aligned}
\mathbf{c}\cdot\mathbf{A} &= \frac{\mu_0 I}{4\pi}\oint_C \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{c}\cdot d\mathbf{r}' \\
&= \frac{\mu_0 I}{4\pi}\int_S \mathbf{n}'\cdot\boldsymbol{\nabla}'\wedge\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{c}\right) dS' \quad \text{(Stokes)} \\
&= \frac{\mu_0 I}{4\pi}\left[\int_S d\mathbf{S}'\wedge\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]\cdot\mathbf{c}.
\end{aligned} \tag{50}$$

Hence

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_S d\mathbf{S}'\wedge\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_S d\mathbf{S}'\wedge\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right). \tag{51}$$

Here, in order to get the leading large r approximation, to $\mathbf{A}(\mathbf{r})$, we simply drop all dependence on $\mathbf{r}'$ from the integrand of (51). So we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[I\int_S d\mathbf{S}'\right]\wedge\mathbf{r} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\mathbf{m}\wedge\mathbf{r}, \tag{52}$$

where we have defined the magnetic dipole moment of the current loop by

$$\mathbf{m} = I \int_S \mathbf{dS} = IS\mathbf{n}. \tag{53}$$

We have reached a result which agrees exactly with (48). To see this recall the usual conversion $\int_V (\ldots)\mathbf{J}(\mathbf{r})\, d\tau \to \int_C (\ldots) I\mathbf{dr}$. Then (49) gives

$$\mathbf{m} = \frac{1}{2} I \int_C \mathbf{r} \wedge \mathbf{dr} = IS\mathbf{n}, \tag{54}$$

upon use of result of example 6 of the vector calculus revision sheet.

We have obtained a result crucial to the understanding of magnetism at all levels: a small current loop gives, via (54), a physical realisation of a magnetic moment.

3.6 Dipole view of m

We now show why, in the previous section, we referred to **m** as a magnetic dipole moment.

At points where there is no charge density $\mathbf{J} = 0$, the magnetic field **B** obeys

$$\boldsymbol{\nabla} \wedge \mathbf{B} = 0. \tag{55}$$

At such points, we can introduce a magnetic scalar potential Ω via

$$\mathbf{B} = -\boldsymbol{\nabla}\Omega. \tag{56}$$

As $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, we have, as in electrostatics,

$$\nabla^2 \Omega = 0, \tag{57}$$

Laplace's equation, of which we know various solutions. The one of relevance here is the analogue of the one for the potential of the electric dipole of moment **p** given as (58) of Sec. 2.1, namely

$$\Omega = -\frac{\mu_0}{4\pi} \mathbf{m} \cdot \boldsymbol{\nabla} \frac{1}{r}. \tag{58}$$

¿From this, we can calculate **B** using (56), and caste the result into the form $\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}$, where **A** is given by (52).

[Given the vector potential (52), we calculate the magnetic field **B**. First evaluate

$$\boldsymbol{\nabla} \wedge \left(\frac{\mathbf{m} \wedge \mathbf{r}}{r^3}\right) = -\boldsymbol{\nabla} \wedge \left(\mathbf{m} \wedge \boldsymbol{\nabla} \frac{1}{r}\right) = -\boldsymbol{\nabla} \wedge (\mathbf{m} \wedge \mathbf{v}), \tag{59}$$

with a temporary abbreviation $\mathbf{v} = \boldsymbol{\nabla} \frac{1}{r}$. Second

$$\begin{aligned} [\boldsymbol{\nabla} \wedge (\mathbf{m} \wedge \mathbf{v})]_k &= \epsilon_{kij} \partial_i \epsilon_{jpq} m_p v_q = (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip}) \partial_i m_p v_q \\ &= \partial_i m_k v_i - \partial_i m_i v_k = m_k \boldsymbol{\nabla} \cdot \mathbf{v} - \mathbf{m} \cdot \boldsymbol{\nabla} v_k \\ &= [\mathbf{m}\, \boldsymbol{\nabla}^2 \tfrac{1}{r} - (\mathbf{m} \cdot \boldsymbol{\nabla}) \boldsymbol{\nabla} \tfrac{1}{r}]_k. \end{aligned} \tag{60}$$

Since we are dealing with non-zero (actually large) r, we can certainly use $\boldsymbol{\nabla}^2 \frac{1}{r} = 0$, so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} (\mathbf{m} \cdot \boldsymbol{\nabla}) \boldsymbol{\nabla} \frac{1}{r} = \frac{\mu_0}{4\pi} \boldsymbol{\nabla} (\mathbf{m} \cdot \boldsymbol{\nabla}) \frac{1}{r} = -\boldsymbol{\nabla} \left[-\frac{\mu_0}{4\pi} (\mathbf{m} \cdot \boldsymbol{\nabla}) \frac{1}{r} \right] = -\boldsymbol{\nabla}\Omega, \tag{61}$$

where Ω is the scalar potential (56).]

We have found a certain analogy between the magnetic dipole moment **m** that determines the leading large r behaviour of the vector potential $\mathbf{A}(\mathbf{r})$ of a current distribution localised near the origin of space, and the electric dipole. Since the 'dipole term' gives the leading contribution to $\mathbf{A}(\mathbf{r})$, this underlines the fact that magnetism has no analogue of the point charge: as far as is known at present magnetic monopoles do not exist. But the small current loop provides a physical realisation of a magnetic dipole.

[**A brief informal aside**

If one considers atoms which possess spin about some axis, one can see roughly that the motion of their electrons approximate to current loops with moments parallel to this axis. If the spin axes of all the atoms, in some material made up of such atoms, can be made to line up parallel, then the material acquires a macroscopic magnetic moment. This offers a little insight into the origin of permanent or (ferro-)magnetism.]

3.7 Forces and couples

¿From (17) of Sec. 1.3, we find that the force, felt by an element of volume δV of medium in which the current density is $\mathbf{J}(\mathbf{r})$, because of a given magnetic field $\mathbf{B}(\mathbf{r})$ is

$$\begin{aligned} \delta\mathbf{F}(\mathbf{r}) &= [\mathbf{J}(\mathbf{r})\delta V]\wedge\mathbf{B}(\mathbf{r}) \quad \text{or} \\ &= I\delta\mathbf{r}\wedge\mathbf{B}(\mathbf{r}). \end{aligned} \tag{62}$$

for an element $\delta\mathbf{r}$ of thin conducting wire carrying current I.

For a loop C_1, carrying current I_1, in a given field $\mathbf{B}$, the total force and couple felt are

$$\mathbf{F} = \oint_{C_1} I_1\mathbf{dr}_1\wedge\mathbf{B}(\mathbf{r}_1) \tag{63}$$

$$\mathbf{G} = \oint_{C_1} \mathbf{r}_1\wedge[I_1\mathbf{dr}_1\wedge\mathbf{B}(\mathbf{r}_1)]\,. \tag{64}$$

It can be shown (see problem set 2) that, for C_1 a current loop of moment $\mathbf{m} = IS\mathbf{n}$ in a uniform magnetic field,

$$\mathbf{F} = 0, \quad \mathbf{G} = \mathbf{m}\wedge\mathbf{B}. \tag{65}$$

If $\mathbf{B}_2(\mathbf{r})$ is the field due to a current loop C_2 carrying current I_2

$$\mathbf{B}_2(\mathbf{r}) = \frac{\mu_0}{4\pi}\oint_{C_2}\frac{I_2\mathbf{dr}_2\wedge(\mathbf{r}-\mathbf{r}_2)}{|\mathbf{r}-\mathbf{r}_2|^3}, \tag{66}$$

then the force $\mathbf{F}_{12}$, exerted on loop C_1 by (the magnetic field due to the current in) the loop C_2, is

$$\mathbf{F}_{12} = \oint_{C_1} I_1\mathbf{dr}_1\wedge\mathbf{B}_2(\mathbf{r}_1) = \frac{\mu_0}{4\pi}I_1I_2\oint_{C_1}\oint_{C_2}\mathbf{dr}_1\wedge(\mathbf{dr}_2\wedge\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}). \tag{67}$$

It is not obvious, but true, that (67) is compatible with Newton's third law. Proof, which requires the application of Stokes's theorem, is asked for in Problem set 2.

Example: parallel wires

Suppose $C_{1,2}$ are infinite wires carrying currents $I_{1,2}$, the former along the x-axis, the latter parallel to it and through $(0, 0, a)$. Use Cartesian coordinates.

Consider the element $I_1\mathbf{dr}_1 = I_1\, dx\mathbf{i}$ at the origin. The force exerted on it by C_2 is

$$\begin{aligned} \mathbf{dF}_1 &= I_1\, dx\mathbf{i}\wedge\mathbf{B}_2(0), \quad \mathbf{B}_2(0) = \frac{\mu_0 I_2}{2\pi a}\,\mathbf{j} \\ &= \frac{\mu_0}{2\pi a} I_1 I_2 \mathbf{k} dx. \end{aligned} \tag{68}$$

This uses the result (38) derived in example a) of Sec. 3.3. It follows that the force per unit length felt by C_1 due to C_2 is

$$\mathbf{F} = \frac{\mu_0}{2\pi a} I_1 I_2 \mathbf{k}. \tag{69}$$

This is a force of attraction for I_1, I_2 of the same sign.

3.8 Proof of (42)

Let S be a surface with unit normal $\mathbf{n}$ which separates regions $V_\pm$ of space, with $\mathbf{n}$ pointing from S into V_+. Let $\mathbf{B}_\pm$ be the magnetic fields in $V_\pm$, and let current $\mathbf{s}$ per unit length flow in S.

Consider an area A of S sufficiently small to be considered plane. Apply Ampére's law

$$\int_C \mathbf{B}.\mathbf{dr} = \mu_0 I \tag{70}$$

to the plane needle shaped contour C shown, in the limit $\delta \to 0$ for finite (small) l. l should be small enough for variation of $\mathbf{B}_\pm$ and $\mathbf{s}$ across it, but non-zero. Also the plane of the needle C contains $\mathbf{n}$, but its orientation, *i.e.* the direction of the normal $\mathbf{b}$, $|\mathbf{b}| = 1$ to the plane, is arbitrary. The direction $\mathbf{t}$, $|\mathbf{t}| = 1$ of the needle is then chosen so that $\mathbf{b}, \mathbf{t}, \mathbf{n}$ form an orthonormal right-handed triad. We get

$$\begin{aligned} (-\mathbf{B}_+ + \mathbf{B}_-).\mathbf{t}l &= \mu_0 \mathbf{s}.\mathbf{b}l \\ (-\mathbf{B}_+ + \mathbf{B}_-).\mathbf{n}\wedge\mathbf{b} &= \mu_0 \mathbf{s}.\mathbf{b} \\ \left[(-\mathbf{B}_+ + \mathbf{B}_-)\wedge\mathbf{n} - \mu_0\mathbf{s}\right].\mathbf{b} &= 0. \end{aligned} \tag{71}$$

Since $\mathbf{b}$ was taken to be in an arbitrary direction in S it follows that

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0 \mathbf{s}. \tag{72}$$

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4 Electromagnetic induction

Recall the paragraph from Sec. 1.5, repeated here: The Maxwell equation

$$\nabla\wedge\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \tag{1}$$

implies

$$\int_C \mathbf{E}\cdot\mathbf{dr}=\int_S \nabla\wedge\mathbf{E}\cdot\mathbf{dS}=-\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{dS}=-\frac{d}{dt}\int_S\mathbf{B}\cdot\mathbf{dS}, \tag{2}$$

by applying Stokes's theorem to a fixed curve $C=\partial S$ bounding a fixed open surface S. If we define the electromotive force (or electromotance) acting in C by

$$\mathcal{E}=\int_C\mathbf{E}\cdot\mathbf{dr}, \tag{3}$$

and the flux of $\mathbf{B}$ through (the open) surface S by

$$\Phi=\int_S\mathbf{B}\cdot\mathbf{dS}, \tag{4}$$

then we get Faraday's Law of induction

$$\mathcal{E}=-\frac{d\Phi}{dt}. \tag{5}$$

This will be studied now.

In chapter two we studied electric fields $\mathbf{E}$ such that

$$\nabla\wedge\mathbf{E}=0,\quad \int_C\mathbf{E}\cdot\mathbf{dr}=0, \tag{6}$$

called conservative, since, in virtue of $\nabla\wedge\mathbf{E}=0$, there exists the electrostatic potential ϕ such that $\mathbf{E}=-\nabla\phi$. In chapter two it was assumed implicitly that there were no magnetic fields in the discussion, but it could equally have been assumed that we were dealing with non-conducting material (*e.g.* the vacuum or free space) and time-independent magnetic fields, since the latter would then be entirely uncoupled from the electrostatics.

Here we study time-dependent magnetic fields and the the non-conservative electric fields that accompany them. The latter may give rise to non-zero electromotive forces (or electromotances, or EMFs for short), and hence cause current flow.

We first make this study in the (pre-Maxwellian) approximation to the full Maxwell theory, in which

$$\nabla\wedge\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0,\quad \nabla\wedge\mathbf{B}=\mu_0\mathbf{J}. \tag{7}$$

In other words, we neglect the displacement current, even though it was seen in Sec. 1.4 to be an essential ingredient of a consistent theory. It can be shown however that this is justified in the practically significant context in which there are alternating currents of low enough frequency flowing in media of high enough conductivity.

We look first at simple situations wherein it can be seen how time-dependent magnetic fields can produce non-zero EMFs and cause current flow.

4.1 Simple examples

If we talk about a bar magnet, we mean a piece of material in which the atomic spins, essentially small current loops, are all lined up, to produce a macroscopic magnetic moment, as in the left hand diagram.

A bar magnet moved relative to a fixed circuit, with a galvanometer, causes a current to flow in the circuit, as motion of the galvanometer needle indicates. There is current flow iff there bar magnet moves.

Suppose the bar magnet in this context is replaced by a second circuit, with a battery, and a current flowing, and with a movable part. Iff there is motion of the latter relative to the first circuit, then will the galvanometer record a current flow. (The magnetic field of the current in the first circuit does the business just as well as did the bar magnet.)

The permanent magnet set-up in the diagram produces magnetic fields in the curved slots in which the loop of a circuit can rotate. If the loop is made to rotate steadily, then an alternating current flows in the circuit. This is the principle of the (AC) generator.

The same set-up can be used to illustrate the principle of the electric motor. Across each slot there is a north and a south pole. Suppose the coil is lying with one side in each slot. When a current is passed through the coil, it flows in opposite directions on the two sides, so these feel equal and opposite forces. In other words a couple is being applied to the coil. If the shaft of the coil is free to rotate, the system can be coupled to pulleys or gears and do work.

4.2 Faraday's law of induction

Let C be either

(a) a fixed closed geometrical curve, or

(b) a physical, possibly moving circuit.

Let S be a surface bounded by $C = \partial S$.
Define the flux, of a possibly time-dependent magnetic field $\mathbf{B}$, through S by

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}. \tag{8}$$

Then Faraday's experimental law, valid in both the contexts (a) and (b), with an appropriate definition in each case of the EMF $\mathcal{E}$ in C, is

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{9}$$

In case (a)

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dr} = \int_S \boldsymbol{\nabla}\wedge \mathbf{E} \cdot \mathbf{dS} \tag{10}$$

and

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}. \tag{11}$$

Consistency of (9 –11) is now assured by means of the Maxwell equation (1), assumed true in general.

For case (b), consider the case of a physical circuit moving with velocity $\mathbf{v}$, possibly dependent on position and time, but $v \ll c$, in a time-dependent magnetic field $\mathbf{B}$.

The force on a particle of charge q moving with velocity $\mathbf{v}$ in the magnetic field $\mathbf{B}$, and therefore also in its accompanying electric field $\mathbf{E}$, is given by eq. (16) of Sec. 1.3:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\wedge \mathbf{B}). \tag{12}$$

Hence we define the electromotance or EMF in C as

$$\mathcal{E} = \frac{1}{q}\oint_C \mathbf{F} \cdot \mathbf{dr} = \oint_C (\mathbf{E} + \mathbf{v}\wedge \mathbf{B}) \cdot \mathbf{dr}. \tag{13}$$

We must show that, in context (b), (9) and (13) are compatible with the Maxwell equation (1).

To achieve this, we set out from an expression for $\dfrac{d\Phi}{dt}$

$$\frac{d\Phi}{dt} = \lim_{\delta t \to 0}\left[\frac{1}{\delta t}\left(\int_{S'} \mathbf{B}(\mathbf{r}', t + \delta t) \cdot \mathbf{dS'} - \int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS}\right)\right]. \tag{14}$$

Then we apply the divergence theorem at time $(t + \delta t)$ to the spatial volume V bounded by S, S' and the curved surface Σ swept out by the circuit C as it moved from position S at time t to position S' at $(t + \delta t)$.

$$\begin{aligned} 0 &= \int_V \boldsymbol{\nabla} \cdot \mathbf{B} d\tau \\ &= \int_{S'} \mathbf{B}(\mathbf{r}', t + \delta t) \cdot \mathbf{dS'} - \int_S \mathbf{B}(\mathbf{r}, t + \delta t) \cdot \mathbf{dS} + \oint_C \mathbf{B}(\mathbf{r}, t + \delta t) \cdot (\mathbf{dr}\wedge \mathbf{v}\delta t). \end{aligned} \tag{15}$$

Here, as the right-hand diagram purports to justify, we have used

$$\mathbf{dS} \approx \mathbf{dr} \wedge \mathbf{v} \delta t, \tag{16}$$

on Σ. Since the third term of (15) is proportional to δt and hence already small, we may neglect δt in the arguments of $\mathbf{B}$ and of $\mathbf{v}$ in it, having already neglected the variation of $\mathbf{B}$ and $\mathbf{v}$ with position across Σ.

The second integral in (15) has the Taylor expansion

$$\int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS} + \delta t \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS}. \tag{17}$$

These remarks allow us to write (15) as

$$0 = \int_{S'} \mathbf{B}(\mathbf{r}', t + \delta t) \cdot \mathbf{dS} - \int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS} - \delta t \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS} + \delta t \oint_C \mathbf{dr} \cdot \mathbf{v} \wedge \mathbf{B}(\mathbf{r}, t). \tag{18}$$

Dividing by δt, we see the first two terms in (18) allow us to bring in $\frac{d\Phi}{dt}$ using (14). So we get

$$0 = \frac{d\Phi}{dt} - \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS} + \oint_C \mathbf{dr} \cdot \mathbf{v} \wedge \mathbf{B}(\mathbf{r}, t). \tag{19}$$

The first term here is related by (9) to $\mathcal{E}$, which is defined in the present context by (13). Hence

$$0 = \oint_C \mathbf{E} \cdot \mathbf{dr} + \int_S \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{dS}, \tag{20}$$

the $\mathbf{v}$-dependent terms having cancelled, so that consistency is assured by the Maxwell equation (1), just as in case (a).

The significance of the minus sign in the definition (5) of the EMF reflects Lenz's law, which states that any EMF induced in a circuit by a change of flux through it tends to oppose any EMF (*e.g.* due to a battery) that already exists in the circuit.

4.3 The Faraday experiment

In the set-up shown the crossbar LM can slide with negligible friction parallel to ON. The uniform time independent magnetic field $\mathbf{B} = (0, 0, B)$ points upwards from the plane of the page.

We shall neglect the resistance of the wire $QMN(\mathcal{E}_0)OLP$ The circuit $C = OLMN(\mathcal{E}_0)$ thus has resistance

$$R, \tag{21}$$

i.e. the resistance R of LM. Also, for large B and R, we neglect magnetic fields arising from any current flowing in the system. The initial conditions are

$$x = x_0, \quad \dot{x} = 0, \quad I = I_0 = \frac{\mathcal{E}_0}{R} \quad \text{at} \quad t = 0. \tag{22}$$

The Biot-Savart law tells us that the force $\delta\mathbf{F}$ acting on the element $\delta\mathbf{r} = \delta y\mathbf{j} = \delta y(0,1,0)$ of LM is given by

$$\delta\mathbf{F} = I\delta\mathbf{r}\wedge\mathbf{B} = I\delta y\, B\mathbf{j}\wedge\mathbf{k} = I\delta y\, B\mathbf{i}. \tag{23}$$

So the total force on LM is

$$\mathbf{F} = Ia\, B\mathbf{i}. \tag{24}$$

By Newton's second law, we have

$$m\ddot{x} = IaB. \tag{25}$$

We cannot assume that I is independent of t, so that we are not yet ready to try to solve (25).

When LM is at x, the flux of $\mathbf{B}$ through $\partial S = C$ is

$$\Phi = \int_S \mathbf{B}\cdot\mathbf{dS} = \text{constant} + B(ax), \tag{26}$$

so that the EMF induced in C in the circuit is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Ba\dot{x}. \tag{27}$$

It follows now that the total EMF in the circuit at time t is

$$\mathcal{E}_0 + \mathcal{E} = \mathcal{E}_0 - Ba\dot{x}, \tag{28}$$

and that

$$\mathcal{E}_0 - Ba\dot{x} = IR. \tag{29}$$

Eqs. (25) and (29) enable the time dependence of I and $\dot{x}$ to be calculated. In view of our neglect of various effects, we have a reasonably simple differential equation for $x(t)$

$$m\ddot{x}R = aB(\mathcal{E}_0 - Ba\dot{x}), \tag{30}$$

indeed soluble quite nicely for small t. This solution exhibits what is expected in general, that the induced EMF opposes the battery EMF, and the current in C is reduced. These are two aspects of Lenz's law.

Lenz's law is a special case of more general belief: le Châtelier's principle. This can be stated as follows: a physical system in a steady state reacts by opposing any change imposed on it from outside.

We neglected the magnetic field due to the current induced in C, which opposes the battery produced I_0. But (using the result (38) from Sec. 3.3 a), we see that the field due to the induced current in LM *e.g.* points downwards on the plane of the diagram, and opposes $\mathbf{B}$. This too exemplifies a Lenz view: flux change of one sign produces currents which create flux of the the opposite sign.

4.4 Coil rotating in a fixed magnetic field

Let C be a closed rectangular curve $PQRS$ of area A. Very thin conducting wire is wrapped N times around the curve C with free ends connected to some external circuit.

Suppose C can rotate rigidly about a fixed axis $\mathbf{j} = (0,1,0)$ with angular velocity ω in the presence of a uniform time-independent magnetic field $\mathbf{B} = (0,0,B)$.

When the normal to the coil makes an angle $\theta = \omega t$ to $\mathbf{B}$ as shown, so that $\mathbf{n} = \cos\theta\mathbf{k} + \sin\theta\mathbf{i}$, then the flux of $\mathbf{B}$ through the coil is

$$\int \mathbf{B}\cdot\mathbf{dS} = N\mathbf{B}\cdot\mathbf{n}A = NB\cos\theta A. \tag{31}$$

Hence the EMF induced in the circuit is

$$\mathcal{E} = -\frac{d\Phi}{dt} = NBA\omega\sin\omega t. \tag{32}$$

If the coil has resistance R, then the current induced in the coil is

$$I = \frac{NBA\omega}{R}\sin\omega t. \tag{33}$$

Using (64) of Sec. 3.7, we know that the couple exerted on the circuit by the magnetic field is

$$\mathbf{G} = N\oint_C \mathbf{r}\wedge(I\mathbf{dr}\wedge\mathbf{B}). \tag{34}$$

It can be shown, with the aid of Stoke's theorem, that this can be caste into the form

$$\mathbf{G} = \mathbf{m}\wedge\mathbf{B}, \tag{35}$$

where the magnetic moment of the plane N-loop coil is given, using (54) of Sec. 3.5, by $\mathbf{m} = NIA\mathbf{n}$. Proof of (35) will be attached to the end of chapter 5. Evaluating (35) we find that $\mathbf{G} = -IANB\sin\theta\mathbf{j}$, which, in the spirit of Lenz's law, tends to counter the torque that applies the angular velocity to the coil.

4.5 Inductance and magnetic energy

We will illustrate these concepts by reference to the long solenoid of Sec. 3.3. First we recall the context and some of the results obtained there.

The solenoid has N turns of wire per unit length and length l very large so that end effects can be neglected. It carries current I. It is cylindrical with axis $\mathbf{k} = (0,0,1)$, and cross-sectional area A. The magnetic field due to the current flow is

$$\mathbf{B} = \mu_0 NI\mathbf{k} \tag{36}$$

inside the solenoid and zero outside. The flux of $\mathbf{B}$ through one turn of the solenoid is

$$\mu_0 NIA \tag{37}$$

and through all Nl turns is

$$\Phi = \mu_0 N^2 lIA. \tag{38}$$

This is proportional to I, and we define the (self)-inductance L of the coil via $\Phi = LI$ giving

$$L = \mu_0 N^2 l A = \mu_0 N^2 V, \tag{39}$$

where $V = Al$ is the volume of the solenoid.

Suppose now that the long solenoid, which remains stationary in this discussion, is attached to a battery of EMF $\mathcal{E}_0$, so that the total EMF in the circuit is

$$\mathcal{E}_0 + \mathcal{E}_{induced} = \mathcal{E}_0 - \frac{d\Phi}{dt} = \mathcal{E}_0 - L\dot{I}, \tag{40}$$

and Ohm's law reads

$$\mathcal{E}_0 = IR + L\dot{I}. \tag{41}$$

Now (*cf.* eq. (17) of Sec. 3.1), the work done by the battery in time δt is

$$\delta W = \mathcal{E}_0 I \delta t = RI^2 \delta t + LI\dot{I}\delta t \tag{42}$$

The first term corresponds to Ohmic heat generation. As it is not of immediate interest, we suppose the circuit is of negligible resistance and neglect it. Hence

$$\delta W = LI\dot{I}\delta t = \delta(\frac{1}{2}LI^2). \tag{43}$$

Assuming the magnetic energy W is zero at $t = 0$ when $I = 0$ also, we have

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi. \tag{44}$$

Next we develop the expression (44)

$$W = \frac{1}{2}I\Phi = \frac{1}{2}I\int_S \mathbf{B}.\mathbf{dS} = \frac{1}{2}I\int_S \boldsymbol{\nabla}\wedge\mathbf{A}.\mathbf{dS} = \frac{1}{2}I\oint_C \mathbf{A}.\mathbf{dr}. \tag{45}$$

Hence we can pass in now familiar fashion to the magnetic energy of a continuous distribution of current density $\mathbf{J}$ confined to a finite region region of space near to which we take our origin.

$$W = \frac{1}{2}\int_V \mathbf{J}.\mathbf{A}d\tau, \quad V = \text{all space}. \tag{46}$$

This leads to an important alternative expression for the magnetic energy.

$$W = \frac{1}{2\mu_0}\int_V \mathbf{A}.\boldsymbol{\nabla}\wedge\mathbf{B}d\tau = \frac{1}{2\mu_0}\int_V \left[-\boldsymbol{\nabla}.(\mathbf{A}\wedge\mathbf{B}) + \mathbf{B}.(\boldsymbol{\nabla}\wedge\mathbf{A})\right] d\tau. \tag{47}$$

We may use the divergence theorem on the first term and see that it vanishes provided that (as can be checked) its integrand goes to zero fast enough as r goes to infinity. Now from (47) we get the important result

$$W = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau. \tag{48}$$

It can easily be seen, using (39) for L and (36) for $B = |\mathbf{B}|$, that the two expressions (44) and (48) for the magnetic energy W give the same result in the case of the long solenoid

$$\frac{1}{2}LI^2 = \frac{1}{2}(\mu_0 N^2 V)I^2 = \frac{1}{2\mu_0}B^2 V = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau. \tag{49}$$

5 Maxwell's equations

5.1 A historical paradox

In magnetostatics, the equation

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}, \tag{1}$$

implies $\nabla \cdot \mathbf{J} = 0$. As $\rho = 0$ in magnetostatics, this is compatible with the continuity equation $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$. However naive application of the integral form of (1)

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS}, \tag{2}$$

to the following situation produced a contradiction, one that Maxwell resolved by generalising (1). The 'capacitor' paradox arises by applying (2) to the two surfaces S_1 and S_2 that are bounded by the same curve C. There is a unique answer for the left-side of (2), but the right-side gives different answers $\mu_0 I$ for S_1 and 0 for S_2.

Maxwell proposed that (1) be changed by addition to a term that made it compatible with $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$. This gives rise (in free space or the vacuum) to

$$\nabla \wedge \mathbf{B} = \mu_0 (\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}), \tag{3}$$

as was shown in Sec. 1.4 to be sufficient to achieve consistency.

How does the use of (3) provide resolution of the paradox? There is an electric field only between the plates. So for S_1, lying outside the plates, we have

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu_0 \int_{S_1} \mathbf{J} \cdot \mathbf{dS} = \mu_0 I. \tag{4}$$

Between the plates, where $\mathbf{J} = 0$, we shall *assume* that $\mathbf{E}$ is uniform so that $\mathbf{E} = \dfrac{\sigma}{\epsilon_0}\mathbf{k}$. Hence

$$\begin{aligned} \frac{1}{\mu_0} \oint_C \mathbf{B} \cdot \mathbf{dr} &= \int_{S_2} \mathbf{J} \cdot \mathbf{dS} + \epsilon_0 \int_{S_2} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS} \\ &= 0 + \epsilon_0 \frac{d}{dt} \int_{S_2} \mathbf{E} \cdot \mathbf{dS} \\ &= \frac{d}{dt}(\sigma A) = \frac{dQ}{dt} = I, \end{aligned} \tag{5}$$

as required for consistency. Here σ is the charge density and A is the plate area. The assumption that $\mathbf{E}$ is uniform is a crude one. It can be avoided by doing a somewhat harder calculation along lines similar to those followed above.

5.2 Energy and energy transport

Recall the field energy formulas

$$W_{el} = \frac{\epsilon_0}{2}\int_V \mathbf{E}^2 d\tau, \quad W_{mag} = \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau, \tag{6}$$

and the expression for the rate of Ohmic heat loss *i.e.* the rate of dissipation of electromagnetic energy as heat

$$\int \mathbf{J}\cdot\mathbf{E}d\tau. \tag{7}$$

The Maxwell equation (3) implies

$$\frac{1}{\mu_0}\mathbf{E}\cdot\boldsymbol{\nabla}\wedge\mathbf{B} = \mathbf{E}\cdot\mathbf{J} + \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}. \tag{8}$$

Now

$$\mathbf{E}\cdot\boldsymbol{\nabla}\wedge\mathbf{B} = -\boldsymbol{\nabla}\cdot(\mathbf{E}\wedge\mathbf{B}) + \mathbf{B}\cdot\boldsymbol{\nabla}\wedge\mathbf{E} \quad = -\boldsymbol{\nabla}\cdot(\mathbf{E}\wedge\mathbf{B}) - \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}. \tag{9}$$

Hence

$$\begin{aligned} -\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} - \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} &= \mathbf{J}\cdot\mathbf{E} + \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot\mathbf{E}\wedge\mathbf{B} \\ -\frac{d}{dt}\left[\frac{\epsilon_0}{2}\int_V \mathbf{E}^2 d\tau + \frac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau\right] &= \int_V \mathbf{J}\cdot\mathbf{E}d\tau + \frac{1}{\mu_0}\int_S \mathbf{n}\cdot\mathbf{E}\wedge\mathbf{B}dS. \end{aligned} \tag{10}$$

For the last term the divergence theorem has been applied to a fixed volume V of space bounded by a surface S. The left side here is the rate of decrease of the total field energy $W = W_{el}+W_{mag}$. The first term on the right side of (10) represents the rate of loss of energy as Ohmic heat, while the second term there is the rate of energy transport out of V through the surface S.

For the latter, define the Poynting vector $\mathbf{S}$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\wedge\mathbf{B}. \tag{11}$$

The flux of $\mathbf{S}$ through a closed surface S, with outward unit normal $\mathbf{n}$, is

$$\int_S \mathbf{S}\cdot\mathbf{n}dS. \tag{12}$$

This is the flux of electromagnetic energy being transported through S out of V.

Eq. (10) thus gives a generally applicable account of energy changes in a conducting medium.

5.3 Decay of charge density in a medium of high conductivity σ

In Sec. 1.4, we derived the continuity equation

$$\boldsymbol{\nabla}\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \tag{13}$$

from Maxwell's equations. In a conducting medium of conductivity σ we have $\mathbf{J} = \sigma\mathbf{E}$ and hence

$$\boldsymbol{\nabla}\cdot\mathbf{J} = \sigma\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\sigma}{\epsilon_0}\rho.$$

Now (13) implies

$$\frac{\sigma}{\epsilon_0}\rho + \frac{\partial\rho}{\partial t} = 0, \quad \text{and hence} \quad \rho(t) = \rho(0)\exp(-\frac{t}{\tau}), \tag{14}$$

where $\tau = \frac{\epsilon_0}{\sigma}$ is the *relaxation time* of the medium. For copper or silver $\tau \approx 10^{-18} sec.$, so that any charge density present – for whatever reason – in the medium at the initial time $t = 0$ quickly goes to zero. It may be expected to flow to the surface of the medium. For a perfect conductor, for which σ is infinite, we have $\rho(t) = 0$ at all times, as has been discussed above.

5.4 Plane wave solutions of Maxwell's equations

We here deal with the vacuum or free-space, *i.e.* $\rho = 0, \quad \mathbf{J} = 0$. We begin as simply as possible by seeking a solution describing a wave propagating in the z-direction with fields that do not depend on x or y.

Looking at $\nabla \cdot \mathbf{E} = 0$, we find that E_z is constant. Looking for linearly polarised solutions of wave type, we put $E_z = 0$, and *assume* we can, for all t, chose axes so that

$$\mathbf{E} = (E, 0, 0). \tag{15}$$

Sec. 1.6 proves that the components of $\mathbf{E}$ each satisfy a wave equation. Hence

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}. \tag{16}$$

The solution of such a wave equation can be written as

$$E(z,t) = f(z - ct) + g(z + ct). \tag{17}$$

The f and g terms here describe waves moving respectively in the positive and negative z-directions with speed c. In particular, we can consider a monochromatic wave, one with a fixed angular frequency ω, in which

$$E = E_0 \, \exp i\omega(\frac{z}{c} - t) = E_0 \, \exp i(kz - \omega t) \tag{18}$$

where we have defined the wave-number k by

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}, \quad . \tag{19}$$

Here $\nu\lambda = \frac{\omega}{2\pi}\lambda = c$ relates the wavelength λ and frequency of the wave in a standard way to other wave variables. Finally, note that the use of complex exponentials is very convenient, but the physical fields must always be identified by taking real parts.

What about the magnetic fields? Looking at $\nabla \cdot \mathbf{B} = 0$, we find that B_z is constant, and take it to be zero. It is natural to assume that $\mathbf{B}$ is of the form

$$\mathbf{B} = \mathbf{B}_0 \exp i(kz - \omega t). \tag{20}$$

Then in $\nabla \wedge \mathbf{E}$ the only no-zero entry is $\frac{\partial E_x}{\partial z}$ so that we have $\mathbf{B}_0 = (0, B_0, 0)$, and hence, from

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{21}$$

we get

$$ikE_0 - i\omega B_0 = 0, \quad B_0 = \frac{E_0}{c}. \tag{22}$$

So our wave solution of Maxwell's equations is

$$\mathbf{E} = (E_0, 0, 0) \, \exp i(kz - \omega t), \quad \mathbf{B} = \frac{1}{c}(0, E_0, 0) \, \exp i(kz - \omega t). \tag{23}$$

It should be checked that (23) satisfies also (the zero current density version of) the Maxwell equation (3), although our use of the fact that each component of $\mathbf{E}$ satisfies a wave equation guarantees it. Thus the simplifying assumptions we have made have led us to the valid and simple wave solution (23) of Maxwell's equations. We could similarly have adopted a choice of axes such that that $\mathbf{E} = (0, E, 0)$, and reached, as above, the solution

$$\mathbf{E} = (0, E_0, 0) \, \exp i(kz - \omega t), \quad \mathbf{B} = (-\frac{1}{c}E_0, 0, 0) \, \exp i(kz - \omega t). \tag{24}$$

The solutions (23) and (24) are linearly independent, and the general monochromatic wave of frequency ω is obtained as a linear superposition of them, has fields $\mathbf{E}$ and $\mathbf{B}$ that are transverse to the direction of propagation of the wave. Also $\mathbf{E} \cdot \mathbf{B} = 0$.

The solutions (23) and (24) are said to be **linearly polarised**, with **polarisation vectors** $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$, giving the directions, for all t, of their electric fields.

To discuss the transport of energy by the wave (23) obtained above, we require the real parts

$$\mathbf{E} = (E_0, 0, 0)\, \cos(kz - \omega t), \quad \mathbf{B} = (0, \frac{1}{c} E_0, 0)\, \cos(kz - \omega t), \tag{25}$$

so that, using (11), we get

$$\mathbf{S} = \frac{1}{\mu_0} \frac{{E_0}^2}{c} \cos^2(kz - \omega t)\, (0, 0, 1). \tag{26}$$

Thus the rate of energy transport across unit area normal to the direction of propagation of the wave (say at $z = 0$) is

$$|\mathbf{S}| = \frac{1}{\mu_0} \frac{{E_0}^2}{c} \cos^2 \omega t. \tag{27}$$

Averaging over one period, $T = \dfrac{2\pi}{\omega}$, of the wave motion, we get for the average rate of energy transport across unit area

$$\langle |\mathbf{S}| \rangle = \frac{\int_0^T |\mathbf{S}|(t) dt}{\int_0^T dt} = \frac{1}{2\mu_0} \frac{{E_0}^2}{c} = \frac{1}{2} \epsilon_0 c E_0^2, \tag{28}$$

since $\epsilon_0 \mu_0 = c^{-2}$. The energy density w of the wave (25) can be calculated using (105) of Chapter 2 for w_{el} and (48) of Chapter 4 for w_{mag}. Thus

$$w = w_{el} + w_{mag} = \frac{1}{2}(\epsilon_0 + \frac{1}{\mu_0 c^2}) E_0^2 \cos^2(kz - \omega t) = \epsilon_0 E_0^2 \cos^2(kz - \omega t). \tag{29}$$

For the time average of this we have

$$\langle w \rangle = \frac{1}{2} \epsilon_0 E_0^2, \tag{30}$$

and hence

$$\langle |\mathbf{S}| \rangle = c \langle w \rangle. \tag{31}$$

For the simple plane wave (25), it follows that the energy density travels at the speed of light across unit area normal to the wave.

Of course, similar results holds for the wave (24).

If we consider a linearly polarised wave with fields

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \, \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t), \quad \mathbf{B}(\mathbf{r}) = \mathbf{B}_0 \, \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t), \tag{32}$$

where $\mathbf{k}$ the wave-vector, with $|\mathbf{k}| = k$, gives the direction of propagation of the wave, (*i.e.* here $\mathbf{k} \neq \mathbf{e}_z$ and the wave number $k \neq 1$). Then $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ implies $\mathbf{E}_0 \cdot \mathbf{k} = 0$, and likewise $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ implies $\mathbf{B}_0 \cdot \mathbf{k} = 0$, so that both these fields are transverse to the direction of propagation. Also (21) implies

$$i\mathbf{k} \wedge \mathbf{E}_0 - i\omega \mathbf{B}_0 = 0, \tag{33}$$

which gives $\mathbf{B}_0$ in terms of $\mathbf{E}_0$. Further the remaining Maxwell equation $\boldsymbol{\nabla} \wedge \mathbf{B} = \dfrac{1}{c^2} \dfrac{\partial \mathbf{E}}{\partial t}$ implies

$$i\mathbf{k} \wedge \mathbf{B}_0 = -i \frac{\omega}{c^2} \mathbf{E}_0, \tag{34}$$

compatibly with (33) iff

$$k^2 = \frac{\omega^2}{c^2}, \quad \text{giving} \quad k = \frac{\omega}{c}. \tag{35}$$

We have merely reproduced our wave in an arbitrary Cartesian basis.

[**Circularly polarised waves**

Take a solution that is (23) minus i-times-(24), with E_0 real. This has physical fields

$$\mathbf{E} = \mathrm{Re}(E_0, -iE_0, 0)\,\exp i(kz-\omega t), \quad \mathbf{B} = \mathrm{Re}\frac{1}{c}(iE_0, E_0, 0)\,\exp i(kz-\omega t) \quad \text{or} \tag{36}$$

$$\mathbf{E} = E_0(\cos(kz-\omega t), \sin(kz-\omega t), 0) \quad , \quad \mathbf{B} = \frac{E_0}{c}(-\sin(kz-\omega t), \cos(kz-\omega t), 0) \quad \text{or} \tag{37}$$

$$\mathbf{E} = E_0\mathbf{e}_s(kz-\omega t) \quad , \quad \mathbf{B} = \frac{E_0}{c}\mathbf{e}_\phi(kz-\omega t, \tag{38}$$

where $\mathbf{e}_s(\phi)$ and $\mathbf{e}_\phi(\phi)$ are the unit vectors of cylindrical polar coordinates (s, ϕ, z) with the z-axis in the direction of propagation of the wave. The wave (38) is said to be (positively) circularly polarised. A wave of negative circular polarisation linearly independent of this can be constructed, using (23) plus i-times-(24) with E_0 real, but we do not need the details contained in this parenthesis].

5.5 Boundary conditions

It seems there is going to be time to cover this in lectures. Sec. 1.7 should perhaps be reviewed at this point.

Suppose a surface S carries either a charge density σ per unit area, or a surface current s per unit length. Let the unit normal $\mathbf{n}$ to S point from the negative $(-)$ to the positive $(+)$ side of S.

We proved in Sec. 2.2, the discontinuity formula

$$\mathbf{n}.\mathbf{E}|_-^+ = \frac{1}{\epsilon_0}\sigma, \tag{39}$$

and in Sec. 3.8, that

$$\mathbf{n}\wedge\mathbf{B}|_-^+ = \mu_0\mathbf{s}. \tag{40}$$

It should be clear that the proofs can be applied to deriving

$$\mathbf{n}.\mathbf{B}|_-^+ = 0, \tag{41}$$

and

$$\mathbf{n}\wedge\mathbf{E}|_-^+ = 0. \tag{42}$$

As an aid to remembering these results, we noted in Sec. 1.7, their exact correspondence with Maxwell's equations themselves.

Note that $\mathbf{n}.\mathbf{v}$ and $\mathbf{n}\wedge\mathbf{v}$ give the **normal** and **tangential** components of any vector $\mathbf{v}$. It is obvious that the tangential component satisfies $\mathbf{n}.(\mathbf{n}\wedge\mathbf{v}) = 0$.

Consider then a perfect conductor $\mathcal{C}$ with surface S and normal $\mathbf{n}$ pointing into the conducting medium, in which $\mathbf{E} = 0$ and $\mathbf{B} = 0$. Then the boundary conditions just inside the free space (negative) side of S demand the vanishing of the normal component of $\mathbf{B}$ and of the tangential component of $\mathbf{E}$. This follows (41,42). Eqs. (39,40) are usually subsequently used to *calculate* σ and s for S.

5.6 Reflection at the surface of a perfect conductor

We consider a monochromatic wave (23) propagating in the z-direction from the half-space $z < 0$, towards perfectly conducting material in $z > 0$, whose surface is the plane $z = 0$. In fact the solution of Maxwell's equations plus the boundary conditions (BC) on $z = 0$ will comprise not only an incident wave but also (at least) a suitably matched reflected wave. The fields of the former will have argument $(kz - \omega t)$, where $kc = \omega$, while those of the latter (moving in the negative z-direction) are $(-kz - \omega t)$. All fields in the problem have the same t-dependence $\propto e^{-i\omega t}$.

We know that the fields **E** and **B** are zero inside perfectly conducting media, it therefore follows the BC are: tangential **E** and normal **B** are zero at $z = 0$. For the wave (23) this just means that $E_x = 0$ at $z = 0$. Thus for the electric fields of the incident and reflected parts of our total wave solution of Maxwell's equations, we take

$$\mathbf{E}_{inc} = (E_0, 0, 0)\,\exp i(kz - \omega t), \quad \mathbf{E}_{ref} = (-E_0, 0, 0)\,\exp i(-kz - \omega t), \tag{43}$$

since their superposition

$$\mathbf{E} = \mathbf{E}_{inc} + \mathbf{E}_{ref}, \tag{44}$$

by construction, gives $E_x = 0$ at $z = 0$. The corresponding magnetic fields are $\mathbf{B} = \mathbf{B}_{inc} + \mathbf{B}_{ref}$ with

$$\mathbf{B}_{inc} = \frac{1}{c}(0, E_0, 0)\,\exp i(kz - \omega t), \quad \mathbf{B}_{ref} = \frac{1}{c}(0, E_0, 0)\,\exp i(-kz - \omega t). \tag{45}$$

We see from this that **B** does have a non-zero tangential component at $z = 0$, namely

$$\mathbf{B} = 2\frac{1}{c}(0, E_0, 0)\,e^{-i\omega t}. \tag{46}$$

But this just tells us that a surface current **s** necessarily accompanies the fields **E** and **B** in a consistent solution of Maxwell's equations and boundary conditions.

Recalling the formula (42) of chapter three for **s**

$$\mathbf{n}\wedge\mathbf{B}|_{-}^{+} = \mu_0\mathbf{s}, \tag{47}$$

we obtain

$$\mu_0\mathbf{s} = -\,\mathbf{n}\wedge\mathbf{B}|_{-} = \frac{2E_0}{c}e^{-i\omega t}\,(1, 0, 0). \tag{48}$$

5.7 The historical paradox revisited

We return to the topic of Sec. 5.1, to provide a treatment which does not make the (crude) assumption that the the electric field **E** beteween the plates is uniform. Assume the plates are circular of radius a, and neglect edge effects. Use cylindrical polars (s, ϕ, z).

We shall treat the case in which

$$\mathbf{E} = E_z(s)\mathbf{k}\exp(-i\omega t), \quad \mathbf{B} = B_\phi(s)\mathbf{e}_\phi\exp(-i\omega t). \tag{49}$$

The Maxwell equation $\nabla\wedge\mathbf{E} + \dfrac{\partial\mathbf{B}}{\partial t} = 0$ has only got a non-trivial $\mathbf{e}_\phi$ component, which gives

$$-\frac{\partial E_z}{\partial s} + (-i\omega)B_\phi = 0. \tag{50}$$

The Maxwell equation

$$\nabla\wedge\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{51}$$

between the plates, where $\mathbf{J} = 0$, has only got a non-trivial z component

$$\frac{1}{s}\frac{\partial}{\partial s}(sB_\phi) = -i\frac{\omega}{c^2}E_z \quad \text{using} \quad \epsilon_0\mu_0 = c^{-2}. \tag{52}$$

Substituting for B_ϕ from (50) into (52), we find

$$\frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial E_z}{\partial s}) + \frac{\omega^2}{c^2}E_z = 0. \tag{53}$$

We set $k = \frac{\omega}{c}$, and recognize (53) as the equation satisfied by the Bessel function $J_0(ks)$. Hence, we write

$$E_z = \alpha J_0(ks), \quad B_\phi = i\frac{1}{\omega}\frac{\partial E_z}{\partial s} = i\frac{\alpha}{\omega}\frac{\partial J_0(ks)}{\partial s}, \tag{54}$$

where α is a constant.

The surface charge density on the the lower plate is

$$\sigma = \epsilon_0 \mathbf{k}\cdot\mathbf{E}|_-^+ = \epsilon_0 \alpha J_0(ks)\exp(-i\omega t), \quad 0 \le s \le a. \tag{55}$$

We now show that the integral form of (51) can be applied consistently to $\oint_C \mathbf{B}\cdot\mathbf{dr}$ whether or not the surface $S, \partial C = S$, chosen passes between the plates or not. Let C be the circumference of the lower plate, S_2 the lower plate itself, and S_1 a surface bounded by C but lying entirely outside the region between the plates and so pierced by the current I. As before, for S_1 $\oint_C \mathbf{B}\cdot\mathbf{dr} = \mu_0 I$. For S_2, on the other hand, we have

$$\begin{aligned}
\mu_0 I &= \mu_0\frac{dQ}{dt} = \mu_0\frac{d}{dt}\int_{S_2}\sigma dS \\
&= 2\pi\mu_0\frac{d}{dt}\int_0^a s\sigma ds \\
&= 2\pi\mu_0(-i\omega)\exp(-i\omega t)\int_0^a s\epsilon_0\alpha J_0(ks)ds \\
&= -2\pi i\frac{1}{\omega}\frac{\omega^2}{c^2}\exp(-i\omega t)\int_0^a s\alpha J_0(ks)ds \\
&= 2\pi i\frac{\alpha}{\omega}\exp(-i\omega t)\int_0^a(-k^2 sJ_0(ks))ds \\
&= 2\pi i\frac{\alpha}{\omega}\exp(-i\omega t)\int_0^a\frac{\partial}{\partial s}(s\frac{\partial J_0(ks)}{\partial s})ds \\
&= 2\pi i\frac{\alpha}{\omega}\exp(-i\omega t)a\frac{\partial J_0(ks)}{\partial s}\Big|_{s=a} = 2\pi a B_\phi(a)\exp(-i\omega t) = \oint_C \mathbf{B}\cdot\mathbf{dr}, \qquad (56)
\end{aligned}$$

as required. The third line here uses (55), the fourth $\epsilon_0\mu_0 = c^{-2}$, the fifth $k = \omega/c$, the sixth Bessel's equation, the seventh (54) for B_ϕ.

5.8 Addition to Sec. 5.6

¿From (43) and (44), we see that the physical field is the real part of $\mathbf{E}$, *i.e.* for E_0 real

$$(\mathbf{E}_{phys})_x = 2E_0 \sin kz \, \sin \omega t, \tag{57}$$

and similarly

$$(\mathbf{B}_{phys})_x = (2/c)E_0 \cos kz \, \cos \omega t. \tag{58}$$

Since the magnitude $|\mathbf{S}|$ of the Poynting vector is proportional to $\sin\omega t \, \cos\omega t$, its mean value over one period of the wave motion is zero (which makes good sense?). (48) implies that the physical surface current $\mathbf{s}$ is given by

$$\mu_0 \mathbf{s} = (2/c) \cos \omega t. \tag{59}$$

We may now use (52) of Sec. 1.7, to calculate the force $\mathbf{f}$ per unit area exerted on the surface $z = 0$ of the conducting medium. It is

$$\mathbf{f} = \frac{1}{2} s_y (\mathbf{B}_{phys})_y = \frac{1}{2\mu_0} \frac{4}{c^2} {E_0}^2 \cos^2 \omega t. \tag{60}$$

Hence the mean force per unit area is

$$\langle f \rangle = \epsilon_0 {E_0}^2, \tag{61}$$

using the result $\langle \cos^2 \omega t \rangle = \frac{1}{2}$. Since the force is normal to the surface, (61) gives the mean pressure (**radiation pressure**) at the surface.

5.9 Proof of the result $\mathbf{G} = \mathbf{m} \wedge \mathbf{B}$

Refer to Sec. 3.7, **Force and couples**, and supply the proof that the couple exerted by a uniform magnetic field $\mathbf{B}$ on a plane current loop, of area A, unit normal $\mathbf{n}$, carrying current I, is given by (65) there, *i.e.*

$$\mathbf{G} = \mathbf{m} \wedge \mathbf{B}, \quad \mathbf{m} = IA\mathbf{n}. \tag{62}$$

This was also quoted as (35) of Sec. 4.4, and used there. Letting $\mathbf{c}$ be an arbitrary constant vector, we have

$$\begin{aligned} \mathbf{c} \cdot \mathbf{G} &= \mathbf{c} \cdot \oint_C \mathbf{r} \wedge (I \mathbf{dr} \wedge \mathbf{B}) = I \oint_C \mathbf{c} \cdot (\mathbf{r} \cdot \mathbf{B} \, \mathbf{dr} - \mathbf{r} \cdot \mathbf{dr} \, \mathbf{B}) \\ &= I \oint_C [\mathbf{c} \cdot (\mathbf{r} \cdot \mathbf{B} \mathbf{dr}) - (\mathbf{c} \cdot \mathbf{B})(\mathbf{r} \cdot \mathbf{dr})] \, . \end{aligned} \tag{63}$$

We now apply Stokes's theorem to each of the terms of (63). For the second term we have

$$\oint_C \mathbf{r} \cdot \mathbf{dr} = \int_S \mathbf{n} \cdot (\boldsymbol{\nabla} \wedge \mathbf{r}) dS = 0. \tag{64}$$

For the first term, moving a scalar product in an allowed way, we have

$$I \oint_C (\mathbf{r} \cdot \mathbf{B} \, \mathbf{c}) \cdot \mathbf{dr} = I \int_S \mathbf{n} \cdot \boldsymbol{\nabla} \wedge (\mathbf{r} \cdot \mathbf{B} \, \mathbf{c}) \, dS = I \int_S \mathbf{n} \cdot \mathbf{B} \wedge \mathbf{c} \, dS = I(\int_S \mathbf{dS}) \wedge \mathbf{B} \cdot \mathbf{c}. \tag{65}$$

Here we have used the elementary result $\boldsymbol{\nabla}(\mathbf{r} \cdot \mathbf{B}) = \mathbf{B}$, for constant $\mathbf{B}$. We may finally detach $\mathbf{c}$ from (65), and get the required result

$$\mathbf{G} = I(\int_S \mathbf{dS}) \wedge \mathbf{B} = (IA\mathbf{n}) \wedge \mathbf{B} = \mathbf{m} \wedge \mathbf{B}. \tag{66}$$

5.10 List of corrections already inserted into webpage version of the Lecture Notes for IB: Electromagnetism

These have been made without changing the page beginnings and endings of the pages circulated during lectures. They include...

P3: 3 lines below (13) ... which, for qv positive ...
P4: (NB) line below (22) ...flux of $\mathbf{J}$ out of...
P9: last line ... (69) satisfies Poisson's equation for ...
P11: (NB) c) third line now begins: with ends at $z = h$ and $z = -h$...
next line ... for some $E = E(h)$ ...
end of (81) $E = \dfrac{1}{2\epsilon_0}\sigma, \qquad$ indept of h.
P12: f) ... σ at end of first sentence changed to ρ.
P13: line below (91) ... $= -\dfrac{\partial\phi_1}{\partial z}\mathbf{k} = ...$
P13: Original wording in early paragraphs of Sec. 2.3 seriously inadequate. Look at replaced text
P18: (7) ...$\phi = d - c\theta, \quad c,\ d$ constants ...
P19: (26) $\qquad 0 = \psi + \boldsymbol{\nabla}\chi$...
P20: j_k has been replaced by J_k in (30) and (31).
P23: (49) should read $\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\wedge\mathbf{J}(\mathbf{r})d\tau$.
P26: $\mathbf{J}$ has been replaced by $\mathbf{j}$ in (68). Here $\mathbf{j} = (0, 1, 0)$.
P31: same correction as on P26 twice near (23).
P32: wording of later sentences of Sec. 4.4 improved.
P35: (NB) LHS of (10) corrected to read $-\dfrac{d}{dt}\left[\dfrac{\epsilon_0}{2}\displaystyle\int_V \mathbf{E}^2 d\tau + \dfrac{1}{2\mu_0}\int_V \mathbf{B}^2 d\tau\right]$
P36: wording before (15) and (24) has been improved.
P37: line below (27) ... period, $T = \dfrac{2\pi}{\omega}$... , and
(30) ... $\langle w\rangle = \frac{1}{2}\epsilon_0 {E_0}^2$.
(34) Correct RHS is $-i\dfrac{\omega}{c^2}\mathbf{E}_0$
P39: RHS (48) ... $\dfrac{2E_0}{c}e^{-i\omega t}(1, 0, 0)$

(NB) Error, not in original, editted carelessly into existing web-page text.
P23: At end of (46) ... $\dfrac{\mu_0}{2}Js$ is correct, *i.e.* J is correct here; I is wrong.

Example Sheet 2.
Q6: no π in denominator of expression for B.
Q7: (i): ... force $\mathbf{F}$ per unit volume, (ii) ... $-\boldsymbol{\nabla}p + \mathbf{F} = 0$ ... , and (iii) $p(s) = \frac{1}{4}\mu_0 J^2(s^2 - a^2)$.