PX436: General Relativity

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Contents

Preface

The aim of this course is to give an account of gravity and general relativity, as I understand it. I hope that you will find the subject as fascinating as I did when I was an undergraduate.

The first four weeks of the course will be devoted to an exposition of the theoretical foundations of general relativity leading to a derivation of the Einstein equations. This will be divided in equal parts into a discussion of the stress-energy-momentum tensor – the source of gravitation – and curvature – its influence in the gravitational field. For the remaining six weeks we will study the predictions and implications of the Einstein equations, covering the classical tests – the motion of celestial bodies and gravitational lensing –, stellar collapse and black holes, gravitational waves and cosmology. The topics chosen, and emphasis of presentation, reflect my own knowledge and expertise and lie very much in the tradition of theoretical physics. At the same time, I have tried to promote and give preference to those parts of the theory that either have been tested against experiment and observation, or are the focus of current experiments.

General relativity is a theory of the structure of space and time and as such makes considerable use of (pseudo-)Riemannian geometry. It is lamentable that though geometry is taught at school before one learns calculus, it is essentially ignored in university level physics courses until one meets general relativity. This is to the diminishment of both subjects and makes the presentation of both in a course such as this difficult. It is my conviction that the essential ideas can be gained from the example of surfaces, of widespread relevance to physics in their own right, which have the benefit of being directly visualisable. In addition to emphasising this, I have also tried to adopt a more modern approach than in many textbooks, using intrinsic descriptions in preference to the old-fashioned tensor calculus.

These lecture notes have been written in an informal and colloquial style. At times the presentation is quite detailed; at others I have taken liberties. As a general rule, there is more material in these notes than I will be able to cover in the lectures, although not too much more. What may appear in the exam should be inferred from the official syllabus and what I say in lectures rather than the fullness of these lecture notes. In preparing these notes I discovered that images taken by NASA are generally not copyrighted and can be used freely for personal or educational purposes. This is a most wonderful thing and I am grateful to be able to use their inspiring images to illustrate these lecture notes. Their image galleries can be found at this url `http://www.nasa.gov/multimedia/imagegallery/index.html`.

I am grateful to the students who took the course in previous years for their feedback and suggestions, and to my colleagues who read my notes and spotted mistakes or points for improvement, especially Nick d'Ambrumenil and George Rowlands.

I shamelessly copy Newton in his Principia (1687): *I heartily beg that what I have here done may be read with candour; and that the defects in a subject so difficult be not so much reprehended as kindly supplied, and investigated by new endeavours of my readers.*

Books and other reading

There are lots of good books. You are encouraged to read as many as you can. Some with particular relevance to the course are listed below.

1. A. Einstein, *The Meaning of Relativity*, Princeton University Press, Princeton, 1945.

   In 1921 Einstein visited the United States and gave a series of lectures at Princeton on his theory of relativity. This text is a reissue of those lectures. Einstein's theory of relativity is universally viewed as amongst the most beautiful and remarkable scientific achievements in recorded history. It is a joy to be able to read his own presentation of it.

   An electronic copy of the book is freely available at this url `https://www.gutenberg.org/files/36276/36276-pdf.pdf`.

2. L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields: Volume 2 of the Course of Theoretical Physics*, 4th edition, Butterworth-Heinemann, Oxford, 1975.

   A wonderful account with the highest standard of theoretical physics. It is moderately dated and gives a traditional 'tensor calculus' approach to the differential geometry, but thereafter the treatment of physical consequences of the theory is excellent. If you consider yourself a theorist, then this is the book I would like to recommend.

3. M.P. Hobson, G.P. Efstathiou, and A.N. Lasenby, *General Relativity: An Introduction for Physicists*, Cambridge University Press, Cambridge, 2006.

   This was the previous lecturer, Elizabeth Stanway's, recommended textbook. Being written relatively recently it is modern in style and up to date. It should go without saying that it can still be recommended highly.

4. J.B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*, Addison Wesley, 2002.

   Hartle's text is one of the more recent ones and it shows in the style of presentation and how up to date the applications and examples are. I am sure some of you will really like it.

5. B. Schutz, *A First Course in General Relativity*, 2nd edition, Cambridge University Press, Cambridge, 2009.

   A standard reference for undergraduate courses in general relativity that is often enthusiastically recommended.

6. C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, W.H. Freeman, 1973.

   The big black coffee table, as it is affectionately known, is perhaps the most widely recommended text at advanced level. Written by three who did so much to advance general relativity, and geometrical thinking in particular, it is a classic. Its major drawback, however, is its length; 1279 pages.

7. R.M. Wald, *General Relativity*, Chicago University Press, Chicago, 1984.

   One of the most widely recommended texts for graduate courses in the US, this is the book that I first read when learning general relativity. It has a more modern style than most, which is what attracted me, adopting an 'abstract index notation' that I believe was advocated by Penrose.

8. W. Rindler, *Relativity: Special, General, and Cosmological*, 2nd edition, Oxford University Press, Oxford, 2006.

   Another text that I read when first learning general relativity. As I recall, it is clear, pedagogical and insightful.

9. S.W. Hawking and G.F.R Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge, 1973.

   Hawking & Ellis is a mixture; certainly it is not the right book to learn the subject from for the first time, but it is not intended for that purpose. There are many things it does not include, for instance any account of the linearised theory and gravitational waves. However, the analysis of the global structure of the main exact solutions is the benchmark for others to aspire to, as is the discussion of the singularity theorems.

10. J. Stewart, *Advanced General Relativity*, Cambridge University Press, Cambridge, 1993.

    For many years John Stewart gave the Part III lectures on general relativity at Cambridge and this book is based on those. I was fortunate enough to attend John's lectures during my brief stint as a relativist; they were excellent. The book can be recommended with one significant caveat; for Cambridge students this course is the second time most of them have seen general relativity, there also being a Part II course.

11. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, New York, 1972.

    Weinberg's book gives a treatment that adopts the style and emphasis of a particle theorist writing in the early 1970s. Symmetry, behaviour under transformations and 'particle' concepts are given preference over geometric constructions. Many like this approach and Weinberg's book is frequently highly recommended, but I have never warmed to it.

12. P.A.M. Dirac, *General Theory of Relativity*, Princeton University Press, Princeton, 1996.

    Dirac's book is now somewhat dated, although there is still much that can be learned from seeing how the great man thought. Perhaps the most remarkable achievement is to give an account of general relativity in only 68 pages. Compare with the 1279 pages of Misner-Thorne-Wheeler!

13. R.P. Feynman, *Feynman Lectures on Gravitation*, Penguin Books, London, 1999.

    This is not a book to learn general relativity from. Having said that, it is fascinating. Written in 1962-1963 at the same time as Feynman was giving his more celebrated set of lectures, he does precisely what you would want the leading quantum field theorist of the day to do; he attempts to reinvent general relativity as a quantum field theory. Of course, he fails, but that is not the point; the insights provided by one of the great theoretical physicists are a delight.

14. H. Hopf, *Differential Geometry in the Large*, 2nd edition, Springer-Verlag, New York, 1989.

    For those who wish to see differential geometry done properly by a first rate mathematician, who originated many important concepts, these lectures are highly accessible to undergraduates. The style is not the modern one, but the clarity of presentation outweighs this in my opinion.

15. D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination*, AMS Chelsea Publishing, New York, 1999.

    One of my graduate students, who shall remain nameless, asked of this reference; "is that Hilbert Hilbert?" I replied "yes, but I believe his first name was David". The remarks made about Hopf's book apply equally to Hilbert & Cohn-Vossen.

16. S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press, Oxford, 1998.

    Everything that you (did not) want to know about black holes. A tour de force of mathematical physics.

General relativity has brought foundationally new concepts to physics, primarily on the largest length scales, from the size of the solar system up to the entire universe. These new insights have been rightly heralded; although Einstein did not receive the Nobel Prize for his theory of general relativity, many others have for results related to, or impacting, its development. Here are links to their Nobel Prize lectures, some closely related to topics in this course, others only more tenuously, all of which you might find enlightening:

Albert Einstein (1921), *"for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect."*

Hans A. Bethe (1967), *"for his contributions to the theory of nuclear reactions, especially his discoveries concerning the energy production in stars."*

Sir Martin Ryle and Anthony Hewish (1974), *"for their pioneering research in radio astrophysics: Ryle for his observations and inventions, in particular of the aperture synthesis technique, and Hewish for his decisive role in the discovery of pulsars."*

Arno A. Penzias and Robert W. Wilson (1978), *"for their discovery of cosmic microwave background radiation."*

Subrahmanyan Chandrasekhar and William A. Fowler (1983), Chandrasekhar: *"for his theoretical studies of the physical processes of importance to the structure and evolution of the stars"*, and Fowler: *"for his theoretical and experimental studies of the nuclear reactions of importance in the formation of the chemical elements in the universe."*

Russell A. Hulse and Joseph H. Taylor (1993), *"for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation."*

Riccardo Giacconi (2002), *"for pioneering contributions to astrophysics, which have led to the discovery of cosmic X-ray sources."*

John C. Mather and George F. Smoot (2006), *"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation."*

Saul Perlmutter, Brian P. Schmidt, and Adam G. Riess (2011), *"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae."*

Great excitement has greeted the direct observation, just over a year ago, of gravitational waves. The achievement of the LIGO experiment is hard to overstate and describing it will be a highlight of these lectures. Their own webpage, with all of their scientific results, can be found here. The original publications of the five events that have been confirmed so far are already recognised as having historic significance; they can be recommended most strongly:

Observation of Gravitational Waves from a Binary Black Hole Merger, Physical Review Letters **116**, 061102 (2016).
GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Physical Review Letters **116**, 241103 (2016).
GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Physical Review Letters **118**, 221101 (2017).
GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence, Physical Review Letters **119**, 141101 (2017).
GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Physical Review Letters **119**, 161101 (2017).

Earthrise as seen from the moon, taken during the Apollo 8 mission on Christmas Eve 1968. Image from NASA.

Chapter 1

Gravity and Relativity

... occasion'd by the fall of an apple, as he sat in a contemplative mood. Why should it not go sideways, or upwards? But constantly to the earth's centre? Assuredly, the reason is that the earth draws it. There must be a drawing power in matter.

William Stukeley (*Memoirs of Sir Issac Newton's Life*, 1726)

1.1 Gravity

Anything that has mass interacts with everything else that has mass. That interaction is always attractive and is known as gravity. It can be felt in the most mundane of everyday things. Jump in the air and you come back to the ground; the rate at which you do so is 9.8 ms^{-2}. It can be seen in the most spectacular of sights; the Milky Way on a cold, dark night.

Observations of the night sky have provided the bulk of our knowledge about gravity. The curious motion of the planets, as viewed from the Earth, was understood first by Copernicus's heliocentric model and later by Newton's theory of gravity. Two bodies of masses M_1 and M_2 and separated by a distance r interact through an attractive force of magnitude

$$F_{\text{gravity}} = \frac{GM_1M_2}{r^2}, \tag{1.1.1}$$

directed along the straight line between them. G is Newton's gravitational constant, equal to 6.67×10^{-11} m^3 s^{-2} kg^{-1} in SI units. On this basis one predicts that the planets orbit about the sun along elliptical trajectories, with the sun near one focus, sweeping out equal areas in equal times and with the square of the orbital period varying as the cube of the semi-major axis of the orbit. The agreement with observations is remarkable. On the basis of this understanding we have been able to send men to the moon and satellites to all parts of the solar system. My favourite result is the discovery of an entirely new planet – Neptune, in 1846 – predicted from the discrepancy between Newton's theory and existing observations.

The agreement, however, is not perfect. There is a regular shift, or precession, in the location of the perihelion of each of the planets. Most of it can be reconciled by accounting for the gravitational pull of Jupiter, and the other planets, but a small discrepancy persists. In the case of Mercury, where the effect is largest, the unaccounted for precession is 43 seconds of arc per century. The precedent of Neptune suggested an undiscovered planet could be the cause and a name – Vulcan – was even chosen. It was never found. When the resolution came, it was by an altogether different conception: In Einstein's theory of general relativity the planets move along geodesics in the space-time geometry created by the mass of the sun.

The source of gravity is mass. This same quantity also controls the dynamic response to all forces via Newton's second law $F = ma$. The striking statement is that this is not true of any other interaction; it applies only to gravity and is what makes the gravitational interaction special. For comparison, consider a charged particle moving in a perpendicular magnetic field. The Lorentz force $q\mathbf{v} \times \mathbf{B}$ gives rise to an acceleration of magnitude qvB/m and the particle moves along a circular trajectory of radius mv/qB. More massive particles move on circles of greater radius. This simple observation allows many particles to be identified from their tracks in accelerator detectors. Gravity behaves differently. In Galileo's fabled experiment objects of different mass, and composition, were dropped from the leaning tower of Pisa and observed to all take the same time to impact the ground. A modern variant[1] was used by MIT professor Walter Lewin in his introductory lectures on mechanics. In an experiment to determine the period of a pendulum, adding his own mass to that of the pendulum bob did not alter the result. Precision tests of the equivalence of gravitational and inertial masses typically employ more sensitive torsional balances. At the time Einstein developed his theory of relativity the most precise measurements of this kind were due to Eötvös. The idea of the experiment is to attach two equal masses to a solid bar suspended by a thread from its midpoint. The apparatus should be isolated from vibrational and thermal disturbances and then carefully monitored. It experiences both Coriolis forces and gravitational forces from the rotation and proximity of the Earth. These couple to the inertial and gravitational masses, respectively, and any difference would give rise to a rotation of the bar. The null results obtained in 1889 demonstrated equivalence to at least 1 part in 10^6. Eötvös extended and improved his experiment over the next 30 years, improving the accuracy with which the equivalence could be established by three orders of magnitude. Today it stands at better than 1 part in 10^{12}.

Einstein's *equivalence principle*, upon which he founded much of his general theory of relativity, is sometimes stated in simple terms as asserting that the inertial and gravitational masses are one and the same thing. However, to my mind this does not convey enough, for Newton was certainly aware of what was implied when he stated that the gravitational interaction between two bodies was proportional to the product of their masses. There is only one mass defined in Newton's Principia, on page 1[2]. Einstein said, and perceived, so much more. He perceived what Newton did not (and knew he did not), that the equivalence of inertial and gravitational masses should be viewed as an equivalence between inertia and gravitation as phenomena; strictly a local equivalence. Einstein was fond of thought experiments; it is because of him that, even in the English language, we refer to them as 'gedanken experiments'. He gave first a simple thought experiment in which a collection of free particles in a(n inertial) frame K are viewed from another frame K' uniformly accelerating with respect to K, say along their common 'up' direction. Being unable to say it better, I quote Einstein in his own words at length:

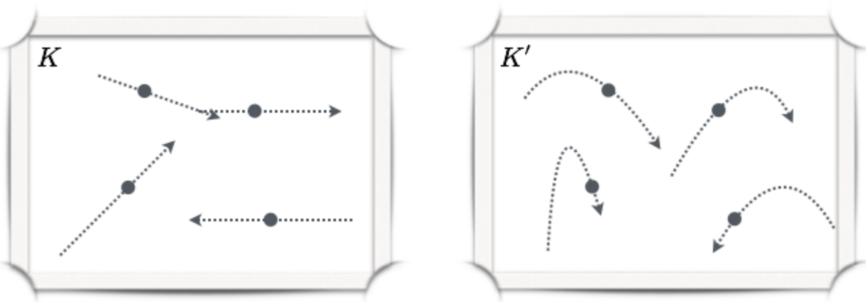


Figure 1.1: Einstein's description of his equivalence principle: the frame K is inertial and K' is accelerating uniformly relatively to it along the vertical direction.

[1]In fact both Galileo and Newton did experiments with pendulums, so it was not so much a 'variant' as a 'reboot'.

[2]It is Definition I. *"The quantity of matter is the measure of the same, arising from its density and bulk conjunctly."* Following the definition Newton writes *"It is this quantity that I mean hereafter everywhere under the name of body or mass. And the same is known by the weight of each body; for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shewn hereafter."*

Relatively to K' all the masses have equal and parallel accelerations; with respect to K' they behave just as if a gravitational field were present and K' were unaccelerated. Overlooking for the present the question as to the "cause" of such a gravitational field, which will occupy us later, there is nothing to prevent our conceiving this gravitational field as real, that is, the conception that K' is "at rest" and a gravitational field is present we may consider as equivalent to the conception that only K is an "allowable" system of coordinates and no gravitational field is present. The assumption of the complete physical equivalence of the systems of coordinates, K and K', we call the "principle of equivalence".

from The Meaning of Relativity, page 57

Einstein's equivalence principle applies not just to phenomena involving massive particles but also to light. Suppose an observer K', experiencing a uniform gravitational field, is sending a light pulse back and forth between two points of their lab. Let us suppose that it is being sent horizontally, as we perceive it. On the grounds of the equivalence principle our situation is identical in nature to the observations of a friend K who sees no gravitational field, but instead the frame K' in a state of constant relative acceleration. To such an inertial observer the trajectory of the light pulse is, as always, along a straight line. But then it cannot be so for the observer K'; in the presence of a gravitational field light must travel along a curved arc, being deflected by the gravitational field. Now suppose we send our light pulse vertically upwards. We are interested in its frequency when emitted and when it reaches the ceiling. Of course, K and K' are in relative motion so that they do not see the same frequency; they differ by the longitudinal Doppler shift. The new insight is that since K' is accelerating relative to K the Doppler shift when the light leaves the floor is not the same as when it arrives at the ceiling. In the presence of a gravitational field light experiences a *gravitational redshift*. If the height of the room is h then the travel time for the light pulse is approximately h/c seconds, during which time the relative speed between K and K' changes by $gh/c = \phi/c$, where ϕ is the gravitational potential of the ceiling relative to the floor. From the expression of special relativity

$$\omega' = \sqrt{\frac{1 - v/c}{1 + v/c}}\, \omega, \tag{1.1.2}$$

for the longitudinal Doppler shift we can estimate the observed change in frequency due to motion up the gravitational potential to be $(\Delta\omega/\omega) \approx -\phi/c^2$. A precision test of this gravitational redshift was performed in 1959 by Pound and Rebka using x-ray emission and absorption from an ^{57}Fe sample and the vibrational motion of a loudspeaker to control the relative velocity of the emitter and absorber. Recoil-less Mössbauer emission, discovered only the year before, was needed to give sufficient sensitivity for observing the tiny gravitational effect.

A further thought experiment conveys the profundity of Einstein's 'principle of equivalence' and how it fundamentally alters our conception of gravity. Suppose you have a large collection of identical metre sticks and you lay them out, at rest in frame K', along a diameter and circumference of a circle, so that they just fit snuggly end-to-end. We will suppose that there are a number D along the diameter and a number C along the circumference. We want to consider how the metre sticks, so laid out, appear in two frames, K and K', in relative acceleration. To set our bearings, first consider that there is no acceleration and both frames are at rest. Then, from what we have learnt and been taught about inertial frames and about circles and so forth, we know that $C/D = \pi$. Now suppose that K' is rotating relative to K with a constant angular velocity. From the point of view of K each of the rods that we have laid out is undergoing an acceleration and there is no gravitational field. But Einstein's principle of equivalence asserts that, from the point of view of K', we may say that the rods[3] are unaccelerated and behave as if a gravitational field were present. At the same time, we know from the special theory of relativity that an observer in the frame K sees each of the rods

[3]Each one, locally, not all of them simultaneously.

laid out along the circumference of the circle travel with some (instantaneous) velocity along their length and so experience a Lorentz contraction, while the rods laid out along the diameter experience no such contraction (along their length). It follows that in this case $C/D > \pi$.

> *We therefore arrive at the result: the gravitational field influences and even determines the metrical laws of the space-time continuum. If the laws of configuration of ideal rigid bodies are to be expressed geometrically, then in the presence of a gravitational field the geometry is not Euclidean.*
>
> from The Meaning of Relativity, page 61

Einstein arrived at these conclusions as early as 1907. It was a further eight years before he fully understood their ramifications and gave a complete formulation of his general theory of relativity.

The motion of celestial bodies tells us about the gravitational influence of the sun. Each of the planets, asteroids and comets moves along their own individual trajectory, conveying the local nature of the sun's gravitational attraction at their positions. But the sun's influence is not restricted to just these locations; it extends everywhere. This is captured by the notion of the sun's *gravitational field*[4]. In Newton's theory it is a scalar quantity, whose gradient measures the gravitational force,

$$\phi_{\text{sun}} = \frac{-GM_\odot}{r}. \tag{1.1.3}$$

Actually, it is the integral of a spherically symmetric mass density – Newton was rightly proud of this result; no matter how close we may be to the surface of the Earth, the force we feel is the same as if its entire mass was concentrated at its centre rather than spread out over a spherical volume. A distribution of matter with mass density $\rho(\mathbf{x}')$ creates at the point $\mathbf{x}$ a gravitational potential $\phi(\mathbf{x})$ given by

$$\phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x', \tag{1.1.4}$$

the integral being taken over the entire distribution of matter generating the gravitational field. In Newton's theory mass acts as the source of gravity; the gravitational pull celestial bodies experience comes from the sun and is generated by it.

A small amount of analysis allows this to be written equivalently as a differential equation for ϕ. There are many ways in which one might do this; one that appeals to me is to observe that the integral on the right-hand-side of (1.1.4) is a convolution and hence the inverse Fourier transform of a product. We can then write the gravitational potential in Fourier space as

$$\begin{aligned} \tilde{\phi}(\mathbf{k}) &= -G\,\frac{4\pi}{k^2}\,\tilde{\rho}(\mathbf{k}), \\ \Rightarrow \quad -k^2\tilde{\phi}(\mathbf{k}) &= 4\pi G\,\tilde{\rho}(\mathbf{k}). \end{aligned} \tag{1.1.5}$$

Converting back to real space gives the differential equation

$$\nabla^2\phi = 4\pi G\,\rho. \tag{1.1.6}$$

It is this equation that Einstein sought an improvement for in his general theory of relativity[5]. The principles that guided him were that while the source of gravity in Newton's theory is

[4]This notion is quite significant, for it turns gravity into a field theory, even if the proper nature of the gravitational field is not fully understood in Newton's formulation, as Newton was fully aware.

[5]Loosely speaking, one may say that Feynman sought an improvement for the Fourier space relation

$$\tilde{\phi} = 4\pi G\,\frac{-1}{k^2}\,\tilde{\rho},$$

in his lectures on gravitation.

the mass density, in any relativistic theory this must be the energy-momentum content of a continuous matter distribution, and from the equivalence principle the gravitational potential should be considered as describing the metric properties of space and time. The equation relating them should be a differential equation of second order, that reduces to Newton's in a weak-field non-relativistic limit, and is consistent with all other known laws of physics. On this basis Einstein arrived at his field equations of general relativity

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.1.7}$$

1.2 Special Relativity

We give a brief review of the basic concepts of Einstein's special theory of relativity, for two purposes: to allow us to introduce notation, of which there is unavoidably a lot; and to describe the geometric character of that theory as a precursor to what we will encounter in general relativity. The Minkowski space-time of special relativity is an exact solution of the Einstein equations of general relativity; it is the vacuum solution in which there is no matter content.

Einstein states the two principles upon which he formulates his special theory of relativity as follows

> i. If K is an inertial system, then every other system K' that moves uniformly and without rotation relatively to K is also an inertial system; the laws of nature are in concordance for all inertial systems. This statement we shall call the "principle of special relativity".
>
> ii. Maxwell's theory of electromagnetism shows that light propagates with a constant speed c, at least in one inertial frame. According to the principle of special relativity, it takes this same value in all inertial frames.

We deduce from these the form of the Lorentz transformations, again following Einstein's own derivation. Suppose K and K' are two observers in relative motion. We may declare the direction of relative motion to be the x-direction of Cartesian coordinate systems for each observer, taken positive in the direction K' is moving away from K. Then K records the motion of K' to be $x = vt$ and likewise K' records the motion of K to be $x' = -vt'$. Now suppose a light pulse moves in the same direction as K' is moving away from K. Then our two observers will record the motion of the light pulse as

$$K: \quad x = ct, \qquad\qquad K': \quad x' = ct'. \tag{1.2.1}$$

A linear transformation between their coordinate systems is consistent with these two observations if it is of the form

$$x' - ct' = \Lambda\big(x - ct\big), \tag{1.2.2}$$

for some Λ, depending only on the speed of their relative motion, v. If instead the light pulse is travelling along the direction in which K is moving away from K' then their observations will be

$$K: \quad -x = ct, \qquad\qquad K': \quad -x' = ct', \tag{1.2.3}$$

and the linear transformation between their coordinate systems is required, by the principle of special relativity, to take the form

$$-x - ct = \Lambda\big(-x' - ct'\big), \tag{1.2.4}$$

for the same Λ. These two relations are equivalent to

$$x' = \frac{1}{2}\left(\Lambda + \Lambda^{-1}\right)x - \frac{1}{2}\left(\Lambda - \Lambda^{-1}\right)ct, \tag{1.2.5}$$

$$ct' = \frac{1}{2}\left(\Lambda + \Lambda^{-1}\right)ct - \frac{1}{2}\left(\Lambda - \Lambda^{-1}\right)x, \tag{1.2.6}$$

Using K's observations of the location of K' we determine Λ to be $(\frac{1+v/c}{1-v/c})^{1/2}$ and then rewrite the Lorentz transformations in their standard form

$$x' = \gamma\big(x - \beta ct\big), \tag{1.2.7}$$

$$ct' = \gamma\big(ct - \beta x\big), \tag{1.2.8}$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. The transformation of the transverse coordinates is the identity, $y' = y, z' = z$. Although there are a number of simple arguments for this, we do not record any of them here. I admire greatly the elegance and simplicity of this argument of Einstein's; there are other approaches, but I have not come across any that I like as much.

From the Lorentz transformation follow the basic concepts of length contraction and time dilation. All that need be done is to consider how a moving observer perceives our metre sticks, or clocks. For instance, if we are holding a clock and a time T passes, the moving observer K' will, according to the Lorentz transformation, record a time

$$ct' = \gamma\big(ct - \beta x\big) = \gamma cT. \tag{1.2.9}$$

From their perspective the time that has elapsed is longer; moving clocks run slow. Likewise, if we are holding a stick of length L with one end at the origin and the other stretched out along the x-direction then a moving observer, K', will measure its length as follows. One end is at the origin, $x' = 0$. At the same instant ($t' = 0$), from their perspective, the other end is at the point

$$x' = \gamma\big(x - \beta ct\big) = \gamma\big(x - \beta^2 x\big) = \gamma^{-1}L. \tag{1.2.10}$$

Thus K' records the length to be contracted by a factor γ^{-1}.

Measurements of time or distance are observer-dependent. However, appropriate combinations of these measurements are not. They are known as the *proper time*, or *proper length*; the time elapsed, or distance travelled, in the rest frame of the object being observed. Looking back at Einstein's derivation of the Lorentz transformation it is easy to see that

$$\big(x' - ct'\big)\big(x' + ct'\big) = \big(x - ct\big)\big(x + ct\big), \tag{1.2.11}$$

so that the quantity $-(ct)^2 + x^2 + y^2 + z^2$ is the same for all observers. This is known as the *invariant*, or *interval.* It plays a special role in elucidating and understanding the nature of space and time. It is the *metric* of special relativity.

Another physical quantity that we can identify as being agreed upon by all observers is the phase of a (plane) wave. The phase of a wave records the number of crests and troughs. Suppose in the passage of a wave a buoy bobs up and down three times. This corresponds to a change in the phase of $3 \times 2\pi$. All observers, regardless of their relative motion, will record this same number, for they cannot claim that 57 waves passed if they are observing the same event. For a plane wave, the phase may be expressed in terms of measurements of position and time as $\phi = kx - \omega t$. The quantity k is called the *wavevector* and ω the *frequency.* For another observer the analogous expression will be $k'x' - \omega' t'$. It follows from the Lorentz transformation, expressing x' and ct' in terms of x and ct, that

$$\frac{-\omega}{c} = \gamma\Big[\frac{-\omega'}{c} - \beta\, k'\Big], \tag{1.2.12}$$

$$k = \gamma\Big[k' - \beta\Big(\frac{-\omega'}{c}\Big)\Big]. \tag{1.2.13}$$

Pay attention to the natural way that this arises. In the Lorentz transformation, reading left to right, measurements of K', ct' and x', are expressed in terms of those of K. Here, reading left to right, measurements of K, $-\omega/c$ and k, are expressed in terms of those of K', but this interchange of the places of K and K' aside the form of the relation is precisely the same. In the 'old language' one says that the quantities $(-\omega/c, k)$ transform *covariantly*, while the quantities (ct, x) transform *contravariantly*[6]. In modern parlance we say that (ct, x, y, z) form the components of a 4-*vector*, while $(-\omega/c, k_x, k_y, k_z)$ form the components of a 1-*form*, or *co-vector*. We will discuss these things more formally in Chapter 2, but before that here is a related and important example.

We learn in quantum mechanics that particles (indeed most things) can be described using wavefunctions. The energy is given by Planck's formula, $E = \hbar\omega$, and the momentum by de Broglie's, $p = \hbar k$. It follows that (minus) the energy and momentum of a particle transform under Lorentz transformations as a 1-form[7]

$$\frac{-E}{c} = \gamma\Big[\frac{-E'}{c} - \beta\, p'_x\Big], \tag{1.2.14}$$

$$p_x = \gamma\Big[p'_x - \beta\Big(\frac{-E'}{c}\Big)\Big]. \tag{1.2.15}$$

It follows from the properties of the Lorentz transformation that the quantity $E^2 - c^2p^2$, where $p = |\mathbf{p}|$ is the magnitude of the momentum, is an *invariant* associated to any particle; a quantity that all observers agree takes the same value. This is the *mass* of the particle, or more correctly m^2c^4.

1.2.1 Notation

Notation is hard won; it takes time and effort to figure out the nicest way to present material, or ideas, or the most natural way to write relations. As such notation should be respected and not 'sniffed at'. However, it is just notation; certainly, it is not physics. For the most part, I do not find it very interesting, so the description I give here is rather terse. It is essential, therefore, that you do not let me befuddle you with something as 'bland' as notation; if you are unsure, ask me to explain – I can assure you that the other students will be most grateful. Either that or they will talk about you behind your back; one of the two.

We write x^1, x^2, x^3 in place of x, y, z, respectively. Note that the indices here are *superscripts*. In the jargon they are called *contravariant*. We may use x^i, with a lower-case Latin index $(i, j, k, \dots)$, as a generic symbol for any spatial component. We write x^0 in place of ct, but continue to refer to it as "time". We will write x^μ, with a lower-case Greek index $(\mu, \nu, \sigma, \dots)$, as a generic symbol for either a point in Minkowski space-time or for a 4-vector. The *summation convention*, in which any repeated index is to be summed over, will be used throughout.

We will write the interval as

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = \eta_{\mu\nu}x^\mu x^\nu, \tag{1.2.16}$$

where the symbol $\eta_{\mu\nu}$ has components

$$\eta_{00} = -1, \quad \eta_{11} = \eta_{22} = \eta_{33} = 1, \tag{1.2.17}$$

and all other components zero. It is called the *Minkowski metric*. The indices on the Minkowski metric are both subscripts: in the jargon we say that they are *covariant*. It is an example of a type $\binom{0}{2}$ tensor. In the same jargon a vector is a type $\binom{1}{0}$ tensor.

[6] Which is covariant and which contravariant is a matter of historical convention.

[7] It is nice to see that the '4-momentum' is naturally a 1-form, as opposed to a vector, in agreement with what we learn in classical mechanics.

It is especially common, even fundamental, to consider the interval between points that are infinitesimally close. If the points are x^μ and $x^\mu + dx^\mu$ then the interval between them is written

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{1.2.18}$$

and again (even more forcefully) called the *Minkowski metric*.

REMARK: There is *no* consensus over the signs in the metric, known as its *signature*. As many people adopt the convention $(-, +, +, +)$, as I have, as adopt the converse $(+, -, -, -)$. Some even switch between the two when they think it is convenient to do so, although this has to be done with great care. Informally speaking the choice of signature depends on whether you prefer to think of the metric (interval) as measuring spatial distances or intervals of time (or masses of particles).

We will write the Lorentz transformation as

$$x'^\mu = \Lambda^\mu_\nu x^\nu. \tag{1.2.19}$$

The symbol Λ^μ_ν has one superscript index and one subscript index. It is *not* a tensor. The interval is invariant under Lorentz transformations. It follows that

$$\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu_\alpha x^\alpha \Lambda^\nu_\beta x^\beta, \tag{1.2.20}$$

from which we conclude the useful identity

$$\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}, \tag{1.2.21}$$

characterising how the Minkowski metric transforms under a Lorentz transformation. In many treatments this is promoted to the status of defining a Lorentz transformation.

If x^μ and y^μ are 4-vectors then

$$-x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 = \eta_{\mu\nu} x^\mu y^\nu = \big(\eta_{\mu\nu} x^\mu\big) y^\nu, \tag{1.2.22}$$

is a Lorentz invariant. The final form is suggestive. We can think of $\eta_{\mu\nu} x^\mu$ as an object that acts on 4-vectors, in this case y^ν, and returns a scalar (Lorentz invariant). It is evidently linear. Linear maps from vectors to scalars are called 1-*forms*; you will also see the terms *dual vectors* or *covectors*. They are also known as type $\binom{0}{1}$ tensors. We create a more compact notation by writing the 1-form $\eta_{\mu\nu} x^\mu$ as x_ν and think of this action of the metric as "lowering indices". A formal inverse operation of "raising indices" can be written as $\eta^{\mu\nu} x_\nu = x^\mu$. It is a linear map taking a 1-form to a vector. For these to be consistent requires that

$$x^\mu = \eta^{\mu\nu} x_\nu = \eta^{\mu\nu} \eta_{\alpha\nu} x^\alpha = \delta^\mu_\alpha x^\alpha, \tag{1.2.23}$$

where δ^μ_α is the *Kronecker delta* with components equal to 1 if $\mu = \alpha$ and zero otherwise. It is a type $\binom{1}{1}$ tensor. The object $\eta^{\mu\nu}$ that raises indices is also a tensor, of type $\binom{2}{0}$, and is called the *inverse metric*. The lowering and raising of indices with the metric and inverse metric extends to all tensors, of any rank, in a more or less obvious manner. In whimsical fashion it is called the *musical isomorphism* and the operations called *flat* and *sharp*.

The transformation for 1-forms is important. Writing the invariant as $x_\nu x^\nu$, and using that it is the same for all observers, we have

$$x'_\mu x'^\mu = x'_\mu \Lambda^\mu_\nu x^\nu = x_\nu x^\nu, \tag{1.2.24}$$

and it follows that the components of a 1-form transform according to

$$x_\nu = \Lambda^\mu_\nu x'_\mu. \tag{1.2.25}$$

Note that this is the "opposite" sense to the coordinates, or the components of a vector, as we saw for (minus) the frequency and wavevector, k_ν, or (minus) the energy and momentum, p_ν.

Remark: The fact that some quantities transform one way and others oppositely is the origin of the terms *co*variant and *contra*variant. Which type is co- and the other contra- is, of course, a matter of historical convention.

Until now, I have not given any definition of a "type $\binom{k}{l}$ tensor". If what you have gleaned so far is that it is an object that looks like this $T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}$, having k contravariant (superscript) indices and l covariant (subscript) indices, then that will be partly right. (Not everything that looks like this is a tensor, Λ^{μ}_{ν} is not.) For the time being, we take the following as a definition: a type $\binom{k}{l}$ tensor is a multilinear map taking k 1-forms and l vectors and returning a number; its components will be denoted $T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}$. Of course, in this and everything else we have written here so far, we have been talking about the "components" of a vector, 1-form, or tensor but have not specified which basis the components refer to. This is bad. We will remedy it in Chapter 2 when we talk properly about geometry and physical quantities. Until then, please do not panic; things may be underspecified but they are not wrong and (in my opinion) still comprehensible.

1.3 Geometry of Minkowski Space-Time

The inner product for Minkowski 4-vectors

$$\vec{U}\cdot\vec{V} = -U^0V^0 + U^1V^1 + U^2V^2 + U^3V^3 = \eta_{\mu\nu}U^\mu V^\nu, \tag{1.3.1}$$

endows Minkowski space-time with a causal structure. 4-vectors can be separated into different types according to whether their magnitude (norm) squared under this inner product is positive, negative, or zero. We say that a 4-vector with positive magnitude is *space-like*, one with negative magnitude is *time-like*, and one with zero magnitude is *light-like*, or *null*

$$\begin{aligned} &\vec{U}\cdot\vec{U} > 0 \qquad && \text{space-like,} \\ &\vec{U}\cdot\vec{U} < 0 && \text{time-like,} \\ &\vec{U}\cdot\vec{U} = 0 && \text{null.} \end{aligned} \tag{1.3.2}$$

Since the Minkowski inner product of two 4-vectors is invariant under Lorentz transformations, the concept of space-like, time-like and null is well-defined and agreed upon by all observers. The following results are fundamental. If $\vec{U}$ is space-like then there exists a frame in which it has no 0-component. Likewise, if $\vec{U}$ is time-like then there exists a frame in which it only has a 0-component. In both cases any frame for which this is true is called a *rest frame* (for the 4-vector). With a thorough understanding of the principle of special relativity these statements are utterly trivial, but they can also be demonstrated algebraically. For instance, suppose the 4-vector $\vec{U} = (U^0, U^1, 0, 0)$ is time-like. Then under a Lorentz transformation the 1-component becomes $\gamma(U^1 - \beta U^0)$ and can be made to vanish by choosing $\beta = U^1/U^0$. (The magnitude of this is guaranteed to be less than 1 because the 4-vector is time-like.) Thus any time-like 4-vector is equivalent to $(U^0, 0, 0, 0)$ under a Lorentz transformation, for some U^0. If U^0 is positive the 4-vector is said to be *future directed*, while if it is negative it is said to be *past directed*. The trajectory of any massive particle is time-like; at each point of its motion its velocity is a time-like vector.

Through each point in Minkowski space one can draw the set of all light rays passing through that point. The tangent vector to any light ray is of the form $(U^0, U^0\mathbf{n})$ for a unit magnitude 3-vector $\mathbf{n}$ corresponding to the direction of propagation. By a suitable rotation we can find a frame in which its components are $(U^0, U^0, 0, 0)$ but there is no 'rest frame'. When U^0 is positive we say the light ray is future directed. Likewise, when U^0 is negative we say the light ray is past directed. In any case, passing through each point of Minkowski there is one light ray for every direction in ordinary 3-space, or for each point on the surface of a 2-sphere. The set of all such sweeps out a three-dimensional surface; the *light cone* through that point.

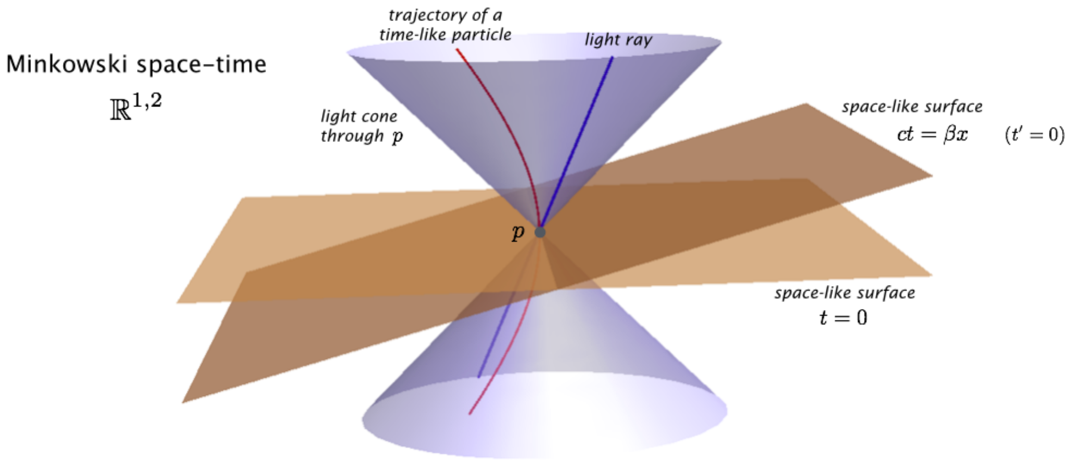


Figure 1.2: The structure of Minkowski space-time in the vicinity of any generic point p.

It separates the rest of Minkowski into disconnected parts. The set of points inside the light cone through p are time-like separated from it and can be reached from p along a time-like trajectory. We say that these points are *causally connected* to p. Points outside the light cone are space-like separated from p; it is not possible for a massive, or massless, particle to pass through both the given point and p.

The set of points $x^0 = 0$ is "space". What this means is that any two points in the set are space-like separated. It is perhaps better to give a local characterisation: at any point p of the space ($x^0 = 0$), the set of tangent vectors are all space-like. Any subset of Minkowski with this property is called *space-like*. The basic examples are the "spaces" $x'^0 = 0$ as seen by different observers, but there are others. Here is a nice example. The set $-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -1$ is a space-like hypersurface. (For simplicity restrict to $x^0 > 0$.) There are several ways of showing this; here is one. The points of the surface can be given the explicit parameterisation

$$x^0 = \cosh(u), \quad x^1 = \sinh(u)\cos(v), \quad x^2 = \sinh(u)\sin(v)\cos(w), \quad x^3 = \sinh(u)\sin(v)\sin(w), \tag{1.3.3}$$

for $u \in [0, \infty), v \in (0, \pi), w \in [0, 2\pi)$. At any point the tangent space is spanned by the three linearly independent vectors

$$\vec{s}_1 = [\sinh(u), \cosh(u)\cos(v), \cosh(u)\sin(v)\cos(w), \cosh(u)\sin(v)\sin(w)]^T, \tag{1.3.4}$$

$$\vec{s}_2 = [0, -\sin(v), \cos(v)\cos(w), \cos(v)\sin(w)]^T, \tag{1.3.5}$$

$$\vec{s}_3 = [0, 0, -\sin(w), \cos(w)]^T, \tag{1.3.6}$$

and one may verify that each of them is space-like.

REMARK: The example we have described is an isometric embedding of the hyperbolic plane $\mathbb{H}^3$. Because of their familiarity with special relativity, physicists often find this picture of the hyperbolic plane easier to grasp than the Poincaré disc model, usually preferred by mathematicians.

REMARK: A time-like hypersurface is any subset of Minkowski with the property that at each point there is a vector tangent to it that is time-like. Likewise, a null hypersurface is any subset of Minkowski with the property that at each point there is one, and only one, null vector tangent to it. This being said, you might like to show that the light cone through any point is a null surface.

1.4 Particle Motion

In many situations we learn about the world by observing the motion and behaviour of *'test particles'*; particles that experience their environment without appreciably disturbing it. On the basis of observations we infer how test particles behave and then use them to deduce the nature of any other environment (gravitational field, electric field, etc.) we put them in. One of the most basic statements in mechanics is that, when left alone, test particles move along straight lines at constant speed; Newton's first law[8]. Recognising that a straight line is the shortest 'distance' between two points, or the path that takes 'least time', we can state this alternatively that the trajectory of a free (test) particle is such as to make

$$S = -mc \int_{\tau_{\text{init}}}^{\tau_{\text{final}}} d\tau, \tag{1.4.1}$$

take a minimum value, where τ is c times the proper time for the particle motion. The quantity, S, is known as the *action*. The prefactor is an arbitrary constant that depends only on the test particle; it is its *mass*. This way of thinking about particle motion, and physics in general, is known as *Hamilton's principle of least action*. Not all of you may be familiar with it, so we will use this example to illustrate and introduce it. The action is a number[9] that depends on the particle motion through its trajectory $x^\mu(\tau)$. Different trajectories yield different values for the action. Hamilton's principle asserts that the observed physical trajectory is the one that corresponds to the 'least' value of the action, or more correctly to a critical point of the action. It is one of the most profound and unifying concepts in all of physics.

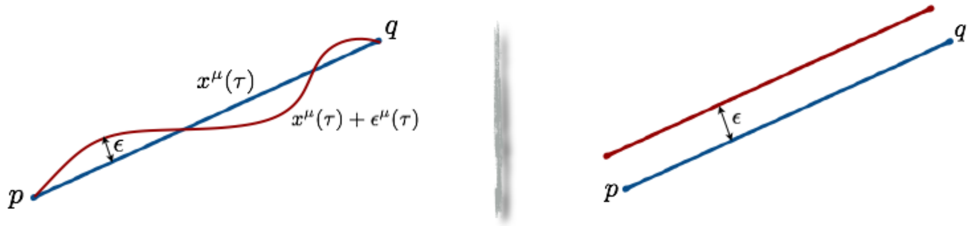


Figure 1.3: Schematic of the motion of free test particles: Left; Newton's first law; such particles travel along straight lines at constant speed, which is entirely equivalent to being critical points of the distance, or time, between two points (the blue curve is shorter than the red, for any ϵ). Right; The symmetry of Minkowski implies that physical quantities are unchanged by uniform translations ϵ. This leads to conservation of energy and momentum.

Let us see what this means for a free particle. For the trajectory $x^\mu(\tau)$ to be a critical point means that the value of the action does not change to first order for all variations of the trajectory. A variation of the trajectory between two points can be written $x^\mu(\tau) + \epsilon^\mu(\tau)$, where ϵ^μ is small and vanishes at the endpoints, but is otherwise arbitrary, and τ continues to denote (c times) the proper time for the trajectory $x^\mu(\tau)$. The action is the 'time taken' along this trajectory. The infinitesimal interval between points of the trajectory $x^\mu(\tau) + \epsilon^\mu(\tau)$ is

$$ds^2 = \eta_{\mu\nu}\big(dx^\mu + d\epsilon^\mu\big)\big(dx^\nu + d\epsilon^\nu\big) = -d\tau^2 + 2\eta_{\mu\nu}\frac{d\epsilon^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau^2 + O(\epsilon^2), \tag{1.4.2}$$

remembering that τ is (c times) the proper time along the trajectory $x^\mu(\tau)$. It follows that

[8] In his Principia, this is Definition III on page 1: *"The vis insita, or innate force of matter, is a power of resisting, by which every body, as much as in it lies, endeavours to persevere in its present state, whether it be of rest, or of moving uniformly forward in a right line."*

[9] Its dimensions are those of Planck's constant $\hbar$.

the change in the value of the action associated to this variation of the trajectory is

$$S[x+\epsilon]-S[x] = mc\int_{\tau_{\rm init}}^{\tau_{\rm final}} \eta_{\mu\nu}\frac{d\epsilon^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau + O(\epsilon^2), \tag{1.4.3}$$

$$= mc\,\eta_{\mu\nu}\frac{dx^\nu}{d\tau}\,\epsilon^\mu\Big|_{\tau_{\rm init}}^{\tau_{\rm final}} - mc\int_{\tau_{\rm init}}^{\tau_{\rm final}} \eta_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\,\epsilon^\mu\,d\tau + O(\epsilon^2). \tag{1.4.4}$$

The first term is zero if the variation ϵ^μ vanishes at the two endpoints, *i.e.* the trajectory is always between the same initial and final points. In that case, we see that the change in the action vanishes for arbitrary variations (ϵ^μ), and the trajectory $x^\mu(\tau)$ corresponds to one of 'least' action, if

$$\frac{d^2x^\nu}{d\tau^2} = 0, \tag{1.4.5}$$

or the test particle moves 'freely' along a straight line at constant speed; precisely Newton's first law.

Minkowski space has a fundamental symmetry; it is homogeneous and isotropic; it looks the same at every point. This corresponds to the basic observation that physical phenomena appear the same wherever you happen to be or whenever you happen to perform your experiment. A pure translational shift $x^\mu \mapsto x^\mu + \epsilon^\mu$, for constant ϵ^μ, has no effect on physical phenomena. In this case, and when x^μ is the physical trajectory so that it satisfies the equation of motion, the previous calculation for the difference in the value of the action (1.4.4) gives

$$S[x+\epsilon]-S[x] = mc\,\eta_{\mu\nu}\frac{dx^\nu}{d\tau}\,\epsilon^\mu\Big|_{\tau_{\rm init}}^{\tau_{\rm final}}. \tag{1.4.6}$$

But this must be zero since a pure translation has no effect on any physical quantities, *i.e.* $S[x+\epsilon] = S[x]$. The conclusion is significant: the quantity

$$mc\,\eta_{\mu\nu}\frac{dx^\nu}{d\tau} \equiv p_\mu, \tag{1.4.7}$$

is conserved in any physical motion; it is called the *4-momentum*[10]. $p_0 = -\gamma mc$ is (minus) the energy divided by c and $p_i = \gamma m\,\partial_t x^i$ is the 3-momentum. Each component is conserved separately. So too is the magnitude of the 4-momentum

$$p_\mu p^\mu = -\big(E/c\big)^2 + p_i p_i = -m^2c^2, \tag{1.4.8}$$

which is the familiar relation between energy, 3-momentum and mass. The general relation between symmetries and conserved quantities, illustrated here, is known as *Noether's theorem.*

Massive particles move along time-like trajectories. There is a frame of reference – the rest frame – in which its 'speed' is zero. Light is different; for light there is no such frame and all observers record the same value c for its speed, regardless of how they are moving. It is the same to say that the photon is massless. Nonetheless, like massive free particles, 'free' photons are observed to move along straight lines. This straight line may be described by its wavevector k_μ, the analogue of the momentum of a massive particle. As the photon is massless, its wavevector is null, $k_\mu k^\mu = 0$. This can also be viewed as the condition that the wavefunction $\sim e^{ik_\mu x^\mu}$ satisfy the wave equation. We do not develop an action principle for the trajectories of massless particles, although this can be done; it is the branch of electromagnetism known as geometrical optics and the action principle is Fermat's principle of least time.

We may say, in summary, that the trajectory of a massive particle is a time-like straight line, while that of a massless particle is a null straight line. We have never observed anything that travels along space-like straight lines; this appears to be a deep property of the world.

[10]Please note that I define this as a 1-form rather than a vector, which is why there is a negative sign for the 0-component; this is absent when the 4-momentum is considered a vector.

REMARK: These basic properties of the motion of test particles – Newton's first law, or the equivalent Hamilton's principle – and the quantities they have that are conserved along their trajectories will continue to apply, in precisely the same form, to describe the motion of test particles in general relativity. The description may seem more striking and profound, but the underlying principles are no different. It is for this reason that we have taken the trouble to describe the basic principles early on, and in a familiar setting.

1.5 The Stress-Energy-Momentum Tensor

In Newton's theory the source of the gravitational interaction is mass. In relativity, there is an equivalence between mass and energy and momentum. Particles, as they move, transport energy and momentum; it is conserved in the process, a consequence of the translational symmetry of Minkowski space as we have just seen. The analogous description for continuous distributions of matter, whether fluids, dust clouds, elementary particle fields or electromagnetic fields, is in terms of the *stress-energy-momentum tensor*. The stress-energy-momentum tensor is the source of the gravitational field in general relativity, replacing the mass of Newton's theory.

We first introduce the stress-energy-momentum tensor for a fluid in the form of an integral conservation law, extending the conservation of energy of a particle. Recall that for particles we have the conservation (the extra factor of c is for later convenience)

$$mc^2\eta_{\mu\nu}\frac{dx^\nu}{d\tau}\,\epsilon^\mu\Big|_{\tau_{\text{init}}}^{\tau_{\text{final}}} = 0, \tag{1.5.1}$$

for any constant translation ϵ^μ [11]. For a continuous gas of particles, or fluid, the mass of a particle, m, should be replaced with a *density*, ρ, that integrates over space to give the total mass in any spatial region. Likewise, the velocity of an individual particle, $dx^\mu/d\tau$, should be replaced with the *fluid velocity*, u^μ, of a material element just as in ordinary fluid dynamics. The statement that the particle trajectory is parameterised by (c times) proper time then becomes

$$\eta_{\mu\nu}u^\mu u^\nu = -1. \tag{1.5.2}$$

That given, the natural replacement of conservation of energy for a particle should be the expression

$$\int_\Sigma \rho c^2 u_\mu\,\text{dvol}_3\Big|_{\tau_{\text{final}}} - \int_\Sigma \rho c^2 u_\mu\,\text{dvol}_3\Big|_{\tau_{\text{init}}} = 0, \tag{1.5.3}$$

where Σ is a region of space, at time τ_{final} or τ_{init}, and $d\text{vol}_3$ is the three-dimensional volume element for that spatial region. A picture can help – see the one below. The surfaces $\Sigma|_{\tau_{\text{final}}}$ and $\Sigma|_{\tau_{\text{init}}}$ are not arbitrary; as for a particle, they are the start and end of the trajectory of the fluid motion; so it is the direction of the fluid velocity that defines the spatial surface that the energy is being transported through, or the unit normal to each surface is $n^\nu = u^\nu$. This allows conservation of energy and momentum to be expressed as an integral over the boundary of a region of space-time

$$\int_{\partial\Omega} \rho c^2 u_\mu u_\nu\, n^\nu \text{dvol}_3 = 0, \tag{1.5.4}$$

where n^ν is the unit normal to the boundary of the region Ω. The integrand $T_{\mu\nu} = \rho c^2 u_\mu u_\nu$ is the *stress-energy-momentum tensor* for a perfect, pressureless fluid. For a perfect fluid with pressure p, the corresponding expression is

$$T_{\mu\nu} = \big(\rho c^2 + p\big)u_\mu u_\nu + p\eta_{\mu\nu}. \tag{1.5.5}$$

[11] We will drop this from the following steps to avoid it being a distraction, but in truth it should really be retained. The issue is: what does it mean for ϵ^μ to be constant in a curved space?

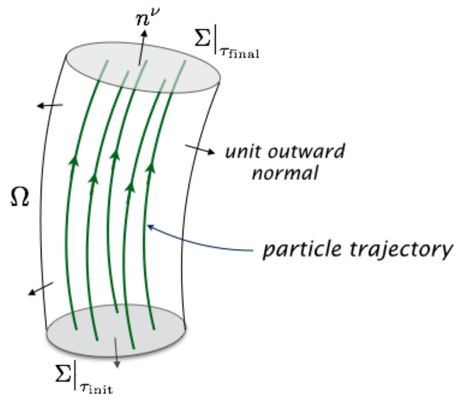


Figure 1.4: Schematic of the motion of dust particles and conservation of energy-momentum. You have to imagine that there are very many more particles than shown.

We will not consider viscous fluids. Note that in its rest frame, $u^\mu = (1,0,0,0)$, we have $T_{00} = \rho c^2$ and $T_{ij} = p\delta_{ij}$. So, T_{00} has an interpretation as the *energy density* and T_{ij} as the 'pressure' or isotropic *stresses*. The components $T_{0i} = T_{i0}$ that are zero in the rest frame have an interpretation as the *momentum density*. Finally, we have developed the expression of conservation of energy and momentum in integral form; an application of Gauss' divergence theorem gives the equivalent differential expression

$$\partial_\alpha T^\alpha_\mu = 0, \tag{1.5.6}$$

which is more commonly encountered in applications.

As a check on all that we have said, let's show that we recover the Navier-Stokes equations for a perfect fluid in the non-relativistic limit. In this limit we may write $u^\alpha = (1, v^i/c)$ to leading order in the small flow speeds v^i and the conservation of energy-momentum becomes

$$c\,\partial_t\Big[\Big(\rho + \frac{p}{c^2}\Big)u_\mu\Big] + c\,\partial_j\Big[\Big(\rho + \frac{p}{c^2}\Big)u_\mu v^j\Big] + \partial_\mu p = 0. \tag{1.5.7}$$

When $\mu = 0$ this reduces to the continuity equation

$$\partial_t \rho + \partial_j\big(\rho v^j\big) = 0, \tag{1.5.8}$$

neglecting terms of order p/c^2. Similarly, when $\mu = i$ we find the Navier-Stokes equations

$$\partial_t\big(\rho v^i\big) + \partial_j\big(\rho v^i v^j\big) + \partial_i p = 0, \tag{1.5.9}$$

after a similar neglect of terms of order p/c^2. The form you may be more familiar with

$$\rho\big(\partial_t v^i + v^j \partial_j v^i\big) + \partial_i p = 0, \tag{1.5.10}$$

follows from this one using the continuity equation.

By way of a summary, let us state that the energy and momentum of a continuous distribution of matter – dust, fluid, elementary particle field, etc. – is conveyed by the stress-energy-momentum tensor $T_{\mu\nu}$. It is conserved, $\partial_\alpha T^\alpha_\mu = 0$, and symmetric $T_{\mu\nu} = T_{\nu\mu}$. The component T_{00} represents the *energy density*, $T_{0i} = T_{i0}$ represents (minus) the *energy flux* or *momentum density*, and the components T_{ij} represent the *spatial stresses*. The stress-energy-momentum tensor will appear as the right-hand-side of the Einstein equations.

1.6 Electromagnetism

It is worth saying a few words about the general structure of the theory of electromagnetism because it played such a formative role in the theory of relativity, because the philosophy of how it is approached has such widespread resonance across all of theoretical physics and because it provides a platform for learning how to approach general relativity. Also, by talking about both general relativity and electromagnetism, which ultimately we formulate in different ways but do not understand why, it is hoped that it will help encourage the creativity of a young person to discover a better way than we currently know of thinking about and understanding both subjects. The following treatment is inspired strongly by Landau.

Maxwell's equations are traditionally written

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\rho, & \nabla\cdot\mathbf{B} &= 0, \\ \nabla\times\mathbf{B} - \epsilon_0\mu_0\partial_t\mathbf{E} &= \mu_0\mathbf{J}, & \nabla\times\mathbf{E} + \partial_t\mathbf{B} &= \mathbf{0}. \end{aligned} \tag{1.6.1}$$

These equations are Lorentz covariant; that observation was one of the strongest motivations for Einstein in his development of special relativity. (His foundational 1905 paper is titled '*Zur Electrodynamik Bewegter Körper*'.) However, this is not most readily apparent the way that they are written. We consider how to rewrite them to make the Lorentz covariance manifest.

There is disparity in the four Maxwell equations; two have 'sources' and two do not. Indeed, we have searched extensively for magnetic monopoles and found none. There are two ways to think about this. The first is that the absence of sources – magnetic charges and currents – for the right-hand-pair of equations is because *a forteriori* there are no sources. The second is that there are sources; we just haven't found them yet. The right-hand-pair of equations should really be extended to produce symmetry with the left-hand-pair by the introduction of magnetic charges and currents. This is bold, but its consequences are profound; it was Paul Dirac who first showed that this would immediately imply the quantisation of electric charge. Thus, there is strong impetus to search for magnetic monopoles; the present status is that we have searched extensively and found none. That being said, we will adopt the first line of thought; *a forteriori* there are no magnetic charges or currents.

It is established in introductory courses on vector analysis that if a vector is divergence free then it can be expressed as the curl of another vector. In other words, the Maxwell equation $\nabla\cdot\mathbf{B} = 0$ is equivalent to $\mathbf{B} = \nabla\times\mathbf{A}$. This result is known as the Poincaré lemma for $\mathbb{R}^3$. There is an analogous statement for the curl of a vector. Namely, if a vector is a gradient then its curl is automatically zero. Again, the Poincaré lemma provides for the converse, that if the curl of a vector is zero then it is the gradient of a function. This means that the Maxwell equation $\nabla\times\mathbf{E} + \partial_t\mathbf{B} = \mathbf{0}$ is equivalent to

$$\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}. \tag{1.6.2}$$

ϕ is called the electrostatic potential and $\mathbf{A}$ the magnetic vector potential. These two potentials form the *electromagnetic gauge field* $A_\mu = (-\phi/c, \mathbf{A})$. It is a natural relativistic object; it is also properly viewed, as here, as a 1-form. From it, the electric and magnetic fields are recovered by forming the antisymmetric derivative

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.6.3}$$

known as the *Maxwell field strength tensor*. It is an antisymmetric type $\binom{0}{2}$ tensor, or in other words a 2-*form*. You are strongly encouraged to verify that $F_{i0} = E_i/c$ and $F_{ij} = \epsilon_{ijk}B_k$.

REMARK: Sometimes you will see it said that the electromagnetic gauge field A is just a mathematical trick and the real physical quantities are the electric and magnetic fields. This is nonsense. The gauge field was deduced in precisely the manner laid out by Newton that in natural philosophy particular propositions should be inferred from phenomena – the observed absence of magnetic charges and currents – and afterwards rendered general by induction. It is pure physics, in the very finest tradition.

The two source-free Maxwell equations are now encoded in the statement that the antisymmetric derivative of the Maxwell field strength tensor is identically zero

$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0. \tag{1.6.4}$$

These are called the *Bianchi identities*. To verify that they are true one simply replaces each F with an antisymmetric derivative of the gauge field to get a sum of second partial derivatives of A. One need then only note that each such derivative occurs twice and with opposite signs.

REMARK: In the language of differential forms, the Maxwell field strength tensor is the derivative of the gauge field, $F = dA$. The Bianchi identities are then the statement that because F is exact it is automatically closed.

1.6.1 The Transformation of Electric and Magnetic Fields

The gauge field A is a 1-form and transforms as such. Thus the components A_μ and A'_μ seen by observers K and K' are related according to

$$A_\nu = \Lambda^\mu_\nu A'_\mu. \tag{1.6.5}$$

Partial derivatives transform similarly; the chain rule gives

$$\partial_\nu = \frac{\partial x'^\mu}{\partial x^\nu}\partial'_\mu = \Lambda^\mu_\nu \partial'_\mu. \tag{1.6.6}$$

It follows at once that the transformation of the Maxwell field strength tensor is

$$F_{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F'_{\mu\nu}. \tag{1.6.7}$$

This allows one to deduce how electric and magnetic fields are perceived by different observers. We describe explicitly only one situation and leave the rest to your own imaginations. Suppose K' is holding an electron. In their frame the electron is at rest and there is an electric field but no magnetic field. What this means is that $F'_{i0} = E'_i/c = -F'_{0i}$ and all other components are zero. But from K's point of view the electron is moving and moving charges generate magnetic fields. We compute this magnetic field

$$\epsilon_{ijk} B_k = F_{ij} = \Lambda^\mu_i \Lambda^\nu_j F'_{\mu\nu} = \left(\Lambda^l_i \Lambda^0_j - \Lambda^0_i \Lambda^l_j\right) E'_l/c, \tag{1.6.8}$$

$$\implies \qquad B_x = 0, \qquad B_y = -\frac{\gamma\beta}{c} E'_z, \qquad B_z = \frac{\gamma\beta}{c} E'_y, \tag{1.6.9}$$

and merely record that the electric field perceived by K is $E_x = E'_x$, $E_y = \gamma E'_y$, $E_z = \gamma E'_z$.

1.6.2 The Dynamical Maxwell Equations

Anything that has charge interacts with everything else that has charge. This interaction is mediated by the *electromagnetic field*[12]. Particles with charge act as sources for the electromagnetic field. The fundamental observation is that the thing that they are sources of is the

[12] Newton called great attention to the fact that his theory of gravity gave no account of how the gravitational interaction was mediated. He wrote "*That one body may act upon another at a distance through a vacuum without the mediation of anything else, by and through which their action and force may be conveyed from one another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it.*" And later "*I have not as yet been able to discover the reason for these properties of gravity from phenomena, and I do not feign hypotheses.*" He was, of course, also pragmatic, writing "*It is enough that gravity does really exist and acts according to the laws I have explained, and that it abundantly serves to account for all the motions of celestial bodies.*" Agreement of Newton's theory with all experimental observations continued for almost two hundred years after his death. In the case of electromagnetism, it took the genius of Faraday and Maxwell to introduce the fields that mediate interaction. For gravity, it was Einstein.

electromagnetic gauge field A and that it is a 1-form. That given, the theory follows from Hamilton's principle of least action.

The way that we learn about electromagnetism is by observing the motion of test charges; particles that have charge, but an amount of charge so small that they experience their environment without disturbing it. We need to determine how their motion relates to the electromagnetic field, which means to give the addition to the free particle action that comes from interaction with the gauge field A. The electromagnetic gauge field A that is generated by charges is a 1-form. A 1-form is precisely the thing that you can integrate along a curve, and the trajectory of a test particle is a curve. Thus the contribution to the action for a test particle moving in an electromagnetic field is the integral of the gauge field A along its trajectory

$$q \int A_\mu \, dx^\mu = q \int_{\tau_{\text{init}}}^{\tau_{\text{final}}} A_\mu(x) \frac{dx^\mu}{d\tau} \, d\tau. \tag{1.6.10}$$

The prefactor is an arbitrary constant that depends only on the test particle; it is its *charge*. A central feature of electromagnetism is already in evidence here. Namely, if A is the derivative of a function, $A_\mu = \partial_\mu \chi$, then it has no influence on the test charge for it may be integrated to the boundary and its contribution to the action depends only on the endpoints of the motion, not on the trajectory. 1-forms that are the derivatives of functions are called *exact*. In electromagnetism one says that the electromagnetic field A is *pure gauge*. This given, we see that more is true. Two electromagnetic fields A and A' that differ by the derivative of a function

$$A'_\mu - A_\mu = \partial_\mu \chi, \tag{1.6.11}$$

exert precisely the same influence on the motion of test charges. Their influence on physical phenomena is identical; they both describe the *same* electromagnetic field. This is known as *gauge invariance*; a transformation between two electromagnetic gauge fields of the form (1.6.11) is called a *gauge transformation*.

Now let us see how the gauge field A influences the motion of test charges, using Hamilton's principle of least action. If the trajectory should be changed to $x^\mu + \epsilon^\mu$ the value of the action will change to

$$q \int A_\nu(x+\epsilon) \, d(x^\nu + \epsilon^\nu) = q \int \Big(A_\nu \, dx^\nu + \epsilon^\mu \partial_\mu A_\nu \, dx^\nu + A_\mu \, d\epsilon^\mu + O(\epsilon^2) \Big), \tag{1.6.12}$$

$$= q \int A_\nu \, dx^\nu + q \, A_\mu \epsilon^\mu \Big|_{\tau_{\text{init}}}^{\tau_{\text{final}}} + q \int \epsilon^\mu \big[\partial_\mu A_\nu - \partial_\nu A_\mu \big] dx^\nu + O(\epsilon^2). \tag{1.6.13}$$

Again, the boundary term vanishes because ϵ^μ is zero at the initial and final points of the trajectory. Combining this variation with the variation of the action for a free particle (1.4.1) we find that the principle of least action gives the trajectory of the test particle as

$$mc \, \eta_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} = q \big[\partial_\mu A_\nu - \partial_\nu A_\mu \big] \frac{dx^\nu}{d\tau} = q \, F_{\mu\nu} \frac{dx^\nu}{d\tau}, \tag{1.6.14}$$

where $F_{\mu\nu}$ is the Maxwell field strength tensor. This is the equation of motion for a charged particle in an electromagnetic field, the covariant version of the *Lorentz force equation*.

REMARK: What is important here is that the motion of test charges is influenced by the electromagnetic field through the Maxwell field strength tensor $F_{\mu\nu}$ rather than the gauge field A_μ directly. Thus, as we learn about electromagnetic phenomena through watching the motion and behaviour of test charges what we will learn about are the classical electric and magnetic fields, **E** and **B**.

Remark: If the gauge field A is exact, or pure gauge, the field strength tensor vanishes identically, as one should expect. One can enquire about the converse; if the field strength tensor vanishes is A necessarily pure gauge? The answer – a version of the Poincaré lemma – is no. A physical manifestation of this intriguing situation is provided by the *Aharonov-Bohm effect*. It has become increasingly important in recent years with the growing interest in topological properties of materials and phases of matter.

Finally, we turn to the dynamical equations for the electromagnetic field itself, *i.e.* we wish to determine the action for the gauge field A. The action is required to be *Lorentz invariant*, *i.e.* the same as measured by all observers, and the expression for it is constrained by the set of invariants that we can find. There are three quantities that are Lorentz covariant

$$\eta^{\mu\nu}A_\mu A_\nu, \qquad \eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}, \qquad \epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}. \tag{1.6.15}$$

Here $\epsilon^{\alpha\beta\mu\nu}$ is the fully antisymmetric *Levi-Civita symbol* with $\epsilon_{0123} = -\epsilon^{0123} = 1$.

Remark: We run through how to establish Lorentz covariance for the second of these, $F_{\mu\nu}F^{\mu\nu}$. One starts with the statement that $x_\mu x^\mu = x_\mu \eta^{\mu\nu} x_\nu$ is an invariant and uses the known transformation properties of the 1-form x_μ to deduce that the inverse metric obeys the relation

$$\eta^{\mu\nu}\Lambda^\alpha_\mu\Lambda^\beta_\nu = \eta^{\alpha\beta}.$$

It is then just a matter of transforming the expression from the frame K to the frame K'

$$\begin{aligned}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} &= \eta^{\mu\alpha}\eta^{\nu\beta}\,\Lambda^\sigma_\mu\Lambda^\tau_\nu F'_{\sigma\tau}\,\Lambda^\gamma_\alpha\Lambda^\delta_\beta F'_{\gamma\delta},\\ &= \eta^{\sigma\gamma}\eta^{\tau\delta}F'_{\sigma\tau}F'_{\gamma\delta},\end{aligned}$$

to verify that it is indeed an invariant.

The third invariant, $\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$, is a total divergence and so does not contribute to the field equations. That is not to say that it is unimportant; it is an invariant and it does characterise the field configuration, but it does not contribute to determining it dynamically. The first invariant, $A_\mu A^\mu$, is perhaps the most subtle one. Naively, it should be discarded on the grounds that it is not invariant under gauge transformations, however, such a term is permissible when it arises from the spontaneous symmetry breaking of a scalar field. This is known as the *Higgs mechanism*. The term $A_\mu A^\mu$ would then give a mass to the gauge field. This is precisely what happens in a superconductor where it is known as the *Meissner-Ochsenfeld effect* and has as its most dramatic consequence magnetic levitation, but in free space the photon remains massless[13]. This given, the action for the electromagnetic field comprises only the term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, to be integrated over the entire extent of the electromagnetic field, *i.e.* all of space-time. Finally, there should also be a coupling between the gauge field A_μ and the 4-current $J^\mu = (c\rho, \mathbf{J})$ describing the sources of the electromagnetic field. The action for the electromagnetic field is then

$$S = \frac{1}{\mu_0}\int_{\mathbb{R}^{1,3}} d^4x \left\{\frac{-1}{4}F_{\mu\nu}F^{\mu\nu} + \mu_0 A_\mu J^\mu\right\}. \tag{1.6.16}$$

Remark: One should check that the coupling between the gauge field and the 4-current is compatible with gauge invariance. A gauge transformation alters the coupling term as follows

$$A_\mu J^\mu \mapsto \big(A_\mu + \partial_\mu\chi\big)J^\mu = A_\mu J^\mu + \partial_\mu\big(\chi J^\mu\big) - \chi\partial_\mu J^\mu.$$

The second term is a total divergence and does not enter the equations of motion, so it can be safely ignored. The third term vanishes by conservation of charge, $\partial_\mu J^\mu = 0$. Thus we see that conservation of charge and gauge invariance are closely linked.

[13] It is the $W^\pm$ and Z bosons that acquire mass.

The form of the action for the electromagnetic field established, the dynamical equations of motion are given by the critical points of the action with respect to variations of the fields, A_μ. We compute directly

$$S[A+\delta A] = \frac{1}{\mu_0}\int_{\mathbb{R}^{1,3}} d^4x \left\{ \frac{-1}{4}\Big[\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big)\big(\partial^\mu A^\nu - \partial^\nu A^\mu\big) \right.$$
$$\left. - 4\partial_\nu \delta A_\mu\big(\partial^\mu A^\nu - \partial^\nu A^\mu\big) + \cdots\Big] + \mu_0\big(A_\mu + \delta A_\mu\big)J^\mu \right\}, \tag{1.6.17}$$

$$= S[A] + \frac{1}{\mu_0}\int_{\mathbb{R}^{1,3}} d^4x\, \delta A_\mu \Big[-\partial_\nu F^{\mu\nu} + \mu_0 J^\mu\Big] + O(2), \tag{1.6.18}$$

and therefore the field equations corresponding to critical points of the action are

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu. \tag{1.6.19}$$

This is the covariant form of Maxwell's first and fourth equations; they should be viewed as a system of linear second order partial differential equations for the gauge field A. I leave it to each of you to check that Maxwell's equations in their traditional form really are reproduced; it is a good exercise.

1.6.3 The Stress-Energy-Momentum Tensor

For continuous fields described by an action, such as the electromagnetic field, there is a general understanding of the origin and conservation of the stress-energy-momentum tensor. It originates from the translational symmetry of Minkowski space-time: the energy-momentum content of any configuration of the electromagnetic field at one location is the same as the energy-momentum content of the same field configuration at any other location. This manifests itself in the Lagrangian depending on the fields A_μ and their derivatives $\partial_\mu A_\nu$ but not explicitly on the position x^μ. Consider two field configurations A and A' related by $A'(x) = A(x+\epsilon)$, *i.e.* A' is the same electromagnetic field as A but at a shifted location. The difference between the value of the action for the two field configurations can be computed in two ways. On the one hand, since the fields in any domain Ω are the same – only their locations have been shifted – any change in the action can only come from the boundary

$$S[A'] - S[A] = \int_{\partial\Omega} \eta_{\mu\nu}\epsilon^\mu L\, n^\nu d\mathrm{vol}_3 = \int_\Omega \partial_\mu\big(\epsilon^\mu L\big) d^4x + O(\epsilon^2), \tag{1.6.20}$$

where n^ν is the unit normal to the boundary of the region Ω, $d\mathrm{vol}_3$ is the three-dimensional volume element for that boundary surface, and we write $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ for the Lagrangian as short-hand. On the other hand, writing that $A_\nu(x+\epsilon) = A_\nu(x) + \epsilon^\mu\partial_\mu A_\nu + O(\epsilon^2)$, we also have

$$S[A'] - S[A] = \int_\Omega \left(\epsilon^\mu \partial_\mu A_\nu \frac{\partial L}{\partial A_\nu} + \partial_\alpha\Big[\epsilon^\mu\partial_\mu A_\nu\Big]\frac{\partial L}{\partial(\partial_\alpha A_\nu)} + O(\epsilon^2)\right) d^4x, \tag{1.6.21}$$

$$= \int_\Omega \left(\epsilon^\mu\partial_\mu A_\nu\left[\frac{\partial L}{\partial A_\nu} - \partial_\alpha\frac{\partial L}{\partial(\partial_\alpha A_\nu)}\right] + \partial_\alpha\left[\epsilon^\mu\partial_\mu A_\nu \frac{\partial L}{\partial(\partial_\alpha A_\nu)}\right]\right) d^4x + O(\epsilon^2), \tag{1.6.22}$$

$$= \int_\Omega \partial_\alpha\left[\epsilon^\mu\partial_\mu A_\nu \frac{\partial L}{\partial(\partial_\alpha A_\nu)}\right] d^4x + O(\epsilon^2), \tag{1.6.23}$$

since the first term in the second line vanishes by the field equations. Equating these two expressions for the difference in the action, it follows that

$$\int_\Omega \epsilon^\mu\partial_\alpha\left[\partial_\mu A_\nu \frac{\partial L}{\partial(\partial_\alpha A_\nu)} - \delta^\alpha_\mu L\right] d^4x = 0. \tag{1.6.24}$$

It is usual to denote the expression in square brackets by $-T^\alpha_\mu$ and call it the stress-energy-momentum tensor. The sign is a convention associated to our choice of signature for the metric. Then, as ϵ is arbitrary, it must be that it is conserved

$$\partial_\alpha T^\alpha_\mu = 0. \tag{1.6.25}$$

However, this is a little too rash and there is in fact a stronger statement. We will write instead

$$\partial_\mu A_\nu \frac{\partial L}{\partial(\partial_\alpha A_\nu)} - \delta^\alpha_\mu L = -T^\alpha_\mu + \partial_\beta B^{\alpha\beta}_\mu, \tag{1.6.26}$$

with $B^{\alpha\beta}_\mu$ antisymmetric in α, β, *i.e.* $B^{\alpha\beta}_\mu = -B^{\beta\alpha}_\mu$ [14]. The upshot is that we still have the conservation law

$$\partial_\alpha T^\alpha_\mu = 0, \tag{1.6.27}$$

but in addition the tensor $T_{\mu\nu} = \eta_{\alpha\nu} T^\alpha_\mu$ can always be chosen to be symmetric[15], *i.e.* $T_{\mu\nu} = T_{\nu\mu}$. It is this symmetric, conserved quantity that is the *stress-energy-momentum tensor*.

For the electromagnetic field, a direct calculation gives

$$\partial_\mu A_\beta \frac{\partial L}{\partial(\partial_\alpha A_\beta)} - \delta^\alpha_\mu L = \frac{1}{\mu_0}\Big[-\big(\partial_\mu A_\beta\big) F^{\alpha\beta} + \frac{1}{4}\delta^\alpha_\mu F_{\gamma\beta} F^{\gamma\beta}\Big], \tag{1.6.28}$$

$$= \frac{1}{\mu_0}\Big[-F_{\mu\beta} F^{\alpha\beta} - \big(\partial_\beta A_\mu\big) F^{\alpha\beta} + \frac{1}{4}\delta^\alpha_\mu F_{\gamma\beta} F^{\gamma\beta}\Big], \tag{1.6.29}$$

$$= \frac{1}{\mu_0}\Big[-F_{\mu\beta} F^{\alpha\beta} - \partial_\beta\Big(A_\mu F^{\alpha\beta}\Big) + A_\mu \partial_\beta F^{\alpha\beta} + \frac{1}{4}\delta^\alpha_\mu F_{\gamma\beta} F^{\gamma\beta}\Big]. \tag{1.6.30}$$

By the field equations (for the Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$) the term $\partial_\beta F^{\alpha\beta}$ vanishes and we can then identify the stress-energy-momentum tensor as

$$T_{\mu\nu} = \frac{1}{\mu_0}\Big[\eta_{\alpha\nu} F_{\mu\beta} F^{\alpha\beta} - \frac{1}{4}\eta_{\mu\nu} F_{\gamma\beta} F^{\gamma\beta}\Big], \tag{1.6.31}$$

$$= \frac{1}{\mu_0}\Big[\eta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4}\eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\Big]. \tag{1.6.32}$$

It is evidently symmetric and also traceless, $\eta^{\mu\nu} T_{\mu\nu} = T = 0$. To get a feeling for what it represents, we write the components in terms of the electric and magnetic fields, using $F_{i0} = E_i/c$ and $F_{ij} = \epsilon_{ijk} B_k$,

$$T_{00} = \frac{\epsilon_0}{2}\big[E^2 + c^2 B^2\big], \tag{1.6.33}$$

$$T_{0i} = -\epsilon_{ijk}\epsilon_0 c E_j B_k, \tag{1.6.34}$$

$$T_{ij} = -\epsilon_0\Big[E_i E_j + c^2 B_i B_j - \frac{1}{2}\delta_{ij}\big(E^2 + c^2 B^2\big)\Big]. \tag{1.6.35}$$

These components have the following interpretation: T_{00} is the *energy density* in the field; T_{0i} is (minus) the *energy flux* of the field (it is the Poynting vector); and T_{ij} is the *stress* in the field. To see more, integrate the conservation law $\partial_\alpha T^\alpha_\mu = 0$ over a region Σ of the space-like surface $x^0 =$ const, *i.e.* any region of 'space'. We find, for the case $\mu = 0$,

$$0 = \int_\Sigma \partial_\alpha T^\alpha_0 \, d^3x = \int_\Sigma \Big[\frac{1}{c}\partial_t\big(-T_{00}\big) + \partial_i T_{0i}\Big] d^3x, \tag{1.6.36}$$

$$= -\bigg(\frac{1}{c}\partial_t \int_\Sigma \frac{\epsilon_0}{2}\big[E^2 + c^2 B^2\big] d^3x + \int_{\partial\Sigma} \epsilon_{ijk}\epsilon_0 c E_j B_k \, dA^i\bigg). \tag{1.6.37}$$

[14] It is known as the Belinfante-Rosenfeld tensor.

[15] The rationale for using the freedom to ensure that the stress-energy-momentum tensor is symmetric comes from the expression for conservation of angular momentum. We will say no more; the interested reader should consult the excellent discussion in Landau's text.

This expresses precisely the conservation of energy for the electromagnetic field in any spatial region.

Of course, there is nothing special about the electromagnetic field; all physical quantities possess energy and fluxes of energy that they contribute to the stress-energy-momentum tensor $T_{\mu\nu}$. This tensor, that expresses the energy, momentum and internal stress content of the 'matter fields' acts as the source for the gravitational field in Einstein's theory of general relativity, much the same way as charge acts as the source of the electromagnetic field in Maxwell's theory. It appears as the 'right-hand-side' of the Einstein equations.

Problems

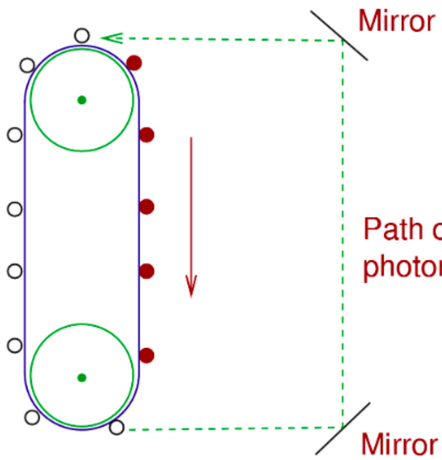
1. Solve the Kepler problem in Newtonian gravity: two bodies of masses M_1 and M_2 interacting through their mutual gravitational attraction.

2. Estimate the perturbation to one planetary orbit caused by gravitational interaction with another planet. If you have relatively little experience with perturbation theory, you might find that this is difficult. And yet it was calculated by Newton: in Book III, Proposition XIV Theorem XIV of his Principia he writes *"[I]f the aphelion of Mars, in the space of a hundred years, is carried* $33'\,20''$ in consequentia, *in respect of the fixed stars, the aphelions of the Earth, of Venus, and of Mercury, will in a hundred years be carried forwards* $17'\,40''$, $10'\,53''$, *and* $4'\,16''$, *respectively. But these motions are so inconsiderable, that we have neglected them in this Proposition"*. You might like to compare your calculations with Newton's.

   [I wish to be clear about this problem: it does not in any way represent the sort of thing that will appear in an exam paper. At the same time, it is precisely the sort of thing that is of foremost importance in giving an accurate description of the motion of the planets and other celestial bodies. Thus, and in keeping with my philosophy, the question is about physics and not exam questions. It should also serve to illustrate the magnitude of what Newton achieved; Einstein too.]

3. Prove Newton's result that the gravitational interaction with a spherically symmetric mass distribution is the same as that with a point object of the same total mass located at the centre of the sphere. [If you need any help, consult Newton's Principia, Book I, Proposition LXXI Theorem XXXI.]

4. The diagram below shows the design of a perpetual motion machine sent by an inventor called Hermann Bondi to a patent office in Switzerland where a certain Albert Einstein works: Atoms attached to a movable belt absorb photons at the top of their travel and



   emit them at the bottom, with the photons directed back to the top-most atoms where they are absorbed. When an atom absorbs a photon of energy $E = \hbar\omega$ its mass increases by $\Delta m = \hbar\omega/c^2$, hence the excited atoms (filled circles) on the right of the belt always outweigh the de-excited atoms (empty circles) on the left and hence perpetual clockwise motion results which can solve all the world's energy problems.

   What flaw does Einstein spot in this madness?

5. In a frame K a photon travels in the x^1x^2-plane at an angle θ with the x^1-axis. Show that in a frame K', moving relatively to K with speed v along the x^1-direction, that the same photon makes an angle θ' relative to the x'^1-axis, where

$$\cos(\theta') = \frac{\cos(\theta) - \beta}{1 - \beta\cos(\theta)}.$$

6. From the following set of expressions involving tensor components, identify those that are meaningless:

$$A^\alpha + B_\alpha, \qquad R^\alpha_\beta A^\beta + B^\alpha, \qquad R_{\mu\nu} = S_\gamma, \qquad T_{\mu\nu} = F_{\mu\nu}, \qquad g_{\mu\nu} g^{\mu\nu} R_{\mu\nu}.$$

7. Write out the equation $\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$ in Cartesian coordinates (ct, x, y, z). How does the equation change under a Lorentz transformation $x^\mu \mapsto x'^{\,\mu} = \Lambda^\mu_\nu x^\nu$?

   Show that the equation

$$\big[\hbar^2 c^2 \eta^{\mu\nu}\partial_\mu\partial_\nu - m^2c^4\big]\phi = 0,$$

   has plane-wave solutions $\phi \sim \mathrm{e}^{i(k_j x^j - \omega t)}$ with $(\hbar\omega)^2 = \hbar^2 c^2 k_j k^j + m^2 c^4$.

8. What is the value of δ^μ_μ? And of $\epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\mu\nu}$?

   If $T^{\mu\nu}$ is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, and $F_{\mu\nu}$ antisymmetric, $F_{\mu\nu} = -F_{\nu\mu}$, show that the contraction $T^{\mu\nu}F_{\mu\nu}$ is zero. Show that if $F_{\mu\nu}$ is antisymmetric and $T^{\mu\nu}$ is an arbitrary type $\binom{2}{0}$ tensor, then $F_{\mu\nu}T^{\mu\nu} = \frac{1}{2}F_{\mu\nu}(T^{\mu\nu} - T^{\nu\mu})$.

   In n dimensions, how many independent components are represented by the symbol $\Gamma^\alpha_{\mu\nu}$? And if the symbol is symmetric in its lower indices, $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$?

9. Show that the subset of Minkowski that is unit space-like distance from the origin

$$\eta_{\mu\nu}x^\mu x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 1,$$

   is a time-like hypersurface.

10. Let p and q be two points of the 2+1-dimensional Minkowski space $\mathbb{R}^{1,2}$ that are space-like separated. Argue (or simply convince yourself) that there exists a frame in which they have coordinates $(0, a, 0)$ and $(0, -a, 0)$ for some number a.

    Consider the two light cones through these points, p and q. Show that their intersection is a hyperbola. Show that it is a hyperbola for any inertial observer, *i.e.* consider how things change under Lorentz transformations. Sketch the situation.

    Now suppose p and q are time-like separated. By running a parallel analysis, show that the light cones through p and q intersect in an ellipse and determine how the semi-major and semi-minor axes depend on the invariant interval between p and q and on the Lorentz transformation relating different inertial frames in which it is viewed. Sketch the situation.

    [This example illustrates a way of understanding the classical conic sections as intersections of light cones. You might like to think of a way of understanding parabolae in this fashion. The generalisation to higher dimensions leads to a description of the beautiful Dupin cyclides, which arise in smectic liquid crystals as the spectacular focal conic textures, and also to the subject of Lie sphere geometry.]

11. Let $A_\mu = (-\phi/c, \mathbf{A})$ be the electromagnetic gauge field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the Maxwell field strength tensor. Show that $E_i = cF_{i0}$ is the electric field and $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ is the magnetic field. What are the components of the contravariant form of the Maxwell tensor, $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$? Express the Lorentz scalar $F_{\mu\nu}F^{\mu\nu}$ in terms of the electric and magnetic fields.

    Let $\epsilon_{\alpha\beta\mu\nu}$ be the fully antisymmetric Levi-Civita symbol with $\epsilon_{0123} = +1$. The dual field strength tensor is defined to be the antisymmetric type $\binom{0}{2}$ tensor (2-form) with components $\star F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$. Express its components in terms of the electric and magnetic fields. Express the pseudoscalar $\star F_{\mu\nu}\, F^{\mu\nu}$ in terms of the electric and magnetic fields.

12. Show that the equations

$$\partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} + \partial_\mu F_{\nu\alpha} = 0, \qquad \partial_\nu F^{\mu\nu} = \mu_0 \, J^\mu,$$

are equivalent to the four Maxwell equations when written in terms of the 3-component vectors **E** and **B**.

13. A charged particle accelerates from rest in a uniform electric field. There is no magnetic field. Describe its motion. Show that its trajectory is a time-like curve.

14. Find the stress-energy-momentum tensor for a scalar Klein-Gordon field ϕ for which the action is

$$S[\phi] = \int \frac{1}{2} \Big[\hbar^2 c^2 \eta^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + m^2 c^4 \phi^2 \Big] \, d^4 x.$$

Show that the components T_{00} and T_{0i} have their usual interpretation as the energy density and momentum density, respectively.

15. A type $\binom{0}{4}$ tensor $R_{\alpha\beta\mu\nu}$ has the symmetry properties

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}, \qquad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \qquad R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}.$$

In n dimensions show that there are

$$\frac{1}{2} \Big(\frac{n(n-1)}{2} \Big) \Big(\frac{n(n-1)}{2} + 1 \Big)$$

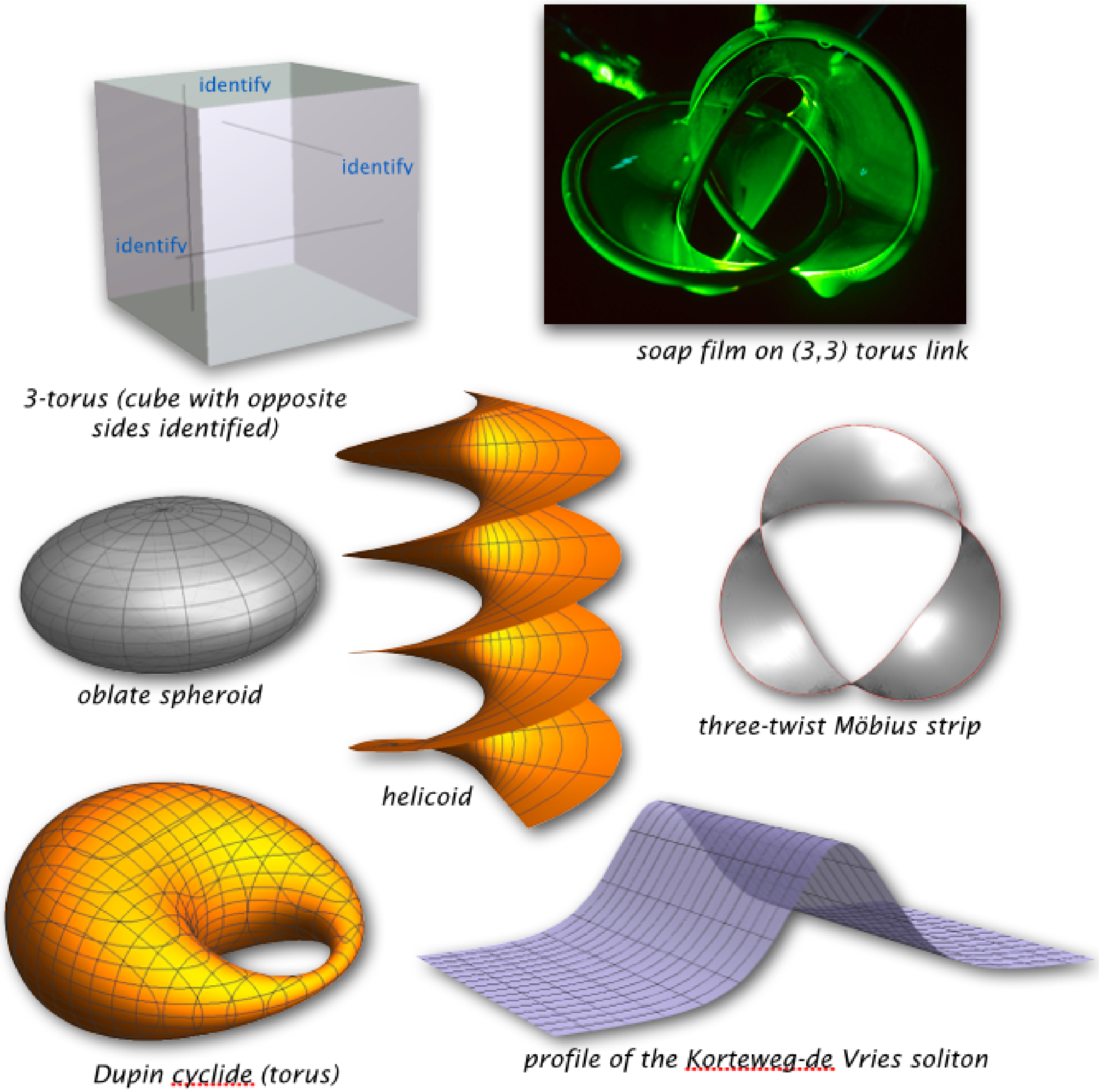
independent components. If, in addition, the tensor also satisfies the identities

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0,$$

show that these represent an additional

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$$

linearly independent constraints. Hence show that the total number of linearly independent components of the tensor is $\frac{1}{12} n^2 (n^2 - 1)$. The Riemann curvature tensor has these properties; this count gives the number of linearly independent curvatures in n dimensions.



Some examples of smooth manifolds for you to think about. Try to supply a variety of others for yourself.

Chapter 2

Differential Geometry

People have very powerful facilities for taking in information visually or kinesthetically, and thinking with their spatial sense. On the other hand, they do not have a very good built-in facility for inverse vision, that is, turning an internal spatial understanding back into a two-dimensional image. Consequently, mathematicians usually have fewer and poorer figures in their papers than in their heads.

William Thurston (1994)

General relativity is a theory about the structure and geometry of space-time. There are many different types of space and their study is a fascinating subject. It separates itself into two main aspects; geometry and topology. In some ways they are very deeply intertwined. It might surprise you, then, to be reminded that geometry dates back at least as far as the ancient Greeks, while topology is essentially a 20th century conception[1]. The disparity is very much less severe if one recalls that a serious definition of curvature is only due to Bernhard Riemann in 1854. In this course, we will focus on geometry and have more or less nothing to say about topology.

Geometry is the study of lengths, areas, volumes and so forth, angles, parallelism and curvature. For us, the aim is to understand what it means to say that space-time is curved, how one describes this curvature and how one detects it through physical phenomena. Let us be direct: we will really need to know two things in this course; what a geodesic is, and what curvature is. A geodesic is a curve of shortest length between two given points. It is the trajectory that a test particle travels along – Newton's first law. Curvature is the following thing. A small circle of geodesic radius r has a circumference $C(r)$ that differs from $2\pi r$. The difference gives a measure of the curvature of the space. It depends on the choice of circle. In fact, there are several notions of curvature, which all turn out to be equivalent; this one is called *sectional curvature*. It is an aim of mine in this chapter to give you a pedagogic account of Riemann's immense achievement in properly understanding curvature.

2.1 Manifolds

There are many different types of space; a variety of examples are shown in the preceeding plate. We shall confine our attention to those that are known as *smooth manifolds*. To see what this is, it suffices to consider a description of the space that we live in, for it is an example; the surface of the Earth is a sphere. One way to describe it is to divide it into four charts; the Antarctic chart, the North American chart, the Chinese chart, and the European chart. Each chart is a piece of paper that contains a detailed depiction of that part of the Earth that the

[1] Although there were several prominent forerunners, most people acknowledge Henri Poincaré's 1895 paper *Analysis Situs* as being foundational for topology in its modern sense. (I learn from Bob MacPherson that all the features of homology theory can be found in Riemann's unpublished notes.)

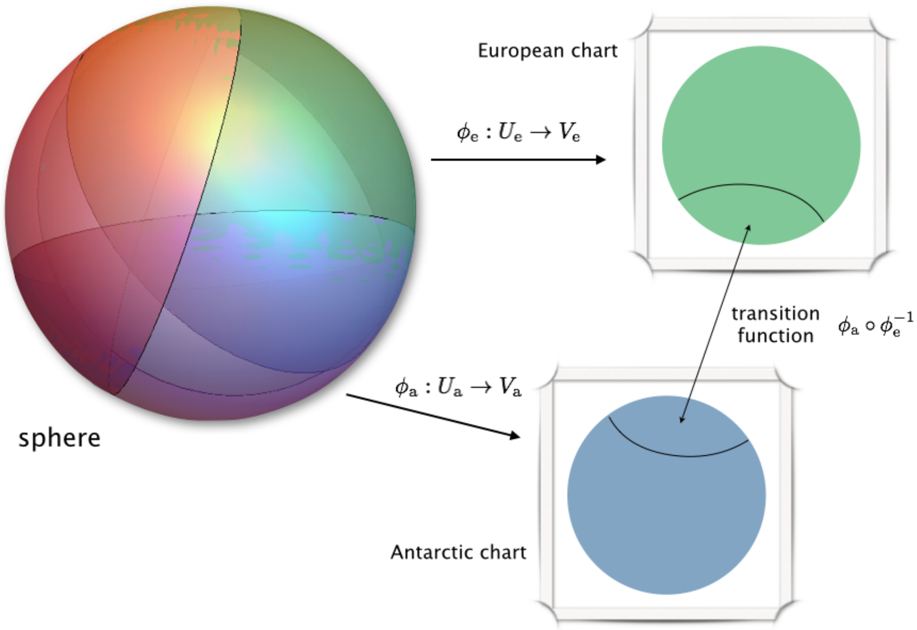


Figure 2.1: Definition of a manifold in terms of local charts.

chart is named after. England is in the European chart and Australia in the Antarctic chart. If you wish to visit Australia you can use the European chart to take you until there is overlap with the coverage of the Antarctic chart at which point you switch to that and complete your journey. That's the basic structure; let's repeat what we have just said in technical language.

A manifold M is a space that locally looks like $\mathbb{R}^n$ for some n, called the dimension of the space. What this means is that every point p of the space is contained in a little region U such that there is a 1-to-1 smooth function, with smooth inverse, between the points of U and the points x of an open subset V of $\mathbb{R}^n$. We write the function as $\varphi : U \to V$ and call it a *local chart* or *local coordinate system*. The inverse construction $\varphi^{-1} : V \to U$ is called a *parameterisation* for a neighbourhood of the point p. A collection of local charts $\varphi_a : U_a \to V_a$ that covers every point of M is called an *atlas*. If a point p lies in two charts, φ_a and φ_b, then one has the obvious *transition function*

$$\varphi_b \circ \varphi_a^{-1} : V_a \to V_b, \tag{2.1.1}$$

which is required to be smooth and have smooth inverse. The collection of triples $\{(\varphi_a, U_a, V_a)\}$, with the stated properties, is the definition of a manifold[2]. If the manifold also comes with something that allows you to measure distances, called a *metric*, then it is a *Riemannian manifold*. The study of Riemannian manifolds and their properties is what is meant by (Riemannian) *geometry*.

Often the sphere is presented differently, as the subset

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = R^2, \tag{2.1.2}$$

of Euclidean $\mathbb{R}^3$. This is known as an *embedding*, a technical term which states that the sphere sits inside the higher-dimensional space with no self-intersections. This situation gives rise to another definition of a manifold. A manifold M is a smooth subset of $\mathbb{R}^N$, for some (large) N, such that every point has a neighbourhood diffeomorphic[3] to an open subset of $\mathbb{R}^n$. This

[2] Strictly, there is also a notion of equivalence to introduce, which I will not do.

[3] The word *diffeomorphic* is a short-hand for what we wrote previously; there exists a smooth 1-to-1 function, with smooth inverse, between the two spaces.

is not a bad definition; in fact it is the definition given by Milnor[4] in his celebrated book *Topology from the Differentiable Viewpoint*. A central result in the theory of manifolds is that the two definitions are equivalent. This is known as the Whitney embedding theorem. More is true; an embedding always exists that is isometric – a result of John Nash, known as the isometric embedding theorems. Loosely, what this means is that any intrinsic geometry that the manifold has is the same as that it acquires from sitting inside a Euclidean space. The upshot of all this is that one is not losing anything by viewing any given manifold as being embedded in a Euclidean space, as every manifold can be viewed in such a way. Enormous pedagogy is gained by thinking about two-dimensional manifolds – surfaces – embedded in $\mathbb{R}^3$. You can make them and look at them with your own eyes. I encourage you to do so. Direct visualisation conveys things in a different way from pure algebra. I, for one, find it endlessly helpful.

2.2 The Metric

We introduce the general notion of a metric, measuring the distance between nearby points of a manifold, by looking at the example of a surface embedded in $\mathbb{R}^3$. The general case is not conceptually different. Strictly speaking, I should first remind you of the fundamental result in measuring distances in Euclidean space. This is Pythagoras's theorem: the square on

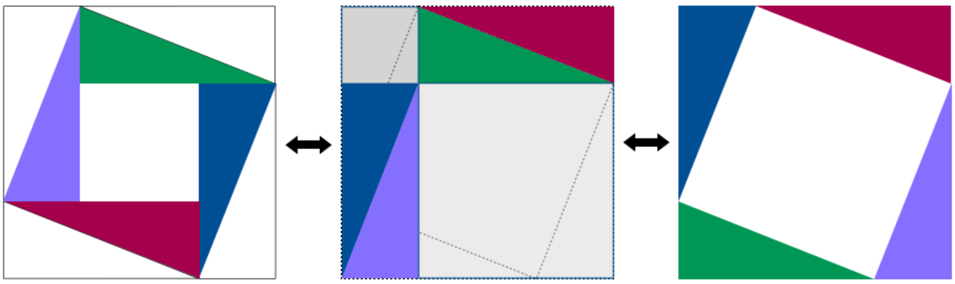


Figure 2.2: Pythagoras' proof of Pythagoras' theorem.

the hypotenuse is the sum of the squares on the other two sides. It leads immediately to the expression for the distance between two points, (x^1, x^2, x^3) and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ in Cartesian coordinates, as

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{2.2.1}$$

This is called the Euclidean metric[5]. This given, we turn to the measure of distances between points of a surface embedded in $\mathbb{R}^3$. The surface can be described explicitly through an *embedding function* $\mathbf{X}(u^1, u^2)$ listing the points of $\mathbb{R}^3$ where the surface is. For instance, for the sphere we might write

$$\mathbf{X}(u^1, u^2) = R\sin(u^1)\cos(u^2)\mathbf{e}_1 + R\sin(u^1)\sin(u^2)\mathbf{e}_2 + R\cos(u^1)\mathbf{e}_3. \tag{2.2.2}$$

The parameters u^1, u^2 in this case are the standard polar angles. The displacement between two nearby points on the surface, $\mathbf{X}(u^1, u^2)$ and $\mathbf{X}(u^1 + du^1, u^2 + du^2)$, is

$$d\mathbf{X} = \partial_{u^1}\mathbf{X}\, du^1 + \partial_{u^2}\mathbf{X}\, du^2 = \partial_{u^i}\mathbf{X}\, du^i, \tag{2.2.3}$$

[4]John Milnor is one the greatest mathematicians still alive; recipient of the Fields Medal, Wolf Prize and Abel Prize amongst many other awards. The bulk of his work has been in topology and understanding the structure of spaces.

[5]The generalisation to $\mathbb{R}^n$ should be obvious

$$ds^2 = (dx^1)^2 + \cdots + (dx^n)^2.$$

and the length squared of an infinitesimal line segment connecting them is

$$\begin{aligned} ds^2 = d\mathbf{X}\cdot d\mathbf{X} &= |\partial_{u^1}\mathbf{X}|^2\,(du^1)^2 + 2\partial_{u^1}\mathbf{X}\cdot\partial_{u^2}\mathbf{X}\,du^1du^2 + |\partial_{u^2}\mathbf{X}|^2\,(du^2)^2, \\ &= g_{11}\,(du^1)^2 + g_{12}\,du^1du^2 + g_{21}\,du^2du^1 + g_{22}\,(du^2)^2, \\ &= g_{ij}\,du^idu^j. \end{aligned} \tag{2.2.4}$$

The object ds^2 is called the *metric* and the quantities g_{ij} are its *components* for the given parameterisation (or choice of coordinates). Please note that $g_{12} = g_{21}$, *i.e.* the metric is *symmetric*. The metric is the square of the distance between nearby points of a space: if the points are labelled by coordinates u^i and $u^i + du^i$, then the square of the distance between them is $g_{ij}du^idu^j$. For our example of the sphere, a simple calculation gives

$$ds^2 = R^2\,(du^1)^2 + R^2\sin^2(u^1)\,(du^2)^2. \tag{2.2.5}$$

For an arbitrary n-dimensional manifold embedded in $\mathbb{R}^N$ it is no different and we write precisely the same thing. The metric is still the square of the distance between nearby points. If u^i are local coordinates for points of the space then we can write it as

$$ds^2 = d\mathbf{X}\cdot d\mathbf{X} = \partial_{u^i}\mathbf{X}\cdot\partial_{u^j}\mathbf{X}\,du^idu^j = g_{ij}\,du^idu^j, \tag{2.2.6}$$

where the quantities g_{ij} are the *components of the metric* in these coordinates.

As a terse but instructive example, consider the 3-sphere; the set of points distance R from the origin in $\mathbb{R}^4$

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = R^2. \tag{2.2.7}$$

It can be parameterised in many ways. For instance, 'Cartesian' coordinates about the point $(0,0,0,R)$ can be introduced by the parameterisation $(u^1,u^2,u^3,\sqrt{R^2-r^2})$, where $r^2 = (u^1)^2 + (u^2)^2 + (u^3)^2$. Such a parameterisation shows that the space is locally Euclidean, to second order accuracy in the u^i. An equivalent, but perhaps more convenient, parameterisation is to take

$$\big(x^1,x^2,x^3,x^4\big) = \big(r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), \sqrt{R^2-r^2}\big). \tag{2.2.8}$$

With this choice, a short calculation determines the metric to be

$$ds^2 = \frac{dr^2}{1-r^2/R^2} + r^2\Big(d\theta^2 + \sin^2(\theta)\,d\phi^2\Big). \tag{2.2.9}$$

Let us take the opportunity to illustrate a few things with this example. First, the parameterisation we have given covers only half the sphere, not the whole space. This is because we have written $x^4 = \sqrt{R^2-r^2}$, which is therefore constrained to be non-negative, with the parameter r taking values in the range $[0,R]$. If not already known, this is something that should be learnt well: parameterisations, or coordinate systems, or local charts, cover only portions of a manifold, not the whole space. Second, the metric is singular when $r = R$; $g_{rr} = (1-r^2/R^2)^{-1}$ is formally infinite. Yet, there is nothing weird about those points of the sphere: the problem is with our coordinate system, not with the space. This is an example of a *coordinate singularity*[6]. It warns us not to jump to unfounded conclusions purely on the basis of the behaviour of coordinates; we must look at the behaviour of physical observables. Finally, if we write $r = R\sin(\chi)$ we can give the metric on S^3 in its usual form, known as the *round metric*,

$$ds^2 = R^2d\chi^2 + R^2\sin^2(\chi)\Big(d\theta^2 + \sin^2(\theta)\,d\phi^2\Big). \tag{2.2.10}$$

One should note that these coordinates cover a larger portion of the sphere and that the metric is no longer singular at $\chi = \pi/2$ (corresponding to $r = R$). It is worth repeating; if not known already, this is a lesson that should be learnt well. In tautological terms, we might say that the way that we see a space depends on the way that we view it.

[6]The eagle-eyed among you should observe that there is also a coordinate singularity at $r = 0$; there is nothing wrong with that point either.

2.2.1 Lorentzian Metrics

The metric for Minkowski space-time

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{2.2.11}$$

is fundamentally different from the metric for a Euclidean space; the difference is the minus sign in front of the time-direction (and all that comes with it). The difference between the number of space-like directions and time-like directions is called the *signature* of the metric. If there is only one time-like dimension the metric is called *Lorentzian* – the signature is then $n-2$. Space-time is a Lorentzian manifold. In general, we will write the metric for a Lorentzian manifold as

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu, \tag{2.2.12}$$

where x^μ are coordinates in a local chart. The components $g_{\mu\nu}$ are symmetric, $g_{\mu\nu} = g_{\nu\mu}$, so that if we think of them as the entries in a matrix then the matrix can be diagonalised at any point of the manifold. That it is Lorentzian means that one of the diagonal entries is negative and the other three positive. We will think of the negative entry as the time direction and conventionally take it to be the component g_{00}.

The Lorentzian version of a sphere serves as a nice example. Lorentzian spheres come in two flavours, 'space-like' and 'time-like'[7]; we describe only the former. The set of points a fixed space-like distance from the origin in $\mathbb{R}^{1,4}$,

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = R^2, \tag{2.2.13}$$

is an isotropic, homogeneous, Lorentzian space-time, called *de Sitter space* or dS_4. It is an exact solution of the Einstein equations. Locally, de Sitter looks like Minkowski space-time. To see this, pick any point, say $(0,0,0,0,R)$, and parameterise a neighbourhood of it by writing

$$x^4 = \sqrt{R^2 + (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}. \tag{2.2.14}$$

One then finds that the metric is the usual Minkowski one

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + O(2), \tag{2.2.15}$$

to second order accuracy in the local coordinates x^μ ($\mu = 0,1,2,3$). This result is general in two respects. First, the same is true at every point of the de Sitter space, a consequence of its homogeneity[8]. Second, it is true at every point of every Riemannian, or pseudo-Riemannian, manifold, although we will not prove this result. Coordinates in which the metric is Minkowski to second order accuracy at a given point are called *Riemann normal coordinates*. In general relativity they are known as *local inertial frames*[9].

Remark: Local inertial frames, or Riemann normal coordinates, are often used extensively in the development of general relativity because Einstein did so. Because I wish to describe things more intrinsically, defining concepts without reference to any coordinates at all, they do not appear as prominently in these lecture notes as in many other textbooks. Of course, the point of particular coordinate systems is to make calculations simple and in this regard local inertial frames, or Riemann normal coordinates, are often invaluable.

[7] They can also be 'null', which correspond to light-cones in the embedding space.

[8] The easiest way to establish this is to note that there is a transitive action of the group $SO(1,4)$ corresponding to the symmetries of the space.

[9] Note that they are not unique; the Lorentz group $SO(1,3)$ acts (transitively) as a 'change of observer' that preserves the local inertial character of the coordinate system.

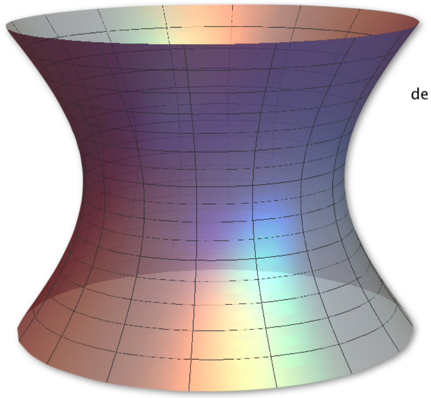


Figure 2.3: The two-dimensional de Sitter space dS_2 can be visualised directly as a hyperboloid in the Minkowski space $\mathbb{R}^{1,2}$.

A parameterisation that covers all of the de Sitter space-time is

$$\begin{aligned} x^0 &= R\sinh(\tau), \\ x^1 &= R\cosh(\tau)\sin(\chi)\sin(\theta)\cos(\phi), \\ x^2 &= R\cosh(\tau)\sin(\chi)\sin(\theta)\sin(\phi), \\ x^3 &= R\cosh(\tau)\sin(\chi)\cos(\theta), \\ x^4 &= R\cosh(\tau)\cos(\chi), \end{aligned} \tag{2.2.16}$$

for τ any real number and χ, θ, ϕ the three standard spherical angles on S^3 [10]. In these coordinates the metric assumes the form

$$ds^2 = -R^2 d\tau^2 + R^2\cosh^2(\tau)\Big[d\chi^2 + \sin^2(\chi)\big(d\theta^2 + \sin^2(\theta)\, d\phi^2\big)\Big]. \tag{2.2.17}$$

τ is a time-like coordinate. For any fixed value of τ, space is a 3-sphere with radius $R\cosh(\tau)$ that depends on the value of the time coordinate. So the picture we have is the following: when τ is negative, but increasing, the space is a contracting 3-sphere; it reaches a minimum size R when $\tau = 0$ and thereafter expands again for positive values of τ. The picture of a hyperboloid in three-dimensional Minkowski (dS_2) conveys this quite nicely. An entertaining problem is to imagine how we would recognise it from observations if the universe that we live in turned out to be de Sitter space.

2.3 Geodesics

A geodesic is a curve of 'shortest length' between two points. Their fundamental significance in general relativity is that test particles move along geodesics. This is Newton's first law of motion – *a particle moves along a straight line at constant speed unless acted upon by an external force* – the only change being to recognise that 'straight line' means the same as 'geodesic'. We first describe some basic properties of curves before deriving the equation obeyed by geodesics and hence that characterises the motion of test particles.

Suppose γ is a curve in our surface. We might describe it explicitly by giving an embedding function $\mathbf{c}(\tau)$, listing the points of $\mathbb{R}^3$ corresponding to the curve. It is clearly equivalent to specify the u^i, that parameterise our surface, as functions of τ, which parameterises our curve,

[10] From this we see that the global topology of de Sitter space-time is $\mathbb{R} \times S^3$, which is different from that of Minkowski, whose topology is $\mathbb{R} \times \mathbb{R}^3$.

since $\mathbf{X}(u^1(\tau), u^2(\tau)) = \mathbf{c}(\tau)$. Distance along the curve is given by either of

$$\begin{aligned} ds^2 &= d\mathbf{c}\cdot d\mathbf{c} = \left|\frac{d\mathbf{c}}{d\tau}\right|^2 d\tau^2, \\ ds^2 &= g_{ij}du^i du^j\Big|_\gamma = g_{ij}\frac{du^i}{d\tau}\frac{du^j}{d\tau}\,d\tau^2. \end{aligned} \tag{2.3.1}$$

It is routine to verify that these are equivalent, as they must be. The length of the curve is

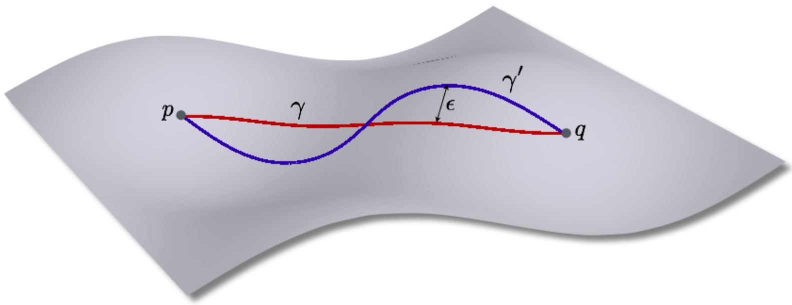
$$\int_\gamma ds = \int_{\tau_{\text{init}}}^{\tau_{\text{final}}} \sqrt{g_{ij}\frac{du^i}{d\tau}\frac{du^j}{d\tau}}\,d\tau. \tag{2.3.2}$$

The same description, and formulae, applies *mutatis mutandis* for a curve in a manifold of any dimension.

In giving an explicit description of our curve we introduced a parameter τ that labelled its points. There is complete freedom over the choice of how we do this; any parameterisation can be used. There is one, however, that is natural, for it has physical significance. That is to parameterise by arc length, *i.e.* choose $\tau = s$, so that the points are labelled by their actual physical distance along the curve[11]. With such a choice the *tangent vector* to the curve, $d\mathbf{c}/d\tau$ or $du^i/d\tau$, is properly normalised and has unit magnitude

$$\left|\frac{d\mathbf{c}}{d\tau}\right|^2 = 1, \qquad \Leftrightarrow \qquad g_{ij}\frac{du^i}{d\tau}\frac{du^j}{d\tau} = 1. \tag{2.3.3}$$

This parameterisation proves convenient in many calculations.



Let p and q be two points on our manifold and γ a curve connecting them. We can measure the length of γ, the distance between p and q along the given curve. If γ' is another curve that also connects p and q then we can measure its length too and ask if that distance is greater or less than the distance along γ. We see that the distance between p and q depends on the choice of curve connecting the two points; it is a function whose argument is a curve. A curve that is a critical point of this distance functional is called a *geodesic*. It is a curve whose length does not change to first order for any small change to the curve. Informally, it is the curve that represents the shortest distance between p and q.

Let us give a local characterisation of a geodesic curve γ. In a local chart it can be expressed as $u^i(\tau)$, where τ is parameterisation by arc length. Then if γ' is a curve close to γ we can describe it locally by the parameterisation $u^i(\tau) + \epsilon^i(\tau)$, where the ϵ^i are all small and vanish at the endpoints, since γ' also passes through the points p and q. We say that γ' is a *variation* of the curve γ. Let us compute its length. The metric for γ' is

$$ds^2 = g_{ij}\Big|_{\gamma'}\left(\frac{du^i}{d\tau}+\frac{d\epsilon^i}{d\tau}\right)\left(\frac{du^j}{d\tau}+\frac{d\epsilon^j}{d\tau}\right)d\tau^2, \tag{2.3.4}$$

$$= \left[g_{ij}\Big|_\gamma + \partial_k g_{ij}\Big|_\gamma \epsilon^k + O(2)\right]\left(\frac{du^i}{d\tau}\frac{du^j}{d\tau} + \frac{du^i}{d\tau}\frac{d\epsilon^j}{d\tau} + \frac{d\epsilon^i}{d\tau}\frac{du^j}{d\tau} + O(2)\right)d\tau^2, \tag{2.3.5}$$

$$= \left(1 + \partial_k g_{ij}\frac{du^i}{d\tau}\frac{du^j}{d\tau}\epsilon^k + 2g_{ij}\frac{du^i}{d\tau}\frac{d\epsilon^j}{d\tau} + O(2)\right)d\tau^2, \tag{2.3.6}$$

[11]We use this choice on all of our road systems, as the helpful roadside signs tell us.

using the fact that γ is parameterised by arc length, the symmetry of g_{ij} and a Taylor expansion of $g_{ij}|_{\gamma'}$ about the corresponding points of γ. It follows that the length of γ' is

$$\int_{\gamma'} ds = \int_{\tau_{\text{init}}}^{\tau_{\text{final}}} \left(1 + \partial_k g_{ij} \frac{du^i}{d\tau}\frac{du^j}{d\tau}\epsilon^k + 2g_{ij}\frac{du^i}{d\tau}\frac{d\epsilon^j}{d\tau} + O(2)\right)^{1/2} d\tau, \tag{2.3.7}$$

$$= \int_{\tau_{\text{init}}}^{\tau_{\text{final}}} \left(1 + \frac{1}{2}\partial_k g_{ij} \frac{du^i}{d\tau}\frac{du^j}{d\tau}\epsilon^k + g_{ik}\frac{du^i}{d\tau}\frac{d\epsilon^k}{d\tau} + O(2)\right) d\tau, \tag{2.3.8}$$

$$= \int_{\gamma} ds + \int_{\tau_{\text{init}}}^{\tau_{\text{final}}} \left[\frac{1}{2}\partial_k g_{ij} \frac{du^i}{d\tau}\frac{du^j}{d\tau} - \frac{d}{d\tau}\left(g_{ik}\frac{du^i}{d\tau}\right)\right]\epsilon^k\, d\tau + O(2), \tag{2.3.9}$$

using the fact that the ϵ^k all vanish at the endpoints. Since the ϵ^k are otherwise arbitrary, the condition for γ to be a curve of extremal length (critical point of the distance functional) is that the integrand vanish[12]. That is

$$g_{ik}\frac{d^2u^i}{d\tau^2} + \partial_j g_{ik}\frac{du^i}{d\tau}\frac{du^j}{d\tau} - \frac{1}{2}\partial_k g_{ij}\frac{du^i}{d\tau}\frac{du^j}{d\tau} = 0, \tag{2.3.10}$$

$$\Rightarrow \quad g_{ik}\frac{d^2u^i}{d\tau^2} + \frac{1}{2}\Big[\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\Big]\frac{du^i}{d\tau}\frac{du^j}{d\tau} = 0, \tag{2.3.11}$$

$$\Rightarrow \quad \frac{d^2u^i}{d\tau^2} + \frac{1}{2}g^{il}\Big[\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}\Big]\frac{du^j}{d\tau}\frac{du^k}{d\tau} = 0, \tag{2.3.12}$$

where in the last step we have used the inverse metric g^{il} and relabelled dummy indices. The quantities that appear in this expression

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\Big[\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}\Big], \tag{2.3.13}$$

are important; they are called the *Christoffel symbols*. To summarise, we have found that if γ is a geodesic then it has a local parameterisation $u^i(\tau)$ that satisfies the *geodesic equation*

$$\frac{d^2u^i}{d\tau^2} + \Gamma^i_{jk}\frac{du^j}{d\tau}\frac{du^k}{d\tau} = 0. \tag{2.3.14}$$

In deriving this equation we made use of the fact that γ was parameterised by arc length, meaning that its tangent vector has unit magnitude

$$g_{ij}\frac{du^i}{d\tau}\frac{du^j}{d\tau} = 1. \tag{2.3.15}$$

The geodesic equation (2.3.14) is a system of second order ordinary differential equations. Standard existence and uniqueness results in the theory of odes then tell us that if we specify an initial point and tangent vector, there is an unique geodesic passing through that point with the given initial tangent vector. Similarly, for points that are close enough together there is an unique geodesic connecting them.

Of course, we are primarily interested in geodesics in a Lorentzian manifold. They are almost entirely the same. If $x^\mu(\tau)$ is a local parameterisation of a geodesic then it satisfies the *geodesic equation*

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{2.3.16}$$

with the *Christoffel symbols* given in terms of the components of the metric by the same formula as above. The only difference is that the geodesic may be space-like, time-like, or null. Particles with non-zero mass travel along time-like geodesics. What this means is that their tangent vectors are time-like

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1. \tag{2.3.17}$$

[12] This is known as the fundamental lemma of the calculus of variations.

That the normalisation is '1' reflects the fact that the geodesic is parameterised by *proper time* (actually proper time times the speed of light), which has been the motivation for using the symbol τ. Massless particles, such as light, still satisfy the geodesic equation (2.3.16) but their tangent vector is null

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{2.3.18}$$

But the magnitude of the tangent vector aside, the description of geodesics in Lorentzian manifolds is the same as in the Riemannian setting.

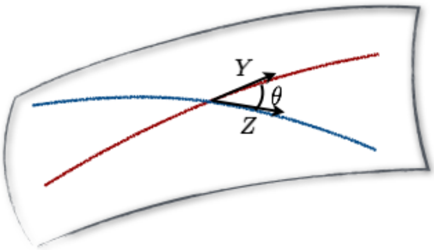
2.4 Angles, Areas, Volumes, etc.

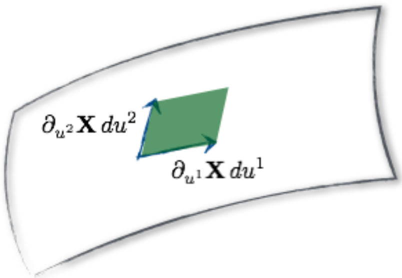
If two curves pass through the same point you can measure the angle between them at the point where they cross. We can write the two curves either as $\mathbf{c}_1(\tau_1), \mathbf{c}_2(\tau_2)$ or as $u^i_1(\tau_1), u^i_2(\tau_2)$. Then, if they are parameterised by arc length, the angle between them is given by

$$\cos(\theta) = \frac{d\mathbf{c}_1}{d\tau_1}\cdot\frac{d\mathbf{c}_2}{d\tau_2}\bigg|_p, \qquad \text{or} \qquad \cos(\theta) = g_{ij}\frac{du^i_1}{d\tau_1}\frac{du^j_2}{d\tau_2}\bigg|_p, \tag{2.4.1}$$

where p is the point at which they cross. If they are not parameterised by arc length then both expressions need to be divided by the magnitude of the tangent vectors to each curve at the point p. Equivalently, if $\mathbf{Y}, \mathbf{Z}$ are vectors at p then $\mathbf{Y}\cdot\mathbf{Z} = |\mathbf{Y}|.|\mathbf{Z}|.\cos(\theta)$ and an intrinsic expression for the same formula is

$$g_{ij}Y^iZ^j = \sqrt{g_{ij}Y^iY^j}.\sqrt{g_{kl}Z^kZ^l}.\cos(\theta). \tag{2.4.2}$$





Think of a surface embedded in $\mathbb{R}^3$; the sphere will do. We can measure the area of a patch of its surface. Consider a small patch of the surface with sides of length $\partial_{u^1}\mathbf{X}\,du^1$ and $\partial_{u^2}\mathbf{X}\,du^2$. Then it is well-known that the area of this patch can be expressed

$$dA = \left|\partial_{u^1}\mathbf{X}\times\partial_{u^2}\mathbf{X}\right| du^1du^2 = |\partial_{u^1}\mathbf{X}|.|\partial_{u^2}\mathbf{X}|.|\sin(\theta)|\,du^1du^2, \tag{2.4.3}$$

where θ is the angle between the two sides. Writing $|\sin(\theta)|$ as $\sqrt{1-\cos^2(\theta)}$, the area element can equivalently be expressed as

$$\begin{aligned} dA &= \sqrt{\left|\partial_{u^1}\mathbf{X}\right|^2\left|\partial_{u^2}\mathbf{X}\right|^2 - \left(\partial_{u^1}\mathbf{X}\cdot\partial_{u^2}\mathbf{X}\right)^2}\,du^1du^2, \\ &= \sqrt{g_{11}g_{22}-g_{12}g_{21}}\,du^1du^2, \\ &= \sqrt{\det g}\,du^1du^2. \end{aligned} \tag{2.4.4}$$

So we see that areas can be expressed in local coordinates in terms of the metric, which is useful if we need to do explicit calculations. Recall that for the sphere (S^2) the metric was $ds^2 = R^2(du^1)^2 + R^2\sin^2(u^1)\,(du^2)^2$ and we find that the area element in these coordinates is

$$dA = R^2\sin(u^1)\,du^1du^2. \tag{2.4.5}$$

The ideas presented here for surfaces in $\mathbb{R}^3$ apply *mutatis mutandis* to arbitrary n-dimensional manifolds embedded in some Euclidean space $\mathbb{R}^N$. The n-dimensional volume element can be written

$$d\text{vol} = \sqrt{\det g}\, du^1 \cdots du^n, \tag{2.4.6}$$

in terms of local coordinates for the manifold. To get at least a correct feeling for this, note that the formula is certainly correct if the metric is diagonal.

In the Lorentzian case, the presence of a time-like direction – minus sign in the metric – means that the determinant $\det g$ is negative. In recognition of this, there is a minor change to how we write the volume element

$$d\text{vol} = \sqrt{|\det g|}\, dx^0 dx^1 dx^2 dx^3, \tag{2.4.7}$$

where, as usual, the x^μ are local coordinates. This aside, there is no difference. This formula, (2.4.7), for the volume element in general coordinates should be remembered; it will be needed.

2.5 Vectors, 1-Forms, Tensors

The tangent to a curve is a vector; it is the velocity of a particle whose trajectory is the given curve. Indeed, in geometry this is taken as a definition: a *vector* at a point p is the tangent to a curve passing through p. If the curve is $\mathbf{c}(\tau)$, or equivalently $u^i(\tau)$, and p is the point corresponding to $\tau = 0$ then we might write

$$\left.\frac{d\mathbf{c}}{d\tau}\right|_{\tau=0} = Z_p, \qquad \text{or} \qquad \left.\frac{du^i}{d\tau}\right|_{\tau=0} = Z^i_p, \tag{2.5.1}$$

to define the vector Z_p. I will try to write consistently Z_p without boldface when I am trying to think of the manifold intrinsically and $\mathbf{Z}_p$ with boldface if I am thinking of the *same* vector from the perspective of some embedding space. The set of all possible tangents to curves passing through p forms the *tangent plane* T_pM to the manifold at p. It is a linear space with dimension n the same as the manifold M. A vector at p is any element of T_pM, an object with magnitude and direction. The collection of all tangent planes for every point of the manifold is called the *tangent bundle* TM. A choice of vector at every point p, varying smoothly with p, is called a *vector field*. We will write it Z. The (surface) fluid velocity in the oceans of the Earth, or the breeze of the air over the entire Earth's surface are highly instructive examples of vector fields on the sphere that are surely well-known to you.

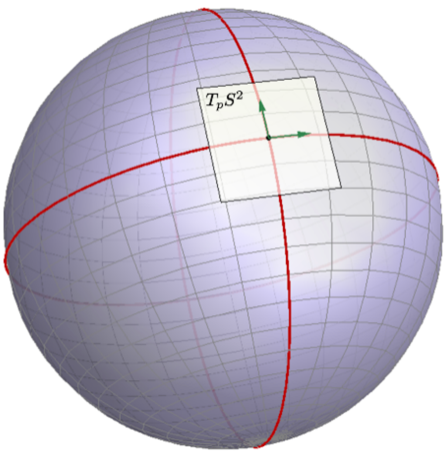


Figure 2.4: Tangent plane to the sphere at a general point p. Coordinate curves are shown, as well as the tangent vectors to these curves at p. Any vector at p can be expressed as a linear combination of these two.

Let u^i be coordinates for a local chart in the neighbourhood of a point p on some surface, say with embedding function $\mathbf{X}(u^1, u^2)$. The curves $\mathbf{X}(\tau, 0)$ and $\mathbf{X}(0, \tau)$ can be called *coordinate curves* passing through p. Their tangent vectors at p, $\partial_{u^1}\mathbf{X}$ and $\partial_{u^2}\mathbf{X}$, form a basis for the tangent space T_pM, since if $\mathbf{c}(\tau)$ is any curve passing through p then its tangent is

$$\left.\frac{d\mathbf{c}}{d\tau}\right|_{\tau=0} = \frac{\partial u^1}{\partial \tau}\,\partial_{u^1}\mathbf{X} + \frac{\partial u^2}{\partial \tau}\,\partial_{u^2}\mathbf{X} = \frac{\partial u^i}{\partial \tau}\,\partial_{u^i}\mathbf{X}. \tag{2.5.2}$$

Evidently this is true in general, for any manifold; the vectors $\partial_{u^i}\mathbf{X}$ that are tangent to coordinate curves form a basis for the tangent space at each point of the local chart covered by the coordinates. It is universal to denote these basis vectors by ∂_i when they are viewed intrinsically, without reference to any embedding space. Although the symbol is the same, and it will be treated much the same in many calculations, it is not the *partial derivative*; it is the tangent vector to the coordinate curve u^i. This notation is universal and has to be acclimatised to. Thus any vector Z_p can be expressed *in a coordinate basis* as

$$Z_p = Z_p^i\,\partial_i, \tag{2.5.3}$$

where the numbers Z_p^i are the *components* of the vector in the coordinate basis.

A function f is a rule that assigns a real number to every point p of a manifold, and varies smoothly with p. We wish to convey what is meant by the derivative of a function. A good question to ask is: "in which direction?" Given a direction Z_p there is a curve $\mathbf{c}(\tau)$ passing through p whose tangent vector at that point is Z_p. We may then compare the values of the function f as we move along the curve $\mathbf{c}$ and compute the derivative

$$\left.\frac{df}{d\tau}\right|_{\tau=0}, \tag{2.5.4}$$

known as the *directional derivative* of f at the point p, in the direction Z_p. Usually this is denoted $Z_p(f)$. The *derivative* of f, therefore, is an object that operates on vectors to obtain directional derivatives. We denote it by

$$\begin{aligned} df:\ & T_pM \to \mathbb{R} \\ & Z_p \to df(Z_p) = Z_p(f) \end{aligned} \tag{2.5.5}$$

It is a linear map from the tangent space at p to the real line; such things are called *1-forms.* They form a linear vector space dual to the tangent space, called the *cotangent space* and denoted T_p^*M. The collection of cotangent spaces for every point p of the manifold is called the *cotangent bundle*. In addition to the derivative of a function, the electromagnetic gauge field A is a commonly encountered example of a 1-form; as are the reciprocal lattice vectors in crystallography; and in classical mechanics one learns that momentum should be properly viewed as a 1-form.

In any local chart the coordinates u^i are functions. Their derivatives du^i are therefore 1-forms. Although the symbol is the same, and in many calculations they will be treated much the same, these are not *integration measures*; they are the derivatives of coordinate functions. This notation is universal and has to be acclimatised to. The 1-forms du^i form a basis for the cotangent space, for

$$df = \frac{\partial f}{\partial u^i}\,du^i. \tag{2.5.6}$$

This basis is dual to the coordinate basis ∂_i for the tangent space, since by unfolding the various definitions

$$du^i\big(\partial_j\big) = \partial_j(u^i) = \frac{\partial u^i}{\partial u^j} = \delta^i_j. \tag{2.5.7}$$

It is a good exercise to go through this result for yourself. In a coordinate basis any 1-form ω may be written $\omega_i\,du^i$, where ω_i are the *components* of the 1-form in the coordinate basis.

All this is exactly the same in a Lorentzian manifold. If x^μ is a local coordinate system then ∂_μ is a basis for the tangent space and dx^μ a basis for the cotangent space. These bases are dual $dx^\mu(\partial_\nu) = \delta^\mu_\nu$. The geometric meaning is exactly the same and the only noticeable change is the use of Greek indices rather than Latin ones[13]. This given, we can define a general tensor. In physics, a tensor is a natural physical quantity[14] that generalises the concept of a vector by exhibiting *multilinearity*. Examples include the stress tensor in a fluid, the dielectric tensor in electromagnetism, the Maxwell field strength tensor, the conductivity tensor (electrical or thermal) and the elasticity tensor in an elastic solid. Mathematically, a type $\binom{k}{l}$ tensor is a multilinear map that at each point p of a manifold takes k 1-forms and l vectors and returns a number, that number varying smoothly with p[15]

$$\begin{aligned} T:\ &\underbrace{T^*_pM\times\cdots\times T^*_pM}_{k\text{ times}}\times\underbrace{T_pM\times\cdots\times T_pM}_{l\text{ times}}\to\mathbb{R} \\ &\omega_p\quad\times\cdots\times\quad\lambda_p\quad\times\quad Y_p\quad\times\cdots\times\quad Z_p\quad\to T(\omega_p,\cdots,\lambda_p,Y_p,\cdots,Z_p). \end{aligned} \tag{2.5.8}$$

As an example of importance, the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ can be thought of as a symmetric type $\binom{0}{2}$ tensor. The metric tells you the distance between two nearby points, $p = x^\mu$ and $q = x^\mu + dx^\mu$ say. It is a quadratic function of the tangent Z_p to a small line segment joining the two points[16]

$$ds^2 = g_{\mu\nu}Z^\mu_p Z^\nu_p\, d\tau^2. \tag{2.5.9}$$

Thus it defines a quadratic form $Q(Z_p) = g_{\mu\nu}Z^\mu_p Z^\nu_p$ and in the usual way a symmetric bilinear on the tangent space

$$g(Y_p, Z_p) = \frac{1}{2}\Big[Q(Y_p + Z_p) - Q(Y_p) - Q(Z_p)\Big] = g_{\mu\nu}Y^\mu_p Z^\nu_p, \tag{2.5.10}$$

that, for obvious reasons, is also called the *metric*. By construction it is symmetric, $g(Y_p, Z_p) = g(Z_p, Y_p) \Leftrightarrow g_{\mu\nu} = g_{\nu\mu}$. That it is linear means

$$g(fX_p + Y_p, Z_p) = f\,g(X_p, Z_p) + g(Y_p, Z_p), \tag{2.5.11}$$

for any three vectors X_p, Y_p, Z_p and function f. Multilinearity of a tensor means the same thing; it is linear with respect to each object it acts on.

In traditional presentations in the physics literature the discussion of tensors is long and centred around their components in a coordinate basis. I find that it is tiresome and does not convey any physical content or understanding. Take any of the examples that I gave, say conductivity[17]. A potential difference is applied to a material so that it experiences an electric field. The response, for small applied fields, is that a current flows through the material. Conductivity is the material property that converts the applied electric field into the current response; symbolically $\mathbf{J} = \boldsymbol{\sigma}\cdot\mathbf{E}$. It acts on a vector ($\mathbf{E}$) and returns a vector ($\mathbf{J}$). So long as the applied electric field is small the response is linear; double the electric field and the current response doubles; superpose two electric fields to create a third and the current response is the linear superposition of the response to each separate field. Thus the conductivity is a linear map from a vector to another vector, or from a vector space to itself. It is naturally defined as a material property entirely independently of any choice of coordinate system or basis for the vector space. This is what is meant, in physics at least, by a type $\binom{1}{1}$ tensor. The same holds for anything that is a tensor, of any type. To my mind it is the physics that is important and

[13]We should remember, of course, that in the Lorentzian setting tangent vectors can be space-like, time-like, or null.

[14]Not every physical quantity is a tensor; spin $\frac{1}{2}$ particles like the electron are not.

[15]When I first started as a postdoc, my supervisor came into my office and exclaimed: "Hey, Gareth! Do you know that tensors are toasters?" It took about 5 seconds to parse what he had said before I replied: "Yes, of course I know that."

[16]That Z_p is the tangent means $dx^\mu/d\tau = Z^\mu_p$.

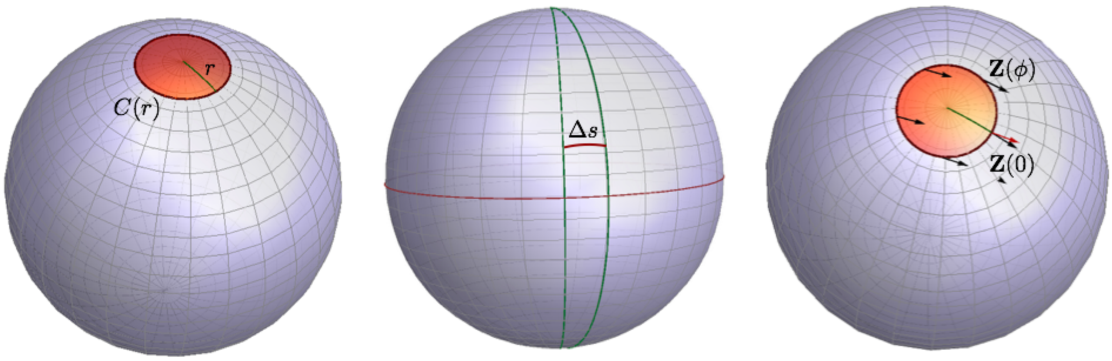
[17]The quantum Hall effect is a beautiful experiment; think of it.

should be emphasised first and foremost. If you understand what the quantity is and what it means to measure it; if you know what it depends on and what it influences; if you can describe how changes in the experimental set-up affect the measurement that is made; if you can explain all this to a fellow student, then you understand what a tensor is. If you cannot, then you don't. Knowledge of raising and lowering of indices, contractions, outer products and transformation laws for components under a change of coordinates seem of pale secondary importance compared to genuine physical understanding.

For the vast majority of physical applications of general relativity only three tensors ever appear; the metric, the stress-energy-momentum tensor, and the Ricci tensor. The latter gives a characterisation of the curvature of space-time. I will try, in the next section, to give a natural geometric description of curvature that emphasises how it manifests itself in physical observations and measurements; the traditional tensor-calculus properties will fall out by themselves, and, in my opinion, in the proper fashion.

2.6 Curvature

In an attempt to introduce curvature in a pedagogic way and try to promote geometric ways of thinking (prior to the deduction of algebraic formulae), I want to describe four concepts of curvature, each of them illustrated with the humble/venerable sphere. These four concepts are: sectional curvature associated to the circumference of a circle of small geodesic radius; Gaussian curvature associated to surfaces embedded in $\mathbb{R}^3$; geodesic deviation, or the failure of parallel straight lines to maintain a fixed separation; and the rotation of a vector under parallel transport. We begin with sectional curvature.



sectional curvature *geodesic deviation* *parallel transport*

Take any point p on a surface and think of the geodesics passing through that point. If a collection of small test particles should move distance r along those geodesics they will form a small circle, of geodesic radius r. Geodesics on the sphere are *great circles* – the intersection of the sphere with planes through the origin in $\mathbb{R}^3$. In standard polar coordinates the lines of longitude are all geodesics passing through the north pole (and south pole). They can be parameterised as

$$R\sin(r/R)\big[\cos(\phi)\mathbf{e}_1 + \sin(\phi)\mathbf{e}_2\big] + R\cos(r/R)\mathbf{e}_3, \qquad (2.6.1)$$

where r is arc length along the geodesic making angle ϕ with the 'Greenwich meridian'. Fixing r, arbitrarily small, and letting ϕ vary through a full 2π sweeps out a circle of geodesic radius r. Its circumference is

$$C(r) = \int_0^{2\pi} R\sin(r/R)\,d\phi = 2\pi R\sin(r/R) = 2\pi r\Big[1 - \frac{1}{6}\frac{1}{R^2}r^2 + O(r^4)\Big]. \qquad (2.6.2)$$

The deviation of the circumference from the value $2\pi r$ detects the curvature of the sphere[18].

[18]Indeed, on the Earth (radius 6371 km) for a circle of geodesic radius 11 km (distance between the University of Warwick and Leamington Spa) the shortfall is 3.4 cm.

This is illustrative of a general formula

$$C(r) = 2\pi r\Big[1 - \frac{1}{6}K\,r^2 + O(r^3)\Big], \tag{2.6.3}$$

for any such circle of geodesic radius r in a manifold of any dimension. The number K is called a *sectional curvature*. It depends on the choice of circle as follows. Each point of the circumference is connected to the centre, p, by an unique geodesic, which we can label by its tangent vector at p. The set of all such tangent vectors associated to the circle define a two-plane in the tangent space at p. Vice-versa, any choice of two-plane in the tangent space at p allows us to construct a circle. The sectional curvature K depends on this two-plane. The collection of all sectional curvatures, for every point of the manifold, gives a complete characterisation of the curvature of the space.

By way of a simple example, geodesics on the 3-sphere, passing through the point $(0, 0, 0, R)$, can be parameterised as

$$R\sin(r/R)\,\mathbf{n} + R\cos(r/R)\,\mathbf{e}_4, \tag{2.6.4}$$

where $\mathbf{n}$ is a unit vector orthogonal to $\mathbf{e}_4$, *i.e.* a point on the 2-sphere. Any plane in the tangent space at $(0, 0, 0, R)$ intersects that 2-sphere in a 'great circle'. If we parameterise $\mathbf{n}$ by spherical polar angles such that the intersection is the equator, then the calculation of the sectional curvature of the 2-plane is identical to the one we have just given, showing that $K = 1/R^2$ for every choice of 2-plane. The result also holds for every point of the 3-sphere; there is nothing special about $(0, 0, 0, R)$. It is not hard to see that the same analysis holds for S^n. Thus the sphere is a space of constant positive curvature. In fact this is an unique characteristic; no other manifold has the same property.

The second concept of curvature I wish to describe is the *Gaussian curvature*, defined for surfaces embedded in $\mathbb{R}^3$. Let p be a point on such a surface and consider the direction that is normal to the surface at p. This is called the *Gauss map*; an association to each point of the surface of a point on S^2 corresponding to the direction of the surface normal at that point. Now consider a small patch of the surface centred at p. To each point of this little patch we associate its corresponding point on the 2-sphere under the Gauss map, its *spherical image*. The ratio of the area of the spherical image[19] to the area of the patch of surface, in the limit where the size of the patch shrinks to nothing, is taken to define the Gaussian curvature

$$K_{\rm G} = \lim_{\text{area}\to 0} \frac{\text{area spherical image}}{\text{area patch of surface}}. \tag{2.6.5}$$

Gauss was so enamoured by his proof that $K_{\rm G} = K$ that he called it the *theorema egregium*[20] – in modern parlance *this theorem is sick!* The demonstration for the sphere is elementary. For a sphere of unit radius the Gauss map is the identity and $K_{\rm G} = 1$. By simple scaling $K_{\rm G} = 1/R^2$ for the sphere of radius R. You are challenged to find for yourself a general proof for any surface embedded in $\mathbb{R}^3$.

The third concept of curvature I wish to describe is *geodesic deviation*. The equator is a great circle on the Earth. Pick two points on it, close to each other. The lines of longitude through those two points are also great circles both of which meet the equator at right angles. They are parallel at the equator. But this is not so if you move along them away from the equator, either north or south. In both directions they bend towards each other and eventually meet at the two poles. The failure of initially parallel geodesics to remain parallel, or to maintain a fixed separation, is a signature of curvature, known as *geodesic deviation*. The analogous convergence, or divergence, of test particles moving along geodesics of the space-time metric in general relativity are known as *tidal forces*. They provide one means, at least

[19]The area of the spherical image may be figured positive if the Gauss map in the neighbourhood of p is orientation preserving and negative if it is orientation reversing.

[20]The theorem is not usually stated as I have implied, but its content is the same; the Gaussian curvature can be computed just using the metric.

in principle, of measuring curvature experimentally, and hence the metric, by observing the motion of test particles. Let us describe the geodesic deviation for the sphere. Let the two geodesics initially tangent at the equator be parameterised as

$$R\cos(r/R)\big[\cos(\phi)\mathbf{e}_1 \pm \sin(\phi)\mathbf{e}_2\big] + R\sin(r/R)\mathbf{e}_3, \tag{2.6.6}$$

where r is arc length along the geodesic measured from the equator. The geodesic connecting corresponding points of these two lines of longitude is

$$R\cos(s/R)\big[\cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_3\big] + R\sin(s/R)\mathbf{e}_2, \tag{2.6.7}$$

where s is arc length along the geodesic measured from the point of latitude α where it crosses the Greenwich meridian. The geodesic separation between corresponding points of our lines of longitude is therefore

$$\Delta s = 2R\arcsin\big(\cos(r/R)\sin(\phi)\big). \tag{2.6.8}$$

If the initial separation Δs_0 is very small, $\phi \ll 1$, then this is approximately

$$\Delta s \approx 2R\phi\,\cos(r/R) = \Delta s_0\,\cos(r/R), \tag{2.6.9}$$

which can be viewed as a solution of

$$\frac{d^2\Delta s}{dr^2} + \frac{1}{R^2}\,\Delta s = 0, \tag{2.6.10}$$

a version of the *geodesic deviation equation* for the sphere. The relative acceleration of nearby geodesics is controlled by the curvature; so observing such tidal forces provides a method for measuring curvature and from it the space-time metric.

The final concept of curvature that I will introduce is the one that we will subsequently adopt with full force as the definition. It is a careful measure of the change in a vector under *parallel transport* around a small closed loop. If a vector should be transported around a closed loop in flat space, in such a manner that it does not undergo any rotation at any time, then it will return to its original location as the same vector; there is no change. This is not so in a curved space. It takes a few concepts to describe the situation properly.

In flat space geodesics are straight lines, meaning that the tangent vector to the geodesic does not change as one moves along it. The same is true in general; geodesics are 'straight lines', suitably interpreted. Consider a sphere embedded in $\mathbb{R}^3$. The geodesics are great circles. The tangent vector to the geodesic, viewed as a vector in $\mathbb{R}^3$, changes as you move along it, but it does so in a very particular way. It only changes in the direction that is normal to the surface; there is no change in the tangential directions. This affords a second definition of geodesic: a geodesic is a curve whose tangent vector only changes in directions normal to the manifold (viewed as embedded in a Euclidean space). We say that the tangent vector is *parallel transported* along itself. You are asked to show in one of the problems that this definition coincides (locally) with the previous one; for now we take it on trust. The notion extends to any vector transported along any curve. If the vector changes only into the normal directions as it is transported along the curve we will say that it is *parallel transported.* This is the notion of parallelism given by Levi-Civita. A final remark that should be emphasised is that the magnitude of the vector should also be preserved under parallel transport; that is, it only experiences a rotation. Likewise, if we parallel transport two vectors we should preserve the angle between them. This condition is called *metric compatibility.*

We can now give a geometric definition of Riemann curvature. At any point p select a vector Z_p in the tangent space at that point. Now consider any circle of geodesic radius r centred on p, just as we did in defining the sectional curvature, and parallel transport Z_p once around this circle. If the space is curved the vector will not come back precisely the same. The change is proportional to the area πr^2 of the geodesic disc. This change is the Riemann curvature. It depends on the 2-plane defined by the choice of circle to transport around.

The Riemann curvature is a sophisticated thing. Let us compute it for the 2-sphere. Choose for p the north pole $(0, 0, R)$ and for Z_p the vector $\mathbf{e}_1$ of $\mathbb{R}^3$ tangent to the sphere at the north pole. A circle of geodesic radius r centred at the north pole can be described by the explicit embedding function

$$\mathbf{c}(\phi) = R\sin(r/R)\big[\cos(\phi)\mathbf{e}_1 + \sin(\phi)\mathbf{e}_2\big] + R\cos(r/R)\mathbf{e}_3. \tag{2.6.11}$$

To get things started we first have to parallel transport the vector Z_p from the north pole to the circle. If we do it along the 'Greenwich meridian' we will obtain the vector $\mathbf{Z}(0) = \cos(r/R)\mathbf{e}_1 - \sin(r/R)\mathbf{e}_3$. To describe its transport around the circle we introduce a set of orthonormal basis vectors (for $\mathbb{R}^3$). The first choice you might think of are the 'spherical polars'

$$\mathbf{e}_\theta = \cos(r/R)\big[\cos(\phi)\mathbf{e}_1 + \sin(\phi)\mathbf{e}_2\big] - \sin(r/R)\mathbf{e}_3, \tag{2.6.12}$$

$$\mathbf{e}_\phi = -\sin(\phi)\mathbf{e}_1 + \cos(\phi)\mathbf{e}_2, \tag{2.6.13}$$

$$\mathbf{n} = \sin(r/R)\big[\cos(\phi)\mathbf{e}_1 + \sin(\phi)\mathbf{e}_2\big] + \cos(r/R)\mathbf{e}_3. \tag{2.6.14}$$

This choice can be used and you will compute the correct value for the change in Z_p under parallel transport, however, the vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are not well-defined at the north pole ($r = 0$) itself. It is better to use a basis without such a defect. One choice is to use

$$\mathbf{s}_1 = \cos(\phi)\mathbf{e}_\theta - \sin(\phi)\mathbf{e}_\phi, \tag{2.6.15}$$

$$\mathbf{s}_2 = \sin(\phi)\mathbf{e}_\theta + \cos(\phi)\mathbf{e}_\phi, \tag{2.6.16}$$

in their place. This basis is still orthonormal. In terms of it we can write the parallel transport of $\mathbf{Z}(0)$ as

$$\mathbf{Z}(\phi) = \cos\big(\alpha(\phi)\big)\mathbf{s}_1 + \sin\big(\alpha(\phi)\big)\mathbf{s}_2. \tag{2.6.17}$$

It remains to determine the function $\alpha(\phi)$. A direct calculation gives how $\mathbf{Z}(\phi)$ changes

$$\begin{aligned}\frac{d\mathbf{Z}}{d\phi} &= \left[-\frac{d\alpha}{d\phi} + \big(1 - \cos(r/R)\big)\right]\sin(\alpha)\,\mathbf{s}_1 + \left[\frac{d\alpha}{d\phi} - \big(1 - \cos(r/R)\big)\right]\cos(\alpha)\,\mathbf{s}_2 \\ &\quad + \Big[\cos(\alpha)\sin(\phi) - \sin(\alpha)\cos(\phi)\Big]\sin(r/R)\,\mathbf{n},\end{aligned} \tag{2.6.18}$$

from which we find that the vector is parallel transported if

$$\frac{d\alpha}{d\phi} - 2\sin^2(r/2R) = 0, \qquad \Rightarrow \qquad \alpha = 2\phi\sin^2(r/2R). \tag{2.6.19}$$

After transport around the entire circle $\mathbf{Z}(0)$ returns as the vector

$$\mathbf{Z}(2\pi) = \cos\big(4\pi\sin^2(r/2R)\big)\mathbf{s}_1 + \sin\big(4\pi\sin^2(r/2R)\big)\mathbf{s}_2. \tag{2.6.20}$$

Finally, we need to parallel transport this back to the north pole, along the Greenwich meridian, which produces the vector

$$\cos\big(4\pi\sin^2(r/2R)\big)\mathbf{e}_1 + \sin\big(4\pi\sin^2(r/2R)\big)\mathbf{e}_2 = \mathbf{e}_1 + \pi r^2\frac{1}{R^2}\mathbf{e}_2 + O(r^4). \tag{2.6.21}$$

As promised, there is a change, proportional to πr^2, with a coefficient that measures the curvature of the sphere.

REMARK: The rotation of a vector under parallel transport around a line of latitude on a sphere is something that you can observe directly. It is the rotation of the plane of oscillation of a Foucault pendulum; transport around a line of latitude is provided by the rotation of the Earth. Hopefully this both piques your interest in Foucault pendulums and helps you to appreciate what curvature means in concrete physical terms for at least one situation.

Taking what we have learnt we can now describe the general situation for any manifold. The change in Z_p depends on the choice of circle to transport it around. As described previously, any such circle can be identified with a 2-plane in the tangent space at p. Any two orthogonal unit vectors X_p, Y_p define a 2-plane at p, so it is just the same to select any two such vectors. The change in the vector Z_p under parallel transport around a circle of geodesic radius r, defined by the vectors X_p, Y_p, can be written as

$$Z_p \mapsto Z_p - \pi r^2 \, R(X_p, Y_p) Z_p + O(r^3), \tag{2.6.22}$$

with $R(X_p, Y_p)Z_p$ being a vector in the tangent space at p. It is linear in each of X_p, Y_p, Z_p and so defines a type $\binom{1}{3}$ tensor at p. This is the *Riemann curvature tensor*. The minus sign that appears in the definition is chosen so that subsequent formulae have their conventional signs. The Riemann tensor is antisymmetric in X_p, Y_p since interchange of these reverses the orientation of the 2-plane they define and hence the direction in which Z_p is parallel transported.

REMARK: For any choice of circle to transport around, the Riemann curvature is a linear transformation on the tangent space at p, defined by $R(X_p, Y_p) : Z_p \to R(X_p, Y_p)Z_p$. This linear transformation depends on two vectors X_p, Y_p and is antisymmetric with respect to interchange of them, $R(X_p, Y_p) = -R(Y_p, X_p)$. Therefore, the Riemann curvature is a 2-form (a skew-symmetric bilinear on the tangent space), whose values when acting on a pair of vectors, instead of being numbers, happen to be a linear transformation on T_pM. One can say that it is an $\mathrm{End}(TM)$-valued 2-form[a]. This way of thinking about curvature – as a 2-form – is common in many modern areas, particularly those involving geometric phases or non-Abelian gauge theories.

[a] The linear transformations of a vector space V to itself are called endomorphisms and the space of all such is denoted $\mathrm{End}(V)$.

The definition given for the Riemann curvature tensor is natural and geometric. It turns out, like all things that are intrinsically geometric, that it can be computed directly from the components of the metric. Knowledge of this is crucial in general relativity where the space-time metric is determined by equating curvature with the stress-energy-momentum tensor of the matter content – the Einstein equations. We show now how curvature is related to the metric. The calculation is identical in structure to what we just did for S^2, so you should bear that in mind to guide you through the details. As before, let Z_p be a vector at p that we wish to parallel transport around a circle of geodesic radius r defined by the 2-plane given by the orthonormal vectors X_p, Y_p. Now let $\mathbf{s}_i$ be an orthonormal basis for the tangent space to our manifold in a local chart about p. Without loss of generality we can take $\mathbf{s}_1, \mathbf{s}_2$ to correspond to X_p, Y_p, respectively, at p. Then, as before, on the circle we can write the parallel transport of Z_p as

$$\mathbf{Z}(\phi) = Z^i(\phi)\, \mathbf{s}_i(\phi), \tag{2.6.23}$$

and compute its derivative along the circle to be

$$\frac{d\mathbf{Z}}{d\phi} = \frac{dZ^i}{d\phi}\mathbf{s}_i + Z^j \frac{d\mathbf{s}_j}{d\phi} = \left[\frac{dZ^i}{d\phi} + Z^j \omega^i_j\right]\mathbf{s}_i + (\cdots)\,\mathbf{n}, \tag{2.6.24}$$

where we have defined

$$\omega^i_j = \mathbf{s}_i \cdot \frac{d\mathbf{s}_j}{d\phi}. \tag{2.6.25}$$

These are (essentially) the components of the *connection*. The change in the normal directions – $(\cdots)\mathbf{n}$ – is unimportant; hence, we have not been careful in how we write it. The condition for parallel transport of the vector is that

$$\frac{dZ^i}{d\phi} + \omega^i_j Z^j = 0. \tag{2.6.26}$$

These equations would be easy to solve if there was only a single equation or the ω^i_j were diagonal, for then we would have $Z^i(\phi) = \mathrm{e}^{-\int^\phi \omega^i_j\, d\phi'} Z^j(0)$. The actual solution is often written in a similar form to this as

$$Z^i(\phi) = \mathcal{P}\,\mathrm{e}^{-\int^\phi \omega^i_j\, d\phi'} Z^j(0), \tag{2.6.27}$$

and called a *path ordered exponential.* The ordering is necessary because the ω^i_j are matrices and matrices do not commute. These things arise in field theory or anything using the Feynman path integral, non-Abelian gauge theory, and even in time-dependent perturbation theory in quantum mechanics. We will not really need them here, but there is no harm in mentioning it in passing. To convey what is meant, we solve the equation for parallel transport perturbatively, thinking of ω^i_j as small, writing $Z^i = Z^i_{(0)} + Z^i_{(1)} + Z^i_{(2)} + \cdots$ and solving order-by-order. At zeroth order we have

$$\frac{dZ^i_{(0)}}{d\phi} = 0, \qquad \Rightarrow \qquad Z^i_{(0)}(\phi) = Z^i(0)\,; \tag{2.6.28}$$

at first order

$$\frac{dZ^i_{(1)}}{d\phi} + \omega^i_j Z^j_{(0)} = 0, \qquad \Rightarrow \qquad Z^i_{(1)}(\phi) = -\int_0^\phi \omega^i_j\, d\phi'\; Z^j(0)\,; \tag{2.6.29}$$

and at second order

$$\frac{dZ^i_{(2)}}{d\phi} + \omega^i_j Z^j_{(1)} = 0, \qquad \Rightarrow \qquad Z^i_{(2)}(\phi) = \int_0^\phi \omega^i_m \left(\int_0^{\phi'} \omega^m_j\, d\phi''\right) d\phi'\; Z^j(0). \tag{2.6.30}$$

We find, then, that the parallel transport of $Z^i(0)$ around a small circle of geodesic radius r returns

$$Z^i(2\pi) = Z^i(0) - \int_{C(r)} \omega^i_j\, d\phi\; Z^j(0) + \int_{C(r)} \omega^i_m \left(\int_0^\phi \omega^m_j\, d\phi'\right) d\phi\; Z^j(0) + O(r^3). \tag{2.6.31}$$

We compute the change, treating each of the two pieces in turn. In doing so, it is convenient to introduce a local Cartesian coordinate system, x^1, x^2, for points of the geodesic disc $D(r)$ whose boundary is the circle $C(r)$. In these local coordinates we write

$$\omega^i_j = \mathbf{s}_i \cdot \frac{d\mathbf{s}_j}{d\phi} = \mathbf{s}_i \cdot \partial_k \mathbf{s}_j \frac{dx^k}{d\phi} = \Gamma^i_{kj} \frac{dx^k}{d\phi}. \tag{2.6.32}$$

The symbols Γ^i_{kj} that appear here are the *components of the connection.* We use the same notation for them as we did for the Christoffel symbols in anticipation of the result that such an identification can be made in the final expression for the Riemann curvature tensor. It then follows immediately from Stokes' theorem that

$$\int_{C(r)} \omega^i_j\, d\phi = \int_{C(r)} \Gamma^i_{kj}\, dx^k = \int_{D(r)} \left[\partial_1 \Gamma^i_{2j} - \partial_2 \Gamma^i_{1j}\right] dx^1 dx^2, \tag{2.6.33}$$

$$= \pi r^2 \left[\partial_1 \Gamma^i_{2j} - \partial_2 \Gamma^i_{1j}\right]\Big|_p + O(r^3). \tag{2.6.34}$$

The second contribution to the parallel transport is a little more subtle. We first evaluate the inner integral as[21]

$$\int_0^\phi \omega^m_j\, d\phi' = \int^x \Gamma^m_{lj}\, dy^l = \Gamma^m_{lj} x^l + O(r^2). \tag{2.6.35}$$

[21] This is integration by parts together with an estimate for the magnitude of the neglected term.

This given, the integral can again be computed with the help of Stokes' theorem and we find

$$\int_{C(r)} \omega^i_m \left(\int_0^\phi \omega^m_j \, d\phi'\right) d\phi = \int_{C(r)} \Gamma^i_{km}\Gamma^m_{lj} x^l \, dx^k + O(r^3), \tag{2.6.36}$$

$$= \int_{D(r)} \left[\partial_1 \left(\Gamma^i_{2m}\Gamma^m_{lj} x^l\right) - \partial_2 \left(\Gamma^i_{1m}\Gamma^m_{lj} x^l\right)\right] dx^1 dx^2 + O(r^3), \tag{2.6.37}$$

$$= \int_{D(r)} \left[\Gamma^i_{2m}\Gamma^m_{1j} - \Gamma^i_{1m}\Gamma^m_{2j}\right] dx^1 dx^2 + O(r^3), \tag{2.6.38}$$

$$= \pi r^2 \left[\Gamma^i_{2m}\Gamma^m_{1j} - \Gamma^i_{1m}\Gamma^m_{2j}\right]\Bigg|_p + O(r^3). \tag{2.6.39}$$

In summary, we have shown that

$$R(X_p, Y_p) Z_p = \left(\partial_1 \Gamma^i_{2j} - \partial_2 \Gamma^i_{1j} + \Gamma^i_{1m}\Gamma^m_{2j} - \Gamma^i_{2m}\Gamma^m_{1j}\right) Z^j_p, \tag{2.6.40}$$

$$= \left(\partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}\right) X^k_p Y^l_p Z^j_p. \tag{2.6.41}$$

The numbers appearing in this formula

$$R^i_{klj} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}, \tag{2.6.42}$$

are the *components of the Riemann curvature tensor* in the given basis. Precisely the same formula gives the components of the Riemann curvature tensor in the Lorentzian setting

$$R^\alpha_{\mu\nu\beta} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\mu\beta}, \tag{2.6.43}$$

the only noticeable change being to use Greek indices rather than Latin ones. From the Riemann tensor we define two other measures of curvature, that ultimately are the ones that appear in the Einstein equations. First, the *components of the Ricci tensor* are defined by *contraction* of the components of Riemann

$$R_{\mu\nu} = R^\alpha_{\alpha\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}. \tag{2.6.44}$$

The Ricci tensor is a symmetric type $\binom{0}{2}$ tensor, $R_{\mu\nu} = R_{\nu\mu}$. Second, the contraction of the Ricci tensor with the inverse metric, $g^{\mu\nu}$, defines the *Ricci scalar*

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.6.45}$$

REMARK: The result (2.6.41) should probably be given the fanfare it deserves, for it is a serious calculation. It represents the result of parallel transport of an arbitrary vector around an arbitrary circle in an arbitrary Riemannian manifold of arbitrary dimension. It is almost incredulous that it is even possible to calculate this.

REMARK: We have, in fact, proved more than we claimed. We have worked exclusively with parallel transport around a *circle* of geodesic radius r, but nowhere was it essential that it was a circle. What was essential was that it is a closed loop associated to a choice of 2-plane at p. The points of the loop do not all have to be at the same geodesic distance from p. The only change this makes in the definition of the Riemann curvature is to replace πr^2 with the area of the geodesic disc that the loop bounds.

2.7 Covariant Derivative

The way that I have introduced the Riemann curvature tensor is not the way you will see in any of the textbooks. There, the following definition is given

$$\begin{aligned} R(X,Y)Z &= \Big(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\Big)Z, \\ \text{or} \qquad R^{\alpha}_{\mu\nu\beta}Z^{\beta} &= \big(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\big)Z^{\alpha}, \end{aligned} \tag{2.7.1}$$

depending on whether it is a maths textbook or a physics textbook. The symbol $\nabla_X Z$ is the *covariant derivative* of Z along the vector field X, to be described shortly. After this definition is given, one then demonstrates its geometric meaning in terms of parallel transport, as I described above. The mathematicians prefer this approach. The reason is that the concept of a connection (covariant derivative) on a manifold can be defined independently of a metric. The expression given above then defines the curvature for any manifold, whether it has a metric or not. This generality and separation of concepts is common in mathematics and it is certainly clear that there is good reason to adopt this approach. I have not felt the need to do so because it is fundamental in general relativity that we can measure distances and intervals of time, *i.e.* that we have a metric. Indeed, the metric is the physical quantity that the theory is all about; it seems reasonable, in this context, to give it a central position in our discussion.

Let us try to outline what (2.7.1) means by defining the covariant derivative. We take a formal mathematical perspective for this purpose. A *connection* ∇ is a rule for differentiating vectors. If Z is a vector then ∇Z is called its *covariant derivative*. Its formal definition is given as follows. For any vector Z the covariant derivative ∇Z is a type $\binom{1}{1}$ tensor. In addition, the connection should satisfy the Leibniz formula

$$\nabla\big(fZ\big) = df \otimes Z + f\nabla Z, \tag{2.7.2}$$

for any smooth function f. *I.e.* the covariant derivative of a function is the ordinary derivative of that function. A vector is said to be *covariantly constant* if $\nabla Z = 0$. The covariant derivative of Z along the vector field X is a type of 'directional derivative'; it is written $\nabla_X Z$. For any pair of vectors X, Z it is another vector whose basic properties are the Leibniz formula

$$\nabla_X\big(fZ\big) = X(f)\,Z + f\nabla_X Z, \tag{2.7.3}$$

and linearity with respect to X, meaning

$$\nabla_{fX+Y}Z = f\nabla_X Z + \nabla_Y Z. \tag{2.7.4}$$

A vector is said to be *parallel transported* along the vector field X if $\nabla_X Z = 0$.

It can be shown that the covariant derivative ∇Z is a local object[22]. It is therefore sufficient to study it in local charts. If $\mathbf{s}_\nu$ are a basis for the tangent space then their covariant derivatives can be written

$$\nabla\mathbf{s}_\nu = \omega^{\alpha}_{\nu}\mathbf{s}_\alpha, \tag{2.7.5}$$

for some 1-forms ω^{α}_{ν}, called the *connection 1-forms*. Given any system of local coordinates x^μ, the tangent vectors to coordinate curves ∂_μ form a basis for the tangent space and the 1-forms dx^μ a dual basis for the cotangent space. Adopting this choice, the covariant derivative of the basis vector ∂_ν can be written

$$\nabla\partial_\nu = \Gamma^{\alpha}_{\mu\nu}dx^\mu\,\partial_\alpha. \tag{2.7.6}$$

The symbols $\Gamma^{\alpha}_{\mu\nu}$ are the *components of the connection* in the coordinate basis. The covariant derivative of any vector $Z = Z^\nu\partial_\nu$ is then

$$\nabla Z = \Big[\partial_\mu Z^\alpha + \Gamma^{\alpha}_{\mu\nu}Z^\nu\Big]dx^\mu\,\partial_\alpha. \tag{2.7.7}$$

[22]It is not hard; you might like to try for yourself.

In books employing the tensor calculus notation, prevalent in the majority of physics texts, you will see this formula written as

$$\nabla_\mu Z^\alpha = \partial_\mu Z^\alpha + \Gamma^\alpha_{\mu\nu} Z^\nu, \tag{2.7.8}$$

and the covariant derivative defined by this expression. I have chosen to give you its modern mathematical definition; it seems appropriate to try to keep with the times.

The final thing that we shall do is show that the coefficients of the connection $\Gamma^\alpha_{\mu\nu}$ are the same as the Christoffel symbols, as our notation has been anticipating. This is a significant result, known as the *fundamental theorem of Riemannian geometry.* It follows from the *metric compatibility* of the connection. The connection ∇ is said to be metric compatible if[23]

$$\nabla_X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{2.7.9}$$

for any three vectors X, Y, Z. I leave you to convince yourselves that this expresses precisely the notion that parallel transport should preserve the magnitude of a vector and the angle between two vectors. Choosing for X, Y, Z the basis vectors $\partial_\alpha, \partial_\mu, \partial_\nu$ this becomes

$$\partial_\alpha g_{\mu\nu} = \Gamma^\beta_{\alpha\mu} g_{\beta\nu} + \Gamma^\beta_{\alpha\nu} g_{\mu\beta}. \tag{2.7.10}$$

By simply permuting indices we write down two more versions of this

$$\partial_\mu g_{\nu\alpha} = \Gamma^\beta_{\mu\nu} g_{\beta\alpha} + \Gamma^\beta_{\mu\alpha} g_{\nu\beta}, \tag{2.7.11}$$

$$\partial_\nu g_{\alpha\mu} = \Gamma^\beta_{\nu\alpha} g_{\beta\mu} + \Gamma^\beta_{\nu\mu} g_{\alpha\beta}. \tag{2.7.12}$$

Adding these latter two and subtracting the first leads to

$$\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} = g_{\alpha\beta}\big(\Gamma^\beta_{\mu\nu} + \Gamma^\beta_{\nu\mu}\big) + g_{\mu\beta}\big(\Gamma^\beta_{\nu\alpha} - \Gamma^\beta_{\alpha\nu}\big) + g_{\beta\nu}\big(\Gamma^\beta_{\mu\alpha} - \Gamma^\beta_{\alpha\mu}\big). \tag{2.7.13}$$

Finally, if the connection coefficients $\Gamma^\alpha_{\mu\nu}$ are taken to be symmetric, $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$, then they are given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\big[\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\big]. \tag{2.7.14}$$

There is an unique, symmetric connection that is metric compatible; its coefficients are the Christoffel symbols that appear in the geodesic equation. This is the fundamental theorem of Riemannian geometry. With it we see that the components of the Riemann curvature tensor (2.6.43) can be expressed purely in terms of the components of the metric and their derivatives.

2.7.1 Continuity, Conservation and Divergence

An important aspect of all physical systems are the conservation laws, or conserved quantities. They are often expressed in local differential form as a *continuity equation*, expressing that a certain quantity is divergence free. The most relevant example for us in this course is the conservation of the stress-energy-momentum tensor, which we have seen expressed locally as

$$\partial_\alpha T^\alpha_\mu = 0. \tag{2.7.15}$$

This expression was obtained in flat Minkowski space, and in 'Cartesian' coordinates. Its general expression, for any manifold and any choice of coordinates has the partial derivative replaced with the covariant derivative. Let us see why by obtaining an expression for the

[23] The analogous expression in tensor calculus notation is

$$\nabla_\alpha g_{\mu\nu} = 0.$$

divergence of a vector field in a general coordinate system. The *divergence* of a vector field is defined in terms of the flux of that vector through the sides of a small volume Ω

$$\begin{aligned} \operatorname{div} Z &= \lim_{\operatorname{vol}\Omega\to 0} \frac{\int_{\partial\Omega}(Z\cdot n)\, d\text{area}}{\operatorname{vol}\Omega}, \\ \text{or} \qquad \int_\Omega \operatorname{div} Z\, d\text{vol} &= \int_{\partial\Omega} g_{ij} Z^i n^j \, d\text{area}, \end{aligned} \tag{2.7.16}$$

where n is the *unit* outward normal to the boundary of the region Ω and $d$area is the area element of this bounding surface. It will be sufficient to work in a local adapted coordinate system where the metric is diagonal

$$g_{ij} = \operatorname{diag}\big(g_1, g_2, \ldots, g_n\big), \tag{2.7.17}$$

and the region Ω is a box whose sides are surfaces given by a constant value of any one coordinate, $x^1 = \pm a$, $x^2 = \pm b$ and so forth. In the coordinate basis, the unit outward normal to the surface $x^1 = \pm a$ is the vector with components

$$n^j = \Big(\frac{\pm 1}{\sqrt{g_1}}, 0, \ldots, 0\Big). \tag{2.7.18}$$

The unit outward normals to the other sides of the box are similar. Now, the induced metric on the surface $x^1 = \pm a$ is

$$ds^2 = g_2\big(dx^2\big)^2 + \cdots + g_n\big(dx^n\big)^2, \tag{2.7.19}$$

so that the area element for this side of the box is

$$d\text{area} = \sqrt{g_2 \cdots g_n}\, dx^2 \cdots dx^n, \tag{2.7.20}$$

with similar expressions for each of the other sides. We can then compute directly

$$\begin{aligned} \int_{\partial\Omega} g_{ij} Z^i n^j \, d\text{area} &= \int_{x^1=a} g_1 Z^1 \frac{1}{\sqrt{g_1}} \sqrt{g_2\cdots g_n}\, dx^2 \cdots dx^n \\ &\quad + \int_{x^1=-a} g_1 Z^1 \frac{-1}{\sqrt{g_1}} \sqrt{g_2\cdots g_n}\, dx^2 \cdots dx^n + \text{other sides}, \end{aligned} \tag{2.7.21}$$

$$= \int_\Omega \partial_1\big(\sqrt{g_1 g_2 \cdots g_n}\, Z^1\big)\, dx^1 dx^2 \cdots dx^n + \text{other sides}, \tag{2.7.22}$$

$$= \int_\Omega \frac{1}{\sqrt{\det g}} \partial_1\big(\sqrt{\det g}\, Z^1\big)\, d\text{vol} + \text{other sides}, \tag{2.7.23}$$

$$= \int_\Omega \frac{1}{\sqrt{\det g}} \partial_i\big(\sqrt{\det g}\, Z^i\big)\, d\text{vol}, \tag{2.7.24}$$

and it follows that the divergence of a vector is given, in local coordinates, by the expression

$$\operatorname{div} Z = \frac{1}{\sqrt{\det g}} \partial_i\big(\sqrt{\det g}\, Z^i\big) = \partial_i Z^i + \bigg(\frac{1}{\sqrt{\det g}} \partial_i \sqrt{\det g}\bigg) Z^i. \tag{2.7.25}$$

It is an exercise to show that the bracketed term is precisely the contracted Christoffel symbol Γ^j_{ji} so that the divergence of a vector is expressed in terms of the covariant derivative as

$$\operatorname{div} Z = \nabla_i Z^i. \tag{2.7.26}$$

Naturally, all this is the same in a Lorentzian manifold, where the divergence of a vector is expressed in local coordinates as $\nabla_\alpha Z^\alpha$. Conservation of the stress-energy-momentum tensor is the statement that it is divergence free; the local expression for a general manifold is

$$\nabla_\alpha T^\alpha_\mu = 0. \tag{2.7.27}$$

REMARK: In obtaining conservation of energy for a particle we appealed to the change in the action under a constant shift ϵ^μ. We did the same for continuous fields, including the electromagnetic field. This really led to the conservation law $\partial_\alpha(\epsilon^\mu T^\alpha_\mu) = 0$ but since ϵ^μ was a constant we could safely drop it. There are no non-zero constant vectors in a general curved manifold; just think of the sphere. The best that we can arrange for is that the vector is *covariantly constant*, so the appropriate generalisation is that the shift ϵ^μ is covariantly constant, $\nabla_\alpha \epsilon^\mu = 0$. From the Leibniz formula for $\nabla_\alpha(\epsilon^\mu T^\alpha_\mu)$ we then arrive at the stated conservation law for the stress-energy-momentum tensor.

2.8 Summary

It seems worthwhile to summarise concisely the equations one uses in describing geometric properties of a space-time in a local coordinate system. So, suppose x^μ are coordinates in a local chart of our space-time. In such a local chart we write the metric as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.8.1}$$

The components $g_{\mu\nu}$ are symmetric in their indices, *i.e.* $g_{\mu\nu} = g_{\nu\mu}$.

The motion of a test particle is along a geodesic. If we parameterise the geodesic curve as $x^\alpha(\tau)$ then it is given by the equation

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{2.8.2}$$

where the $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols. They are given in terms of the components of the metric by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\big(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\big). \tag{2.8.3}$$

Geodesics defined in this way are parameterised by arc length if they are space-like or proper time if they are time-like. What this means is that for a time-like geodesic its tangent vector has 'unit magnitude'

$$g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1. \tag{2.8.4}$$

If the geodesic is null, corresponding to the path of light, then its tangent vector is a null vector and has 'zero magnitude'.

The complete description of the curvature of the space-time is conveyed by the Riemann tensor. From it one can deduce the Ricci tensor and the Ricci scalar, appearing in the Einstein equations. The curvature can be determined from the components of the metric. In a coordinate basis the components of the Ricci tensor are

$$R_{\mu\nu} = R^\alpha_{\alpha\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}, \tag{2.8.5}$$

and the Ricci scalar is

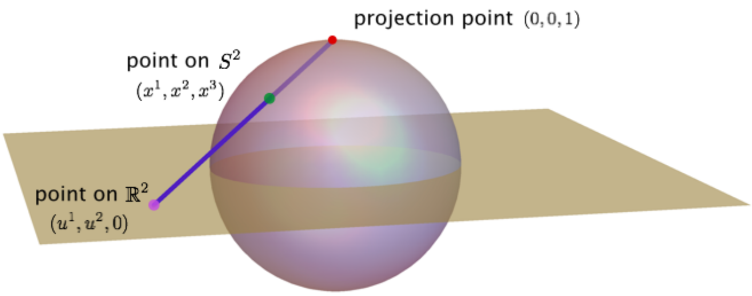
$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.8.6}$$

Problems

1. A torus can be described explicitly as the subset of Euclidean $\mathbb{R}^3$ given by
$$\left(\sqrt{(x^1)^2+(x^2)^2}-R\right)^2+(x^3)^2-\rho^2=0,$$
where R and ρ are the two 'radii' of the torus. Sketch the surface, or plot it with a program such as Mathematica. Give an explicit parameterisation of the surface. Determine the metric for your parameterisation and find the total surface area of the torus.

2. A torus can be explicitly embedded in the three-dimensional sphere, itself thought of as a subset of $\mathbb{R}^4$, through the parameterisation
$$\mathbf{X}(u^1,u^2)=\Big(\frac{1}{\sqrt{2}}\cos(u^1),\frac{1}{\sqrt{2}}\sin(u^1),\frac{1}{\sqrt{2}}\cos(u^2),\frac{1}{\sqrt{2}}\sin(u^2)\Big).$$
Determine the metric and find the total surface area. This embedding is known as a *flat torus.*

3. The diagram below defines *stereographic projection* between the 2-sphere S^2 and $\mathbb{R}^2$.



It comes in two forms; given a point (u^1,u^2) of $\mathbb{R}^2$ it can be mapped to an unique point of the 2-sphere. Show that the coordinates (x^1,x^2,x^3) of this point are
$$x^1+ix^2=\frac{2(u^1+iu^2)}{(u^1)^2+(u^2)^2+1},\qquad x^3=\frac{(u^1)^2+(u^2)^2-1}{(u^1)^2+(u^2)^2+1}.$$
The second form is that any point (x^1,x^2,x^3) of $S^2\subset\mathbb{R}^3$ can be mapped to an unique point (u^1,u^2) of $\mathbb{R}^2$. Show that the coordinates of this point are
$$u^1+iu^2=\frac{x^1+ix^2}{1-x^3}.$$
The use of complex numbers in these expressions is supposed to suggest to you that there is something to be gained by identifying $\mathbb{R}^2$ with $\mathbb{C}$ in these relations. Doing so gives (part of) the description of the 2-sphere as a complex manifold, the *Riemann sphere.*

Show that the metric on the 2-sphere in stereographic coordinates is
$$ds^2=\frac{4}{\big((u^1)^2+(u^2)^2+1\big)^2}\Big[(du^1)^2+(du^2)^2\Big].$$
Compute the area of the unit 2-sphere using these coordinates.

Describe the 3-sphere using stereographic projection to $\mathbb{R}^3$.

4. A soap film spanning two identical coaxial circular rings adopts the shape of a catenoid, a famous minimal surface. It can be described by the explicit embedding
$$\mathbf{X}(u^1,u^2)=\Big(\cosh(u^1)\cos(u^2),\cosh(u^1)\sin(u^2),u^1\Big).$$
Make the surface as a soap film. Sketch the surface, or plot it with a program such as Mathematica. Determine the metric.

5. Any surface embedded in $\mathbb{R}^3$ can be described locally as the graph of a function, say with the explicit embedding
$$\mathbf{X}(u^1, u^2) = \big(u^1, u^2, h(u^1, u^2)\big),$$
where $h(u^1, u^2)$ is the 'height function'. This description was first introduced by Gaspard Monge and is sometimes known as 'Monge gauge'. Determine the metric in Monge gauge and the area element for the surface. Give a local description of the unit 2-sphere in this way. How much of the surface can you cover with a single height function?

6. Consider the 'unit time-like sphere' in Minkoswki space-time, *i.e.* the subset defined by
$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -1.$$
Restrict attention to the component with $x^0 > 0$. Parameterise the three-dimensional surface, determine the metric and the volume element.

7. Explain why geodesics on the 2-sphere are great circles; or at least convince yourself that they are. Parameterising the sphere by standard spherical polar coordinates, determine the geodesic equations in these coordinates.

8. Consider the cylinder of radius a with parameterisation $(a\cos(u^1), a\sin(u^1), u^2)$. Determine the geodesic equation. Find and describe the geodesics.

9. Consider again the flat torus
$$\mathbf{X}(u^1, u^2) = \Big(\frac{1}{\sqrt{2}}\cos\big(u^1\big), \frac{1}{\sqrt{2}}\sin\big(u^1\big), \frac{1}{\sqrt{2}}\cos\big(u^2\big), \frac{1}{\sqrt{2}}\sin\big(u^2\big)\Big).$$
Find and describe the geodesics.

10. Determine the geodesic equations for the de Sitter space dS_4. Try to describe the null geodesics. [Hint: do not try to solve the geodesic equations; instead recall that de Sitter is a space-like sphere in a 4+1-dimensional Minkowski space-time ($\mathbb{R}^{1,4}$) and use what you know about geodesics on ordinary spheres in Euclidean spaces.]

11. Anti-de Sitter space, AdS_4, can be defined as the subset of $\mathbb{R}^{2,3}$ given by
$$-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = -a^2,$$
for a positive constant a. Construct a parameterisation of AdS_4 and determine the metric in your parameterisation. Show that the curve $x^0 = a\cos(\tau/a)$, $x^1 = a\sin(\tau/a)$, $x^2 = x^3 = x^4 = 0$ is a time-like geodesic.

12. Make sure you can explain what a geodesic is and derive the geodesic equation for a general metric. It is important.

13. Consider the embedding of the hyperbolic plane $\mathbb{H}^2$ in 2+1-dimensional Minkowski space-time as the subset
$$-(x^0)^2 + (x^1)^2 + (x^2)^2 = -a^2,$$
with $x^0 > 0$ and for any positive constant a. Using this description, determine the sectional curvature. [Hint: don't forget to exploit the fact that the space is isotropic and homogeneous, so you need only do the calculation at any one point.]

    Consider the circle of geodesic radius r (on the hyperbolic plane embedded in $\mathbb{R}^{1,2}$ as above) given by
$$\mathbf{c}(\phi) = \Big(a\cosh(r/a), a\sinh(r/a)\cos(\phi), a\sinh(r/a)\sin(\phi)\Big).$$
Find the result of parallel transport of the vector $\mathbf{Z} = (0, 1, 0) = \mathbf{e}_1$, tangent to the surface at the point $(a, 0, 0)$, around this circle.

14. Consider the three metrics

$$ds^2 = \begin{cases} dr^2 + a^2 \sin^2(r/a)\, d\phi^2, \\ dr^2 + r^2\, d\phi^2, \\ dr^2 + a^2 \sinh^2(r/a)\, d\phi^2, \end{cases}$$

where a is a fixed, positive constant. Determine the components of the Ricci tensor R_{ij} and hence the value of the Ricci scalar $R = g^{ij} R_{ij}$ for each metric.

15. The Ricci tensor is symmetric, $R_{\mu\nu} = R_{\nu\mu}$. We use this property but did not establish it in the notes. Here is a way to do so, using our approach. We gave an expression for the Ricci tensor as

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\beta\alpha} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\alpha\nu},$$

where $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols and are symmetric in μ, ν, *i.e.* $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. This means we need only show symmetry for the second and fourth terms in the expression for the Ricci tensor; the first and third are explicitly symmetric.

Show that $\Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\alpha\nu} = \Gamma^\alpha_{\nu\beta} \Gamma^\beta_{\alpha\mu}$.

So we need only show that $\partial_\mu \Gamma^\alpha_{\alpha\nu}$ is symmetric. This follows from the fact that

$$\Gamma^\alpha_{\alpha\nu} = \partial_\nu \ln \sqrt{|\det g|}.$$

From the definition of the Christoffel symbols in terms of the metric, show that

$$\Gamma^\alpha_{\alpha\nu} = \frac{1}{2} g^{\alpha\beta} \partial_\nu g_{\alpha\beta}.$$

Now let M be a square, invertible matrix. We will suppose that it is symmetric and hence diagonalisable. By diagonalising M, or otherwise, establish the matrix identity

$$\det M = \mathrm{e}^{\mathrm{tr}\, \ln M},$$

and hence show that

$$\partial_\nu \det M = \det M \ \mathrm{tr}\big(M^{-1} \partial_\nu M\big).$$

Taking for M the metric g, complete the demonstration that the Ricci tensor is symmetric.

A different strategy is as follows: Recall that the Ricci tensor can be expressed in terms of the Riemann tensor as

$$R_{\mu\nu} = R^\alpha_{\alpha\mu\nu} = \eta^{\alpha\beta} R_{\alpha\mu\nu\beta}.$$

Show that the symmetries of the Riemann tensor

$$R_{\alpha\mu\nu\beta} = -R_{\mu\alpha\nu\beta}, \qquad R_{\alpha\mu\nu\beta} = -R_{\alpha\mu\beta\nu}, \qquad R_{\alpha\mu\nu\beta} = R_{\nu\beta\alpha\mu},$$

imply the symmetry of the Ricci tensor, $R_{\mu\nu} = R_{\nu\mu}$. [Of course, the lengthy part of this approach, which we have skipped entirely, is to establish the given properties of the Riemann tensor.]

16. In the lectures an expression was given for the components (in a coordinate basis) of the covariant derivative of a vector

$$\nabla_\mu Z^\alpha = \partial_\mu Z^\alpha + \Gamma^\alpha_{\mu\nu} Z^\nu.$$

Show that the corresponding expression for the components of the covariant derivative of a 1-form A, with components A_ν in a coordinate basis, is

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha.$$

Hence show that the components of the Maxwell field strength tensor, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, are still given by $\partial_\mu A_\nu - \partial_\nu A_\mu$.

What is the corresponding expression for the components (in a coordinate basis) of the covariant derivative of a type $\binom{0}{2}$ tensor, $\nabla_\alpha F_{\mu\nu}$?

Write out in full the local continuity equation for the stress-energy-momentum tensor $\nabla_\alpha T^\alpha_\mu = 0$.

17. In the notes, an expression was obtained for the divergence of a vector $\operatorname{div} Z = (\det g)^{-1/2}\partial_i(\det g)^{1/2} Z^i$. If the vector is the gradient of a scalar, $Z^i = \partial^i\psi = g^{ij}\partial_j\psi$, then this leads to an expression for the Laplacian in a general coordinate system

$$\nabla^2 = \frac{1}{\sqrt{\det g}}\partial_i\big(\sqrt{\det g}\, g^{ij}\partial_j\big).$$

Use this formula to give the form of the Laplacian in cylindrical and spherical coordinates.

In the Lorentzian setting, precisely the same formula yields the wave operator (often denoted by the symbol $\Box$ and frequently called the d'Alembertian)

$$\Box = \frac{1}{\sqrt{|\det g|}}\partial_\mu\big(\sqrt{|\det g|}\, g^{\mu\nu}\partial_\nu\big).$$

Verify this first for the usual Minkowski metric. What is the wave operator for the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) c^2 dt^2 + \frac{1}{1-\frac{2m}{r}} dr^2 + r^2\left[d\theta^2 + \sin^2(\theta)\, d\phi^2\right]?$$

Can you find any (non-zero) solutions of the wave equation $\Box\psi = 0$ in the Schwarzschild space-time? [Hint: you should consider this to be very hard!!]

The original image of the Eagle Nebula taken by the Hubble Space Telescope was released in 1995 and called *Pillars of Creation*. It is one of the most iconic images of space and caught the imagination of the general public, including myself. This image was taken in 2015 for the 20th anniversary. Image from NASA.

Chapter 3

Einstein's General Theory of Relativity

The possibility of explaining the numerical equality of inertia and gravitation by the unity of their nature gives to the general theory of relativity, according to my conviction, such a superiority over the conceptions of classical mechanics, that all the difficulties encountered in development must be considered as small in comparison.

Albert Einstein (1921)

I present in this chapter two derivations of the Einstein equations of general relativity; the first using an action principle due to Hilbert, and the second an imitation of the line of reasoning that led Einstein to his own derivation. Einstein's concept in general relativity is that the matter content of the universe determines the space-time metric. How it does so can be described by an action principle. The action should be a scalar quantity constructed out of the metric and its derivatives; the field equations are obtained from the condition that the actual metric corresponds to a critical point of the action. Now, in Newton's theory the gravitational potential satisfies a differential equation of second order, so we look for an action that depends on derivatives of the metric of at most second order and is linear in the second order derivatives. There is only one scalar quantity with these properties; the Ricci scalar R. So the action must be proportional to it. The Einstein-Hilbert action in general relativity is[1]

$$S = \frac{c^4}{16\pi G}\int_M R\,d\mathrm{vol} = \frac{c^4}{16\pi G}\int_M R\sqrt{|\det g|}\,d^4x, \tag{3.1}$$

where R is the Ricci scalar, g is the metric, and the integral is taken over the entire space-time manifold M. The prefactor turns out to be what is needed to produce the Newtonian theory in the weak field regime with G Newton's gravitational constant. The Einstein equations are a system of equations for the metric of space-time. They are obtained from the action by the usual principle that the physical metric corresponds to a critical point of the action. Let $g_{\mu\nu} + h_{\mu\nu}$ be a variation of the metric $g_{\mu\nu}$, *i.e.* the $h_{\mu\nu}$ are infinitesimally small and vanish on the boundary of the space-time. The variation of the measure factor $\sqrt{|\det g|}$ can be determined by writing the perturbed metric as

$$g_{\mu\nu} + h_{\mu\nu} = g_{\mu\alpha}\big(\delta^\alpha_\nu + h^\alpha_\nu\big). \tag{3.2}$$

The determinant of the perturbed metric is then a product – recall $\det AB = \det A \det B$ – with the second factor being the determinant of a near identity matrix, $\delta^\alpha_\nu + h^\alpha_\nu$. By diagonalising the matrix h^α_ν we see that this latter determinant is equal to

$$\det\big(\delta^\alpha_\nu + h^\alpha_\nu\big) = 1 + h^\alpha_\alpha + O(2) = 1 + g_{\mu\nu}h^{\mu\nu} + O(2). \tag{3.3}$$

[1]If the manifold has a boundary an additional term, known as the Gibbons-Hawking-York boundary term, should be added to make the variational problem well-posed.

It follows that the variation of the measure factor is given by

$$\sqrt{|\det g|} \;\mapsto \sqrt{|\det g|}\sqrt{1 + g_{\mu\nu}h^{\mu\nu} + O(2)}, \tag{3.4}$$

$$= \sqrt{|\det g|} + \frac{1}{2}\sqrt{|\det g|}\, g_{\mu\nu}h^{\mu\nu} + O(2). \tag{3.5}$$

The variation of the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$, can be broken into two parts; the variation of the inverse metric $g^{\mu\nu}$, and the variation of the Ricci tensor $R_{\mu\nu}$. The former is easy. Since, by definition of the inverse metric, $g^{\mu\nu}g_{\mu\nu} = 4$, the variation of the inverse metric must be

$$g^{\mu\nu} \mapsto g^{\mu\nu} - h^{\mu\nu} + O(2). \tag{3.6}$$

We write the variation of the Ricci tensor as $R_{\mu\nu} \mapsto R_{\mu\nu} + \delta R_{\mu\nu} + O(2)$. The variation of the Einstein-Hilbert action is then

$$S[g_{\mu\nu} + h_{\mu\nu}] = S[g_{\mu\nu}] + \frac{c^4}{16\pi G}\int_M \left[g^{\mu\nu}\delta R_{\mu\nu} - h^{\mu\nu}\Big(R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}\Big)\right]\sqrt{|\det g|}\, d^4x + O(2). \tag{3.7}$$

It turns out that the term involving the variation of the Ricci tensor makes no contribution; it is a total divergence and integrates to the boundary where the variation vanishes. The argument that is always given for this is very slick, making use of the existence of local inertial frames, or Riemann normal coordinates. These are coordinates, adapted to any particular point p, where the metric is the Minkowski one to second order accuracy about p. It follows that the Christoffel symbols all vanish at the point p, although their variation $\delta\Gamma^{\alpha}_{\mu\nu}$ does not. The variation of the Ricci tensor can then be written, at p, as

$$g^{\mu\nu}\delta R_{\mu\nu}\Big|_p = \eta^{\mu\nu}\left[\partial_\alpha \delta\Gamma^{\alpha}_{\mu\nu} - \partial_\mu \delta\Gamma^{\alpha}_{\alpha\nu}\right], \tag{3.8}$$

$$= \partial_\alpha\left[\eta^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - \eta^{\alpha\nu}\delta\Gamma^{\mu}_{\mu\nu}\right], \tag{3.9}$$

$$= \operatorname{div}\left[g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\alpha\nu}\delta\Gamma^{\mu}_{\mu\nu}\right]\Bigg|_p. \tag{3.10}$$

Since this last expression is a natural geometric object the relation must be valid in any coordinate system. This establishes that the variation of the Ricci tensor is a divergence and therefore makes no contribution to the field equations. We find, therefore, that the condition for $g_{\mu\nu}$ to be a critical point of the Einstein-Hilbert action is that

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = 0. \tag{3.11}$$

These are the *Einstein equations* in the absence of matter content. If the action for the matter content is

$$S_{\text{matter}} = \int_M L\, d\text{vol} = \int_M L\,\sqrt{|\det g|}\, d^4x, \tag{3.12}$$

where L is the Lagrangian, then its change under a variation of the metric can be written, formally, as

$$S_{\text{matter}}[g_{\mu\nu} + h_{\mu\nu}] - S_{\text{matter}}[g_{\mu\nu}] \equiv \int_M h^{\mu\nu}\,\frac{1}{2}T_{\mu\nu}\,\sqrt{|\det g|}d^4x, \tag{3.13}$$

which serves as a *definition* of the stress-energy-momentum tensor. To see that it is the same object as we introduced previously, we compute it explicitly for the electromagnetic field. There the action is

$$S_{\text{em}} = \frac{1}{\mu_0}\int_M \frac{-1}{4}F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}\,\sqrt{|\det g|}\, d^4x, \tag{3.14}$$

where the field strength tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ depends on the metric through the connection ∇. Computing the variation explicitly we find

$$\begin{aligned} S_{\rm em}[g_{\mu\nu} + h_{\mu\nu}] = \frac{1}{\mu_0} \int_M \Big[\frac{-1}{4} F_{\alpha\beta} F_{\gamma\delta} \Big(g^{\alpha\gamma} - h^{\alpha\gamma} \Big) \Big(g^{\beta\delta} - h^{\beta\delta} \Big) - \frac{1}{2} F^{\alpha\beta} \delta F_{\alpha\beta} + O(2) \Big] \\ \times \Big(1 + \frac{1}{2} g_{\mu\nu} h^{\mu\nu} + O(2) \Big) \sqrt{|\det g|}\, d^4x, \end{aligned} \tag{3.15}$$

$$\begin{aligned} = S_{\rm em}[g_{\mu\nu}] + \frac{1}{\mu_0} \int_M \Big\{ h^{\mu\nu} \frac{1}{2} \Big[g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} \Big] \\ - \frac{1}{2} F^{\alpha\beta} \delta F_{\alpha\beta} + O(2) \Big\} \sqrt{|\det g|}\, d^4x. \end{aligned} \tag{3.16}$$

Here the variation $F^{\alpha\beta}\delta F_{\alpha\beta}$ is that associated with the metric dependence of the connection ∇. It can be shown that this vanishes on account of Maxwell's equations for the electromagnetic field, $\nabla_\alpha F^{\mu\alpha} = 0$. Thus, the stress-energy-momentum tensor for the electromagnetic field is

$$T_{\mu\nu} = \frac{1}{\mu_0} \Big[g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} \Big], \tag{3.17}$$

precisely the expression that we obtained previously. This all being said, combining the variation of the Einstein-Hilbert action with that of the matter content the final *Einstein equations* that determine the metric of space-time are

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{3.18}$$

REMARK: The derivation I have given of the Einstein equations is due to Hilbert (1915). I would say that it is a 'modern' approach, but given that it first appeared at almost exactly the same time as Einstein's own paper deriving his field equations, Einstein included it in his 1916 review article, and Landau used the same derivation in his *Course of Theoretical Physics* from the 1930s, you would be justified in not believing me. It is not 'modern'. Nonetheless, the approach to physics based on action principles is not as widely accessible to undergraduates in physics as I would like; it is to the enormous discredit of theoretical physics as a profession that we have not managed to make our most central and unifying concept more widely accessible to undergraduates. Inspired by the example laid out by Landau, and Feynman amongst others, I have made a small effort to remedy this.

We also now give a stripped down and simplified version of Einstein's line of reasoning in arriving at an improvement of Newton's theory of gravity, showing explicitly that Einstein's equations reduce to Newton's in an appropriate limit. In Newton's theory we have the equations

$$\frac{d^2x^i}{dt^2} = -\partial_i\phi, \tag{3.19}$$

$$\partial_i\partial_i\phi = 4\pi G\,\rho, \tag{3.20}$$

for the motion of a test particle, along trajectory $x^i(t)$, in the gravitational field produced by a mass distribution ρ. According to Einstein's equivalence principle, gravity and inertia should be considered equivalent as phenomena. The motion of the test particle, viewed in Newton's theory as non-inertial motion in a gravitational field, can equally be viewed as free, meaning geodesic, motion in a curved space. It should therefore be replaced by the geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \tag{3.21}$$

In a non-relativistic regime where gravitational effects are weak and velocities of motion small, we can expect the components $dx^i/d\tau$ of the particle velocity to be $O(v/c)$ and therefore small compared to $dx^0/d\tau$, which is $O(1)$. Therefore, to leading order the geodesic equation is

$$\frac{1}{c^2}\frac{dx^i}{dt^2} + \Gamma^i_{00} = 0, \tag{3.22}$$

so that we may identify $c^2\Gamma^i_{00}$ with the gradient of the Newtonian potential. We can say more, if we assume that the metric deviates only by small quantities from that of Minkowski space. Then the Christoffel symbol may be approximated as

$$\Gamma^i_{00} = \frac{1}{2}\big[\partial_0 g_{i0} + \partial_0 g_{0i} - \partial_i g_{00}\big] = \frac{-1}{2}\partial_i g_{00}, \tag{3.23}$$

with the assumption that spatial derivatives of the metric are more significant than time derivatives, as is appropriate for 'static' gravitational situations like the solar system. From this we identify

$$g_{00} = -1 - \frac{2}{c^2}\phi, \tag{3.24}$$

where ϕ is the Newtonian potential, in the weak field regime. The Newtonian potential appears as the '00-component' of the space-time metric.

Now we look for a replacement of Newton's field equation. The source is the mass density in Newton's theory. From considerations of special relativity, we have seen that this should be replaced with the stress-energy-momentum tensor $T_{\mu\nu}$. Its 00-component is the energy density, or mass density times c^2, so that such a replacement matches exactly in terms of character with the identification of the Newtonian potential with the component g_{00} of the space-time metric. The left-hand-side of Newton's field equation involves second derivatives of the potential; these are now naturally viewed as representing the curvature of space-time. What we need to do is find a symmetric type $\binom{0}{2}$ tensor, corresponding to curvature, that can be identified with the stress-energy-momentum tensor. The properties that guide us in finding it are that it should involve derivatives of no greater than second order of the metric, it should be linear and homogeneous in these second derivatives and it should be compatible with the conservation property of the stress-energy-momentum tensor, $\nabla_\alpha T^\alpha_\mu = 0$. There is only one quantity that can be constructed from the Riemann curvature tensor with these properties:

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}. \tag{3.25}$$

We therefore postulate the field equation

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = K\, T_{\mu\nu}, \tag{3.26}$$

for a constant K, to be determined by matching to Newton's theory in the weak field limit. It will be convenient to make as much use of the stress-energy-momentum tensor as possible, as we have more knowledge of it than of the components of the Ricci tensor. Taking the trace of the field equations gives $-R = K\,T$ and therefore

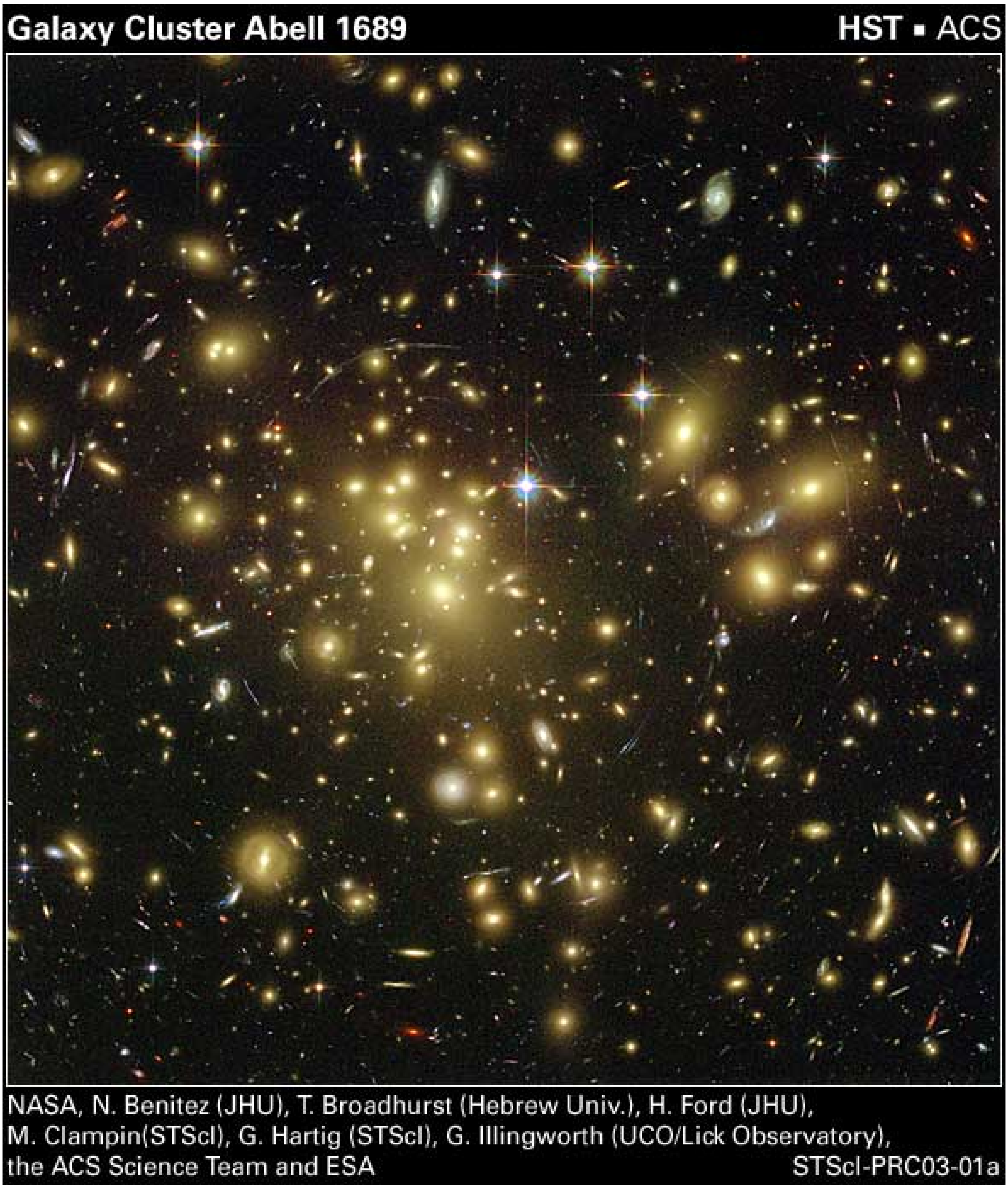
$$R_{\mu\nu} = K\left(T_{\mu\nu} - \frac{1}{2}T\, g_{\mu\nu}\right). \tag{3.27}$$

In the weak field limit we can approximate the 00-component of the right-hand-side as $\frac{1}{2}K\rho c^2$, so we only need an approximation for R_{00} to determine K. With the assumption that the metric is 'static' so that time derivatives can be neglected we find

$$R_{00} = \partial_i\Gamma^i_{00} = \frac{1}{c^2}\partial_i\partial_i\phi, \tag{3.28}$$

to leading order, and it follows that $K = 8\pi G/c^4$. Thus we arrive at the *Einstein field equations of general relativity*

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}. \tag{3.29}$$



A beautiful example of gravitational lensing in this Hubble Space Telescope image of the galaxy cluster Abell 1689, taken in 2003. Such images vividly bring to life the basic content of Einstein's theory, that mass distorts the structure of space and hence the path along which light, or anything else, travels. Image from NASA.

Chapter 4

The Schwarzschild Solution

As you see, the war treated me kindly enough, in spite of the heavy gunfire, to allow me to get away from it all and take this walk in the land of your ideas.

Karl Schwarzschild (*Letter to Einstein*, 1915)

It is always nice to have exact solutions to work with. The first of these, for general relativity, was provided by Karl Schwarzschild in late 1915. It is a solution for the vacuum region of space-time outside a single, central massive body. As such, it can be used to model the motion of the planets and other celestial bodies in the gravitational field of the sun and so determine the corrections to Newton's theory predicted by general relativity; these are the classical tests. It also contains a singularity, a point in the space-time at which the curvature is infinite, and so is the progenitor of all studies of black holes.

4.1 Derivation of the Metric

The gravitational field produced by a massive central body, such as the sun, is, in Newton's theory, spherically symmetric. The analogous, spherically symmetric solution of the Einstein equations of general relativity was first obtained by Karl Schwarzschild in 1915. We may take for the metric the following form[1]

$$ds^2 = -f(r)c^2dt^2 + g(r)dr^2 + r^2\Big[d\theta^2 + \sin^2(\theta)\, d\phi^2\Big], \tag{4.1.1}$$

with only two unknown functions $f(r)$ and $g(r)$, that depend only on the radial coordinate. That they are independent of the time-coordinate t corresponds to the space-time being *static*. It turns out that this is not an assumption; spherical symmetry is strong enough to imply this property, a result known as Birkhoff's theorem. Therefore, (4.1.1) represents the most general form of the metric compatible with spherical symmetry.

The unknown functions f and g are found by inserting this form of the metric into the Einstein equations. So we need to determine the curvature of space-time represented by (4.1.1). The usual formula for the components of the Ricci tensor gives them in terms of the Christoffel symbols, so we begin by determining those. Although there is a formula in terms of derivatives of the metric, it usually turns out to be more economical to read them off from the equation for the geodesics, so what we really do is determine the equation for the geodesics. As usual, let $\gamma = \big(ct(\tau), r(\tau), \theta(\tau), \phi(\tau)\big)$ be a curve parameterised by arc length (c times proper time)

[1] I will not describe properly what it means to say that a space-time has a certain symmetry, or isometry. Therefore, I ask you to be pragmatic and take that spherical symmetry implies that the metric can be written in such a way that part of it looks like the standard round metric for a sphere.

and consider a variation of it. By direct calculation we have that the metric for the variation of γ is

$$ds^2 = g_{\mu\nu}(x+\epsilon)\,\frac{d(x^\mu+\epsilon^\mu)}{d\tau}\frac{d(x^\nu+\epsilon^\nu)}{d\tau}\,d\tau^2, \tag{4.1.2}$$

$$\begin{aligned} &= \Big[-1 - 2f(r)c\partial_\tau t\,\partial_\tau\epsilon^0 + 2g(r)\,\partial_\tau r\,\partial_\tau\epsilon^r + \Big(-f'(r)\big(c\partial_\tau t\big)^2 + g'(r)\big(\partial_\tau r\big)^2 \\ &\qquad + 2r\big[\big(\partial_\tau\theta\big)^2 + \sin^2(\theta)\big(\partial_\tau\phi\big)^2\big]\Big)\,\epsilon^r + 2r^2\partial_\tau\theta\,\partial_\tau\epsilon^\theta \\ &\qquad + 2r^2\sin(\theta)\cos(\theta)\big(\partial_\tau\phi\big)^2\,\epsilon^\theta + 2r^2\sin^2(\theta)\partial_\tau\phi\,\partial_\tau\epsilon^\phi + O(2)\Big]\,d\tau^2, \end{aligned} \tag{4.1.3}$$

recalling that τ is (c times) proper time for γ, *i.e.* $g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$, and where the prime denotes differentiation with respect to argument. The geodesics are critical points of the 'distance' function, or in other words curves γ for which

$$\int_{\gamma'} ds - \int_{\gamma} ds = O(2), \tag{4.1.4}$$

for all variations γ' of the curve preserving the endpoints. Using the form of the metric for the variation γ' we find that the geodesics are given by the equations

$$0 = \partial_\tau\big[-f(r)c\partial_\tau t\big], \tag{4.1.5}$$

$$0 = \partial_\tau\big[g(r)\partial_\tau r\big] + \frac{1}{2}f'(r)\big(c\partial_\tau t\big)^2 - \frac{1}{2}g'(r)\big(\partial_\tau r\big)^2 - r\Big[\big(\partial_\tau\theta\big)^2 + \sin^2(\theta)\big(\partial_\tau\phi\big)^2\Big], \tag{4.1.6}$$

$$0 = \partial_\tau\big[r^2\partial_\tau\theta\big] - r^2\sin(\theta)\cos(\theta)\big(\partial_\tau\phi\big)^2, \tag{4.1.7}$$

$$0 = \partial_\tau\big[r^2\sin^2(\theta)\partial_\tau\phi\big]. \tag{4.1.8}$$

From the geodesic equations we can read off the Christoffel symbols; we record the complete set that are non-zero (recall we denote ct by x^0)

$$\begin{aligned} &\Gamma^0_{0r} = \Gamma^0_{r0} = \frac{f'}{2f}, \quad \Gamma^r_{00} = \frac{f'}{2g}, \quad \Gamma^r_{rr} = \frac{g'}{2g}, \quad \Gamma^r_{\theta\theta} = \frac{-r}{g}, \quad \Gamma^r_{\phi\phi} = \frac{-r\sin^2(\theta)}{g}, \\ &\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta), \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)}. \end{aligned} \tag{4.1.9}$$

Next we want to determine the components of the Ricci tensor in our coordinate system. Recall that they are given in terms of the Christoffel symbols by the general formula

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\beta\alpha} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}. \tag{4.1.10}$$

The calculation is tedious, but not difficult. It turns out that only the diagonal components are non-zero and they are given by

$$R_{00} = \frac{1}{2g}\Big[f'' - \frac{1}{2f}\big(f'\big)^2 - \frac{1}{2g}f'g' + \frac{2}{r}f'\Big], \tag{4.1.11}$$

$$R_{rr} = \frac{-1}{2f}\Big[f'' - \frac{1}{2f}\big(f'\big)^2 - \frac{1}{2g}f'g' - \frac{2f}{rg}g'\Big], \tag{4.1.12}$$

$$R_{\theta\theta} = 1 - \frac{1}{g} - \frac{r}{2fg}f' + \frac{r}{2g^2}g', \tag{4.1.13}$$

$$R_{\phi\phi} = \sin^2(\theta)R_{\theta\theta}. \tag{4.1.14}$$

The vacuum Einstein equations, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$, imply that the Ricci scalar vanishes; just take the trace of the Einstein equations. It follows that the components of Ricci must themselves be zero. The vanishing of R_{00} and R_{rr} then combine to imply that

$$\partial_r\big(fg\big) = 0, \tag{4.1.15}$$

or $fg = 1$. You might think that the equation only implies $fg = \text{const}$, but the constant can be taken to be unity without loss of generality by the freedom to rescale t by any multiplicative constant. Using this to simplify the equation $R_{00} = 0$ we find

$$f'' + \frac{2}{r} f' = 0, \tag{4.1.16}$$

$$\Rightarrow \qquad f = a - \frac{2m}{r}, \tag{4.1.17}$$

for arbitrary integration constants a and m. Finally, the condition $R_{\theta\theta} = 0$ demands that $a = 1$ but places no constraints on m. At last we can write the Schwarzschild metric in its standard form

$$ds^2 = -\left(1 - \frac{2m}{r}\right) c^2 dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 \Big[d\theta^2 + \sin^2(\theta)\, d\phi^2\Big]. \tag{4.1.18}$$

The sole free parameter m is the *mass* of the Schwarzschild black hole. The distance $r = 2m$, where our coordinate system is badly behaved, is known as the *Schwarzschild radius.* It is the *event horizon* of the black hole. However, it is not a singularity in space-time; it is only a break-down of our coordinate system, called a *coordinate singularity.* On the other hand the origin, $r = 0$, is a singularity of space-time. It is called a *black hole.*

REMARK: The parameter m is always referred to as the *mass* of the Schwarzschild black hole, even though it has dimensions of length; and $r = 2m$ is properly called the Schwarzschild radius. m is the mass in 'natural units'[a], being equal to GM/c^2 in SI units, where M is the actual mass. For an object with the mass of the sun, $m \approx 1.48$ km.

[a]Natural units are those where the speed of light c, Newton's gravitational constant G and Planck's constant $\hbar$ are all taken to be unity; it is always possible to restore the appropriate factors by dimensional analysis.

4.2 The Event Horizon

By far the most striking feature of the solution (4.1.18) is the surface $r = 2m$, called the *event horizon.* Superficially, the metric is singular here, but we have seen this before and learnt not to be too hasty. So what is it really like? How do we find out? We have to consider actual physical phenomena and how they appear to us. We will observe the motion of an intrepid explorer[2] as they approach the event horizon, and the light signals they send back to us. Both move along radial geodesics (θ, ϕ fixed constants); the explorer along a time-like geodesic and the light signals along null geodesics. We wrote down the geodesic equations as the first step in solving for the Schwarzschild metric, so looking back we can say that the 't-equation' gives

$$\left(1 - \frac{2m}{r}\right) c\partial_\tau t = \gamma, \tag{4.2.1}$$

for a constant γ. In addition, we know that the geodesic is parameterised by (c times) proper time, or is null, so that

$$-\left(1 - \frac{2m}{r}\right) \big(c\partial_\tau t\big)^2 + \frac{1}{1 - \frac{2m}{r}} \big(\partial_\tau r\big)^2 = -\kappa, \tag{4.2.2}$$

where $\kappa = 1$ for the time-like geodesic of our explorer, and $\kappa = 0$ for the null geodesics of the light signals they send to us. Eliminating $\partial_\tau t$ in favour of the constant γ this becomes

$$\big(\partial_\tau r\big)^2 = \gamma^2 - \kappa + \frac{2m\kappa}{r}. \tag{4.2.3}$$

[2]In my head she is called Jolene, but you may prefer to supply your own narrative.

Taking a square root, the trajectory of our friend (who is time-like so set $\kappa = 1$) is given implicitly by the expression

$$\tau = \int_r^R \frac{dr'}{\sqrt{|\gamma^2 - 1 + 2m/r'|}} = \int_{1/R}^{1/r} \frac{du}{u^2\sqrt{|\gamma^2 - 1 + 2mu|}}, \tag{4.2.4}$$

where τ is (c times) the proper time since our explorer set out at radius R. An explicit expression for the explorer's position r as a function of the proper time τ is not nearly as important as the simple observation that this expression is perfectly regular through the event horizon at $r = 2m$ (or $u = 1/2m$). In fact, the explorer reaches the singularity at $r = 0$ in *finite proper time*. A person falling freely inwards, approaching the horizon radially, will pass straight through it; it is not a singularity of the space-time.

What do we see, from a distant position well away from the horizon? Expressing the motion of the infalling explorer in terms of the coordinates we measure as $r(t)$, (4.2.2) becomes

$$\begin{aligned} \frac{-1}{1 - \frac{2m}{r}}\,\gamma^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)^3}\,\frac{\gamma^2}{c^2}\left(\partial_t r\right)^2 &= -1, \\ \Rightarrow \quad \frac{\gamma}{c}\,\partial_t r &= -\left(1 - \frac{2m}{r}\right)\left|\gamma^2 - 1 + \frac{2m}{r}\right|^{1/2}, \end{aligned} \tag{4.2.5}$$

and we obtain the formula

$$cT = \int_r^R \frac{\gamma\, dr'}{(1 - 2m/r')\sqrt{|\gamma^2 - 1 + 2m/r'|}} = \int_{1/R}^{1/r} \frac{\gamma\, du}{u^2(1 - 2mu)\sqrt{|\gamma^2 - 1 + 2mu|}}, \tag{4.2.6}$$

for the time that we record it takes for the explorer to get to the radial distance r. The main conclusion here is that this time diverges (logarithmically) as the explorer approaches the event horizon at $r = 2m$. In other words, we never see them reach the horizon itself, according to our own measure of time. It implies that we can never receive any information that such an explorer obtains from the event horizon, or from the region of space-time inside it. It is 'hidden' from us. Even light does not make it to us in a finite time as we measure it. For suppose our explorer pauses at radial position r to send us a signal, say a radio communication, that travels to us along a radial null geodesic. We record the trajectory of the light signal as satisfying

$$\left(1 - \frac{2m}{r}\right) c^2 dt^2 = \frac{1}{1 - \frac{2m}{r}}\, dr^2, \tag{4.2.7}$$

so that the time we measure for it to reach us is

$$cT = \int_r^R \frac{dr'}{1 - \frac{2m}{r'}} = R - r + 2m \ln\left|\frac{R - 2m}{r - 2m}\right|, \tag{4.2.8}$$

and this diverges as our explorer friend approaches the event horizon at $r = 2m$. The event horizon is a barrier, concealing a region of space-time that we can never have any knowledge of so long as we stay in the comfort and safety of the asymptotic far-field of the Schwarzschild metric.

Some additional insight is gained from a change of perspective, or coordinates. The *Eddington-Finkelstein coordinates* replace the asymptotic Schwarzschild time coordinate with an affine coordinate parameterising the radial null geodesics. The *ingoing Eddington-Finkelstein coordinate* u is defined by

$$du = cdt + \left(1 - \frac{2m}{r}\right)^{-1} dr, \tag{4.2.9}$$

so that each trajectory $u = \text{const.}$ (and $\theta = \phi = \text{const.}$) corresponds to an ingoing radial null geodesic. By eliminating cdt, one finds directly that the Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{2m}{r}\right)du^2 + 2du dr + r^2\left[d\theta^2 + \sin^2(\theta)\, d\phi^2\right]. \tag{4.2.10}$$

In these coordinates the components of the metric no longer diverge on the surface $r = 2m$, a reflection of what we learnt above that the event horizon is not a singularity of the space-time. From these coordinates, however, we do learn something new, and interesting, about the event horizon. This surface, $r = 2m$, is covered by Eddington-Finkelstein coordinates (u, θ, ϕ). The metric on the event horizon is

$$ds^2\Big|_{r=2m} = (2m)^2\Big[d\theta^2 + \sin^2(\theta)\, d\phi^2\Big]. \tag{4.2.11}$$

What this means is that the interval between two points of the event horizon that only differ in the value of the u coordinate is zero, or in other words the event horizon is a surface that contains a null direction; it is a *null surface*.

Analogous to the ingoing coordinates we have described, there are also *outgoing Eddington-Finkelstein* coordinates (v, r, θ, ϕ) with v defined by

$$dv = cdt - \left(1 - \frac{2m}{r}\right)^{-1} dr, \tag{4.2.12}$$

and such that each trajectory $v =$ const. (and $\theta = \phi =$ const.) corresponds to an outgoing radial null geodesic. In terms of these coordinates the Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dv^2 - 2dvdr + r^2\Big[d\theta^2 + \sin^2(\theta)\, d\phi^2\Big]. \tag{4.2.13}$$

The description of the Schwarzschild space-time that follows from a study with these coordinates is an exact time-reversed version of that for the ingoing coordinates. This is to be expected given the static nature of the space-time, *i.e.* invariance under $t \to -t$.

4.3 Kruskal-Szekeres Coordinates

REMARK: This section is important; it conveys much about coordinates and the difference between coordinates for a local chart of a manifold and the manifold itself.

The Schwarzschild and Eddington-Finkelstein coordinates, both ingoing and outgoing, show us that how we see a space-time depends on how we view it[3]. It is natural to ask if there are other coordinate systems that reveal more about the Schwarzschild space-time. Indeed there are; the most complete[4] view is given by the Kruskal-Szekeres coordinates, which we now introduce. First, we adopt both Eddington-Finkelstein coordinates, u and v, in terms of which the metric takes the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dudv + r^2\Big[d\theta^2 + \sin^2(\theta)\, d\phi^2\Big]. \tag{4.3.1}$$

Here, r should be interpreted as a function of u and v, the inversion of the relation

$$\frac{u - v}{4m} = \frac{r}{2m} + \ln\left(\frac{r}{2m} - 1\right). \tag{4.3.2}$$

There remains enormous freedom in the choice of coordinates; any new coordinates U, V defined by relations of the form $u = u(U)$, $v = v(V)$ take the metric to

$$ds^2 = -\left(1 - \frac{2m}{r}\right)\frac{du}{dU}\frac{dv}{dV}\, dUdV + r^2\Big[d\theta^2 + \sin^2(\theta)\, d\phi^2\Big]. \tag{4.3.3}$$

So even preserving the general structure of the metric there is as much freedom as the choice of two arbitrary functions[5]. Which coordinates should we use? The idea is to choose the

[3] An essentially tautological statement, but one that can be forgotten.

[4] I will not give any real notion of what this means; for those who are interested a thorough treatment can be found in Hawking & Ellis, who refer to it as 'maximal coordinate extension'.

[5] Not quite arbitrary; to serve as coordinates their derivatives must not vanish.

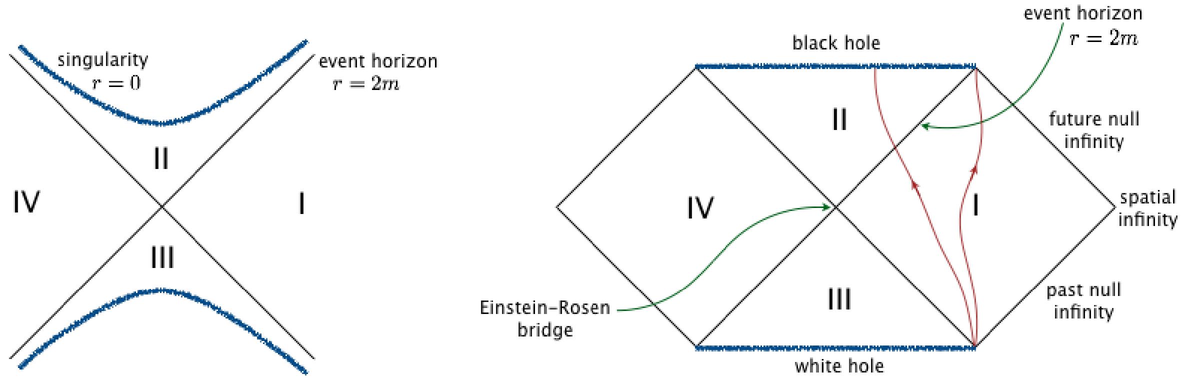


Figure 4.1: Two views of the Schwarzschild space-time. Left: the Kruskal coordinates described in the text, and Right: a compact version of the same picture, known as a Carter-Penrose diagram. Its key feature is that null directions are preserved in the compactification so that the causal structure can be correctly inferred from the diagram. Two typical time-like trajectories are indicated; one that remains outside the event horizon and one that passes through it.

functions $u(U)$ and $v(V)$ so as to maximally extend the coverage of the space-time and, in particular, to remove any coordinate singularities. Kruskal's choice[6] was to take

$$U = \mathrm{e}^{u/4m}, \qquad V = -\mathrm{e}^{-v/4m}, \tag{4.3.4}$$

for which we find

$$dUdV = \frac{r}{32m^3}\,\mathrm{e}^{r/2m}\left(1 - \frac{2m}{r}\right) dudv. \tag{4.3.5}$$

It is traditional to make a final change by defining

$$cT = \frac{U+V}{2}, \qquad X = \frac{U-V}{2}, \tag{4.3.6}$$

giving the metric for the Schwarzschild space-time in *Kruskal-Szekeres coordinates*

$$ds^2 = \frac{32m^3}{r}\,\mathrm{e}^{-r/2m}\Big[-c^2dT^2 + dX^2\Big] + r^2\Big[d\theta^2 + \sin^2(\theta)\,d\phi^2\Big]. \tag{4.3.7}$$

Tracing through the various definitions we find that the coordinates cT, X are related to the original Schwarzschild coordinates ct, r by the expressions

$$ct = 2m\,\ln\left|\frac{cT+X}{cT-X}\right|, \tag{4.3.8}$$

$$\left(\frac{r}{2m} - 1\right)\mathrm{e}^{r/2m} = -c^2T^2 + X^2. \tag{4.3.9}$$

It follows that the entirety of the Schwarzschild time-coordinate $-\infty < ct < \infty$ is contained within the subspace bounded by $cT > -X$ and $cT < X$ of the Kruskal-Szekeres coordinates. This corresponds to only one of four 'quadrants', separated by the boundaries $cT = \pm X$ – see figure (particularly left). The singularity at $r = 0$ corresponds to the set of points $-c^2T^2 + X^2 = -1$ in the Kruskal-Szekeres coordinates. It is a space-like set that exists at the very end of space-time, as it should, being the end of all things[7]. Region I is the asymptotic far-field of the Schwarzschild coordinates. A radially infalling explorer crosses the event horizon $cT = X$ (both positive) and enters region II. The boundary – the event horizon – is a null

[6]A different choice of coordinates, that also gives a maximal extension, is described by Landau & Lifshitz.

[7]It is not the end of all things – an observer that remains outside the event horizon does not fall into the singularity – but I could not resist the prose, even if it is inaccurate.

surface; even light emitted from within region II cannot escape it and necessarily arrives (in the infinite future) at the singularity at $-c^2T^2+X^2=-1, T>0$. For this reason it is called a *black hole*. Any time-like observer in region II reaches the singularity in finite proper time. Region III is an exact 'time-reversed' copy of region II. It contains a singularity at $-c^2T^2+X^2=-1$, $T<0$. Any light signal emitted from within region III will cross the event horizon and leave it; for this reason it is referred to as a *white hole*. The two regions are 'mirror images' of each other, a reflection of the symmetry $t \to -t$ associated with the space-time being static; we give them different names because of the difference in the way that we perceive them and because we have no experience of anything other than a particular direction for 'time'. Finally, region IV is a duplicate of region I, a second asymptotically flat region of space-time. They are connected by a space-like 'portal' or 'wormhole'; a 2-sphere of 'radius' $2m$ called the Einstein-Rosen bridge. It is impossible for any time-like observer to travel between them or even to communicate with the other side.

4.4 The Classical Tests of General Relativity

The classical predictions of general relativity can be presented using the motion of test particles in the Schwarzschild space-time, this capturing the behaviour of a small body in the spherically symmetric gravitational far-field of a larger mass. The basic statement is one we have used many times; test particles move along geodesics. Thus the classical tests of general relativity – the precession of the perihelion of Mercury, gravitational lensing and the Shapiro delay – all come from a knowledge, and understanding, of the geodesics of the Schwarzschild space-time. We study them now.

We obtained the geodesic equations in the (t,r,θ,ϕ) coordinates as part of our derivation of the Schwarzschild metric. We reproduce them now, simplified by using the explicit forms for the functions $f(r), g(r)$ in the metric

$$0 = \partial_\tau\left[-\left(1-\frac{2m}{r}\right)c\partial_\tau t\right], \tag{4.4.1}$$

$$0 = \partial_\tau\left[\frac{\partial_\tau r}{1-2m/r}\right] + \frac{m}{r^2}\big(c\partial_\tau t\big)^2 + \frac{m/r^2}{\big(1-2m/r\big)^2}\big(\partial_\tau r\big)^2 - r\Big[\big(\partial_\tau\theta\big)^2 + \sin^2(\theta)\big(\partial_\tau\phi\big)^2\Big], \tag{4.4.2}$$

$$0 = \partial_\tau\big(r^2\partial_\tau\theta\big) - r^2\sin(\theta)\cos(\theta)\big(\partial_\tau\phi\big)^2, \tag{4.4.3}$$

$$0 = \partial_\tau\big(r^2\sin^2(\theta)\partial_\tau\phi\big). \tag{4.4.4}$$

The first of these integrates immediately to give a first integral of the motion

$$c\partial_\tau t = \frac{\gamma}{1-2m/r}, \tag{4.4.5}$$

for a constant γ, that looks a bit like the asymptotic γ-factor of special relativity. This constant of the motion γ is associated to the time-translation invariance ($t \to t + \text{const.}$) of the metric, and, as such, is properly referred to as the 'energy' (in natural units and per unit mass). As in the non-relativistic Kepler problem, the two angular geodesic equations have the solution

$$\theta = \pi/2, \qquad\qquad r^2\partial_\tau\phi = \ell, \tag{4.4.6}$$

where ℓ is a constant, and this one particular solution is enough to understand all of the geodesics. The reason is the same; it is a consequence of the spherical symmetry, and typically thought of in terms of conservation of angular momentum. This all given, the equation for $r(\tau)$ reduces to

$$0 = \partial_{\tau\tau}r + \frac{\gamma^2 m/r^2}{1-2m/r} - \frac{m/r^2}{1-2m/r}\big(\partial_\tau r\big)^2 - \left(1-\frac{2m}{r}\right)\frac{\ell^2}{r^3}. \tag{4.4.7}$$

However, it proves easier not to deal with this equation directly but to recall that the geodesic is parameterised by (c times) proper time, which contributes the first integral

$$-1 = -\left(1 - \frac{2m}{r}\right)c^2\big(\partial_\tau t\big)^2 + \frac{1}{1-2m/r}\big(\partial_\tau r\big)^2 + r^2\Big[\big(\partial_\tau \theta\big)^2 + \sin^2(\theta)\big(\partial_\tau \phi\big)^2\Big], \tag{4.4.8}$$

$$= \frac{-\gamma^2}{1-2m/r} + \frac{1}{1-2m/r}\big(\partial_\tau r\big)^2 + \frac{\ell^2}{r^2}. \tag{4.4.9}$$

This equation rearranges to the form

$$\big(\partial_\tau r\big)^2 - \frac{2m}{r} + \frac{\ell^2}{r^2}\left(1 - \frac{2m}{r}\right) = \gamma^2 - 1, \tag{4.4.10}$$

and can then be thought of as an energy balance 'kinetic energy plus potential energy equals total energy' for a one-dimensional dynamical system with effective potential

$$2V_{\text{eff}} = -\frac{2m}{r} + \frac{\ell^2}{r^2} - \frac{2m\ell^2}{r^3}. \tag{4.4.11}$$

The first two terms are those that appear in the Kepler problem of Newtonian gravity; it is only the last term $-2m\ell^2/r^3$ that is new in general relativity.

4.4.1 Precession of Perihelia

To find the orbits, we proceed in the same fashion as one does in the usual Kepler problem, switch to describing the geodesic (orbit) by the radial distance as a function of angle, *i.e.* $r = r(\phi)$, and introduce a new variable $u = 1/r$. This leads to the equation

$$\big(\partial_\phi u\big)^2 - \frac{2m}{\ell^2}u + u^2 - 2mu^3 = \frac{\gamma^2 - 1}{\ell^2}. \tag{4.4.12}$$

It can be solved exactly as an elliptic integral. We will treat it approximately and find that the easiest thing to do is to differentiate with respect to ϕ to give the second order equation

$$\partial_{\phi\phi}u - \frac{m}{\ell^2} + u - 3mu^2 = 0, \tag{4.4.13}$$

in which we consider the term $-3mu^2$ as a small perturbation[8]. We factor the quadratic polynomial to the form

$$\partial_{\phi\phi}u + \big(1 - 6mu_c\big)\big(u - u_c\big) - 3m\big(u - u_c\big)^2 = 0, \tag{4.4.14}$$

where

$$u_c = \frac{1}{6m}\left[1 - \left(1 - \frac{12m^2}{\ell^2}\right)^{1/2}\right] = \frac{m}{\ell^2} + \frac{6m^3}{\ell^4} + O(m^5\ell^{-6}), \tag{4.4.15}$$

and has an interpretation as the inverse radius of a circular orbit. Neglecting the quadratic term, $-3m(u - u_c)^2$, the solution is

$$u = u_c\Big[1 + e\cos\big(\sqrt{1 - 6mu_c}\,\phi\big)\Big], \tag{4.4.16}$$

where e is the eccentricity of the orbit. This should be contrasted against the Keplerian orbit, obtainable by neglecting the general relativistic effects entirely

$$u_{\text{Kepler}} = \frac{m}{\ell^2}\Big[1 + e\cos(\phi)\Big]. \tag{4.4.17}$$

[8]Our presentation is mildly idiosyncratic, being specially adapted to the particular problem; the formal method for what we are doing is known as multiple time scales perturbation theory. I am grateful to George Rowlands for explaining this to me.

The precession of the perihelion arising from general relativistic effects is given by the difference from 2π of the change in angle between successive perihelia and is readily seen to be

$$\Delta\phi = \frac{2\pi}{\sqrt{1-6mu_c}} - 2\pi = 6\pi m u_c + O(m^2 u_c^2), \tag{4.4.18}$$

$$= \frac{6\pi m^2}{\ell^2} + O(m^4 \ell^{-4}). \tag{4.4.19}$$

The orbit of the planet Mercury is not close to circular, the eccentricity being $e = 0.21$. Nonetheless, for the purpose of estimates it suffices to take the value of the semi-major axis, 5.79×10^7 km, and average orbital speed, 47.4 km s^{-1}, in estimating the value of the constant ℓ. These values give $\ell = 9.15 \times 10^3$ km. Recall that the Schwarzschild mass m is equal to GM/c^2, where M is the actual mass. For the sun $m = 1.48$ km. The predicted contribution of general relativistic effects to the precession of the perihelion of Mercury is therefore $\Delta\phi = 4.93 \times 10^{-7}$ radians, or 0.102 arcseconds. This is the shift per Mercurian year. Mercury orbits the sun with a period of 0.24 years, so that the precession amounts to

$$0.102 \times \frac{100}{0.24} = 42.5 \quad \text{seconds of arc per century.}$$

This prediction, which accurately accounts for the discrepancy between the observed precession and the predictions of Newton's theory, was in 1915 the only experimental test of Einstein's theory of general relativity.

4.4.2 Gravitational Lensing

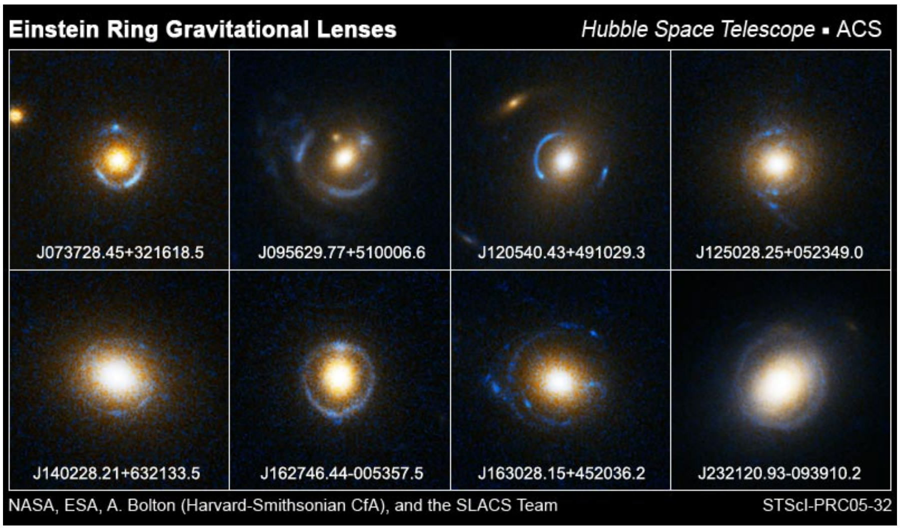


Figure 4.2: A gallery of Einstein ring images taken by the Hubble Space Telescope. They are the result of the gravitational lensing of light from a distant object by one closer to us along the same line-of-sight. Image from NASA.

We have all seen the beautiful images of *gravitational lensing* of distant galaxies by those closer to us along the same 'line of sight'. They are wonderous and truly inspiring. The original observation of gravitational deflection of light (or lensing) was made for stars 'behind' the sun by Sir Arthur Eddington during the total solar eclipse of 1919. The premise for it is much the same as for gravitational lensing of distant galaxies; we are observing light travelling along null geodesics and passing near to a massive object. The simplest account of this phenomenon is therefore provided by the null geodesics of the Schwarzschild metric.

The derivation we gave for time-like geodesics does not pass over directly to null geodesics. Nonetheless, they are geodesics and satisfy the same geodesic equation – their tangent vector is parallel transported along itself – with the only difference being that this tangent vector is a null vector. Therefore they are still characterised by constants

$$\left(1-\frac{2m}{r}\right)c\partial_\tau t=\gamma, \qquad\qquad r^2\partial_\tau\phi=\ell, \tag{4.4.20}$$

much as before, with the help of which the condition that the tangent vector is null gives

$$\left(\partial_\tau r\right)^2+\frac{\ell^2}{r^2}-\frac{2m\ell^2}{r^3}=\gamma^2. \tag{4.4.21}$$

Switching to a parameterisation in terms of ϕ, as before, and rewriting in terms of the inverse radius $u=1/r$, as before, we arrive at the equation

$$\begin{aligned}\left(\partial_\phi u\right)^2+u^2-2mu^3&=\frac{\gamma^2}{\ell^2},\\ \text{or}\qquad \partial_{\phi\phi}u+u-3mu^2&=0.\end{aligned} \tag{4.4.22}$$

The equation can be integrated exactly as an elliptic integral; as before we shall adopt a perturbative approach, writing $u=u^{(0)}+u^{(1)}+\cdots$ and solving order by order. The zeroth order solution neglecting general relativistic effects is $u^{(0)}=b^{-1}\cos(\phi)$ and corresponds to the straight line $x=b$ – just use $x=r\cos(\phi)$ and $u=1/r$. The first order perturbation to this is given by the particular integral of

$$\partial_{\phi\phi}u^{(1)}+u^{(1)}=3m\left(u^{(0)}\right)^2=\frac{3m}{2b^2}\left[1+\cos(2\phi)\right], \tag{4.4.23}$$

which one may verify is

$$u^{(1)}=\frac{3m}{2b^2}\left[1-\frac{1}{3}\cos(2\phi)\right]. \tag{4.4.24}$$

Thus the photon trajectory, incorporating the first order correction coming from general relativity, is described by

$$u=\frac{1}{r}=\frac{1}{b}\cos(\phi)+\frac{3m}{2b^2}\left[1-\frac{1}{3}\cos(2\phi)\right]+O(m^2b^{-3}). \tag{4.4.25}$$

The closest approach of the photon to the origin ($r=0$) still occurs when $\phi=0$ and is given by the distance $b(1+m/b^2)^{-1}$. The asymptotic directions can be found by setting $u=0$, leading to

$$\cos(\phi_\pm)=\frac{b}{2m}\left[1-\sqrt{1+\frac{8m^2}{b^2}}\right]=\frac{-2m}{b}+O(m^3b^{-3}). \tag{4.4.26}$$

The total deflection angle is therefore given approximately by

$$\Delta\phi=\frac{4m}{b}, \tag{4.4.27}$$

to leading order in general relativistic effects.

Let us estimate this deflection for light passing close to the sun as in the original observations of Eddington. Recall that the Schwarzschild mass m, in SI units, is equal to GM/c^2 where M is the (proper) stellar mass and G is Newton's gravitational constant. For the sun $GM/c^2\approx 1.48$ km, while the solar radius is approximately 6.96×10^5 km. It follows that light that just grazes the solar surface undergoes a total deflection according to general relativity of approximately

$$\Delta\phi\approx 8.5\times10^{-6}\ \text{radians}, \tag{4.4.28}$$

or about 1.75 seconds of arc.

* * *

Gravitational lensing has become an important observational tool for inferring mass distributions of intervening objects that act as lenses for more distant galaxies or quasars. The idea is to relate the deflection angle to the mass distribution of the 'lens' from theory and then by 'fitting' an experimental image infer the mass distribution. The simplest description models the lens by a 'weak-field' metric

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right)\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right], \tag{4.4.29}$$

where, as we learned in Chapter 3, Φ may be identified with the Newtonian potential. Since the light that reaches us travels along null geodesics we can say

$$\begin{aligned} c^2 dt^2 &= \frac{1 - 2\Phi/c^2}{1 + 2\Phi/c^2}\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right], \\ \text{or} \qquad cdt &= \left(1 - 2\Phi/c^2\right) d\ell, \end{aligned} \tag{4.4.30}$$

where $d\ell^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ and assuming Φ/c^2 is small. We see that the Newtonian potential acts much like an effective refractive index for the light as it travels to us through the intervening matter distribution. With this in mind, the path that we observe the light ray to follow will be the same as if the light were passing through a medium with refractive index $n(\mathbf{x}) = 1 - 2\Phi(\mathbf{x})/c^2$, and is hence given by Fermat's principle of least time – the path followed by the light is such as to make the time taken as short as possible.

Let us orient our coordinate system so that the x^3-axis corresponds to the 'line of sight' to the distant quasar. Then, parameterising the path of the light by $(x^1(z), x^2(z), z)$, the time that it takes to reach us is

$$ct = \int_{\text{quasar}}^{\text{Earth}} \left(1 - 2\Phi/c^2\right)\left[\left(\partial_z x^1\right)^2 + \left(\partial_z x^2\right)^2 + 1\right]^{1/2} dz, \tag{4.4.31}$$

and by Fermat's principle the path should be chosen so as to correspond to a critical point of this functional; a geodesic of the so-called *optical metric*. Carrying out the variation we find

$$\begin{aligned} ct[x+\epsilon] - ct[x] = \int_{\text{quasar}}^{\text{Earth}} \Big\{ &\left(1 - 2\Phi/c^2\right) \frac{\partial_z x^i\, \partial_z \epsilon^i}{\left[(\partial_z x^1)^2 + (\partial_z x^2)^2 + 1\right]^{1/2}} \\ &- \frac{2}{c^2}\epsilon^i \partial_i \Phi \left[\left(\partial_z x^1\right)^2 + \left(\partial_z x^2\right)^2 + 1\right]^{1/2} \Big\} dz + O(\epsilon^2), \end{aligned} \tag{4.4.32}$$

$$\begin{aligned} = -\int_{\text{quasar}}^{\text{Earth}} \epsilon^i \Big\{ &\partial_z\left(\frac{\left(1 - 2\Phi/c^2\right)\partial_z x^i}{\left[(\partial_z x^1)^2 + (\partial_z x^2)^2 + 1\right]^{1/2}}\right) \\ &+ \frac{2\partial_i \Phi}{c^2}\left[\left(\partial_z x^1\right)^2 + \left(\partial_z x^2\right)^2 + 1\right]^{1/2} \Big\} dz + O(\epsilon^2), \end{aligned} \tag{4.4.33}$$

and it follows that the path taken by the light satisfies

$$\partial_z\left(\frac{\left(1 - 2\Phi/c^2\right)\partial_z x^i}{\left[(\partial_z x^1)^2 + (\partial_z x^2)^2 + 1\right]^{1/2}}\right) = -\frac{2\partial_i \Phi}{c^2}\left[\left(\partial_z x^1\right)^2 + \left(\partial_z x^2\right)^2 + 1\right]^{1/2}. \tag{4.4.34}$$

Integrating over z along the line of sight, from the quasar to ourselves, and recalling that the distance element is $d\ell = [(dx^1)^2 + (dx^2)^2 + dz^2]^{1/2}$, we can then say

$$\left(1 - 2\Phi/c^2\right)\frac{\partial x^i}{\partial \ell}\bigg|_{\text{quasar}}^{\text{Earth}} = \frac{-2}{c^2}\int_{\text{quasar}}^{\text{Earth}} \partial_i \Phi \, d\ell. \tag{4.4.35}$$

For the left-hand-side we will assume that the Newtonian potential Φ vanishes at both source and observer, which are situated far from the matter distribution of the lensing galaxy it is accounting for. It follows that the left-hand-side is the difference in the angle the light ray makes with the 'line of sight' at Earth and at the quasar, or in other words, the total deflection angle $\Delta\alpha_i$. For the right-hand-side we use Newton's formula for the gravitational potential and find

$$\Delta\alpha_i = \frac{2G}{c^2}\int_{\text{quasar}}^{\text{Earth}} \partial_i \left(\int_{\text{lens}} \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \right) d\ell, \tag{4.4.36}$$

$$= \frac{-2G}{c^2}\int_{\text{quasar}}^{\text{Earth}} \int_{\text{lens}} \frac{(x-x')^i \rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x'\, d\ell. \tag{4.4.37}$$

The integral over the line of sight can be done with a 'thin lens' approximation for the lensing galaxy, which seems completely reasonable, and that the path distance ℓ is very nearly the same as the 3-component $z \approx \ell$. This then leads to the lensing formula

$$\Delta\alpha_i = \frac{-4G}{c^2}\int_{\text{lens}} \frac{(x-x')^i \rho_2(x')}{|x-x'|^2}\, d^2x', \tag{4.4.38}$$

where $\rho_2(x') = \int \rho(x', z)\, dz$ is the integral of the mass density over the 'thickness' of the lensing galaxy to give an effective two-dimensional average. Fitting observational images to predictions from this formula allows lensing matter distributions to be inferred.

Problems

1. Calculate the general relativity contribution to the precession of the perihelion of the Earth.

2. Consider a satellite, of mass M_s, in circular orbit about the Earth and neglect all other gravitational influences. Determine the constants ℓ and γ characterising the orbit.

   Show that the rate at which time passes as measured by the satellite differs from the proper time (τ/c) measured by an observer at infinite distance by a factor

$$\alpha = \frac{1}{c}\frac{\partial \tau}{\partial t} = \left(1 - \frac{3m}{r}\right)^{1/2}.$$

   Compute this factor for a satellite at distance $r = 2.66 \times 10^4$ km from the centre of the Earth. Compute it also for the same satellite when it was being calibrated on the Earth's surface.

   The GPS system gives highly accurate positional information for locations on the Earth's surface. In part, this depends on timing data sent between the satellites and the Earth's surface. Compute the total time discrepancy that accumulates over one day and estimate the positional error that would be made if this was not accounted for in the operation of the GPS system.

3. In the notes we obtained the equations describing geodesics of the Schwarzschild space-time in the form

$$\left(1 - \frac{2m}{r}\right)\frac{d\,ct}{d\tau} = \gamma, \qquad r^2\frac{d\phi}{d\tau} = \ell,$$
$$\left(\frac{dr}{d\tau}\right)^2 - \frac{2m}{r} + \frac{\ell^2}{r^2} - \frac{2m\ell^2}{r^3} + 1 - \gamma^2 = 0,$$

   where γ and ℓ are constants, and with $\theta = \pi/2$ without loss of generality. Sketch the form of the effective potential for different values of the 'angular momentum' ℓ. Show that the effective potential has critical points (maximum and minimum) at values of the radial coordinate given by

$$r = \frac{\ell^2}{2m}\left(1 \pm \left[1 - \frac{12m^2}{\ell^2}\right]^{1/2}\right),$$

   and hence that there are no (stable) circular orbits if $\ell^2 \geq 12m^2$.

   Show that the maximum in the effective potential corresponds to the value $V = 0$ when $\ell^2 = 16m^2$. Non-relativistic particles with angular momenta less than this value therefore fall into the black hole and are captured by it. Show that the capture cross section is $\sigma = 16\pi m^2$.

4. In the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)c^2dt^2 + \frac{1}{1 - \frac{2m}{r}}\,dr^2 + r^2\left[d\theta^2 + \sin^2(\theta)\,d\phi^2\right],$$

   make the change of variables $r \mapsto \rho$ defined by

$$r = \left(1 + \frac{m}{2\rho}\right)^2 \rho,$$

   and show that the metric then takes the form

$$ds^2 = -\left(\frac{1 - \frac{m}{2\rho}}{1 + \frac{m}{2\rho}}\right)^2 c^2dt^2 + \left(1 + \frac{m}{2\rho}\right)^4\left[dx^2 + dy^2 + dz^2\right],$$

where x, y, z are standard Cartesian coordinates for the spatial sections ($t = \text{const}$), with $x^2 + y^2 + z^2 = \rho^2$.

Show that the coordinate range $\frac{m}{2} \leq \rho < \infty$ covers the region $r \geq 2m$ of the original Schwarzschild coordinates. What region of the original Schwarzschild coordinates is covered by the range $0 < \rho \leq \frac{m}{2}$?

5. In 1964 Irwin Shapiro proposed a new solar system test of general relativity based on the time taken to send radar signals to the inner planets – Venus and Mercury – and back. This time should depend on the gravitational field that the signal passes through, so the idea is to compare the time taken when Venus or Mercury are at different points of their orbit so that the signal has a varying distance, b, of closest approach to the sun. Denote the radial distance of the Earth from the sun by R_{E} and that of Venus/Mercury by R_{P}. We may neglect the motion of either planet during the entire journey time of the radar signal. (Why?) Equally, it suffices to take the trajectory of the radar signal as a straight line.

   Using the Cartesian coordinate system of the previous problem, show that along the trajectory of the radar signal

$$cdt = \left(1 - \frac{m}{2\rho}\right)^{-1}\left(1 + \frac{m}{2\rho}\right)^{3} d\ell,$$

   where $d\ell$ is 'spatial distance' in the Cartesian coordinate, and hence that the time taken when the inner planet (Venus or Mercury) is near superior conjunction is given by

$$\begin{aligned} c\Delta t &\approx 2\int_{\mathrm{E}}^{\mathrm{P}} \left(1 + \frac{2m}{\rho}\right) d\ell, \\ &= 2\left\{\sqrt{R_{\mathrm{E}}^2 - b^2} + \sqrt{R_{\mathrm{P}}^2 - b^2} + 2m\left[\operatorname{arcsinh}\sqrt{(R_{\mathrm{E}}/b)^2 - 1} + \operatorname{arcsinh}\sqrt{(R_{\mathrm{P}}/b)^2 - 1}\right]\right\}, \\ &\approx 2\left\{R_{\mathrm{E}} + R_{\mathrm{P}} + 2m\ln\frac{4R_{\mathrm{E}}R_{\mathrm{P}}}{b^2}\right\}. \end{aligned}$$

   [For the last step use the formula $\operatorname{arcsinh} x = \ln(x + \sqrt{x^2+1})$, either with or without proof.]

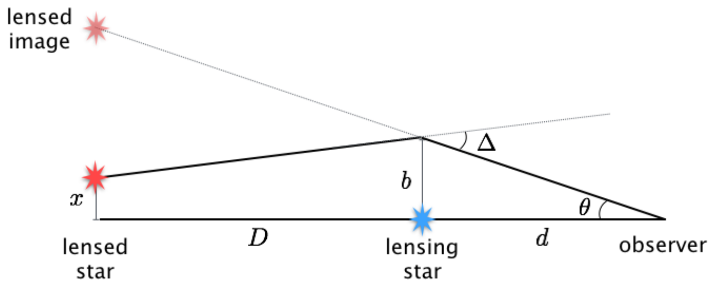
   The second term may be thought of as a delay due to the gravitational potential of the sun. Estimate its magnitude. What is the analogous expression for the delay at inferior conjunction?

   In Shapiro's original paper the time delay was calculated in Schwarzschild coordinates rather than the Cartesian coordinates suggested here. Calculate the delay in Schwarzschild coordinates.

   [The variation in the delay as both Venus and Mercury moved between inferior and superior conjunction was measured experimentally in 1967 and again in 1970. The latter results are presented here.]

6. In 1936 Einstein published a short calculation on the 'Lens-like Action of a Star by the Deviation of Light in the Gravitational Field', which is linked to here. This problem works through that calculation. The figure (not to scale) illustrates the general setup: a star at distance d from the observer acts as a lens for a more distant star, displaced off the direct line of sight by a distance x. The deflection angle of light passing close to the lens is Δ and the observed angle of the lensed star, relative to the direction of the lensing star, is θ. Show that the deflection angle is given by

$$\Delta = \frac{b}{d} + \frac{b - x}{D},$$



and hence that the observation angle θ is (m is the usual parameter appearing in the Schwarzschild metric)

$$\theta = \frac{x}{2(D+d)} \pm \left[\frac{4mD}{d(D+d)} + \left(\frac{x}{2(D+d)}\right)^2\right]^{1/2}.$$

Why are there two values? What happens to the two images as x gets steadily larger? Describe (qualitatively) what is seen when $x \to 0$ and estimate the magnitude of the effect in this case.

7. Consider an observer stationary at Schwarzschild coordinate r_0, so that their trajectory is given by $(\gamma\tau, r_0, \theta_0, \phi_0)$. Show that τ is c times proper time if the constant γ is given by

$$\gamma = \left(1 - \frac{2m}{r_0}\right)^{-1/2}.$$

If $u^\mu = dx^\mu/d\tau$ is the velocity of the observer (divided by c), show that their acceleration (divided by c^2) is

$$a = \frac{du}{d\tau} = \left(\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu\right)\partial_\alpha,$$

and hence that the magnitude of the acceleration (divided by c^2) is

$$\frac{GM}{c^2 r_0^2}\left(1 - \frac{2m}{r_0}\right)^{-1/2}.$$

What is its direction?

8. How large would a sphere of material with the same density as air have to be for its Schwarzschild radius to exceed its own radius?

Were you to fall into a black hole, would you see the singularity at $r = 0$ once you crossed the event horizon?

9. Consider the Einstein equations with cosmological constant

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

and look for a solution corresponding to a static, spherically symmetric space-time external to a central body with metric

$$ds^2 = -f(r)\, c^2 dt^2 + g(r)\, dr^2 + r^2\big[d\theta^2 + \sin^2(\theta)\, d\phi^2\big],$$

analogous to the Schwarzschild one. Given that

$$R_{00} - \frac{1}{2}R g_{00} = \frac{f}{r^2}\frac{d}{dr}\Big[r\big(1 - g^{-1}\big)\Big], \qquad R_{rr} - \frac{1}{2}R g_{rr} = \frac{f'}{rf} - \frac{g}{r^2}\big(1 - g^{-1}\big),$$

show that f and g are given by

$$f = g^{-1} = 1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2.$$

Show that the time-like geodesics of this space-time, for the particular case $\theta = \pi/2$, are characterised by

$$\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)\frac{d\,ct}{d\tau} = \gamma, \qquad r^2\frac{d\phi}{d\tau} = \ell,$$
$$\left(\frac{dr}{d\tau}\right)^2 - \frac{2m}{r} + \frac{\ell^2}{r^2} - \frac{2m\ell^2}{r^3} - \frac{1}{3}\Lambda\big(r^2 + \ell^2\big) + 1 - \gamma^2 = 0,$$

where γ and ℓ are constants. Thinking of the last of these in terms of an effective potential, describe qualitatively the influence of the cosmological constant on planetary motion. Given current estimates for the value of the cosmological constant of 10^{-52} m^{-2}, comment on whether or not these effects are detectable in our solar system.

10. Allow for possible time dependence of the functions f and g in the metric for a spherically symmetric space-time

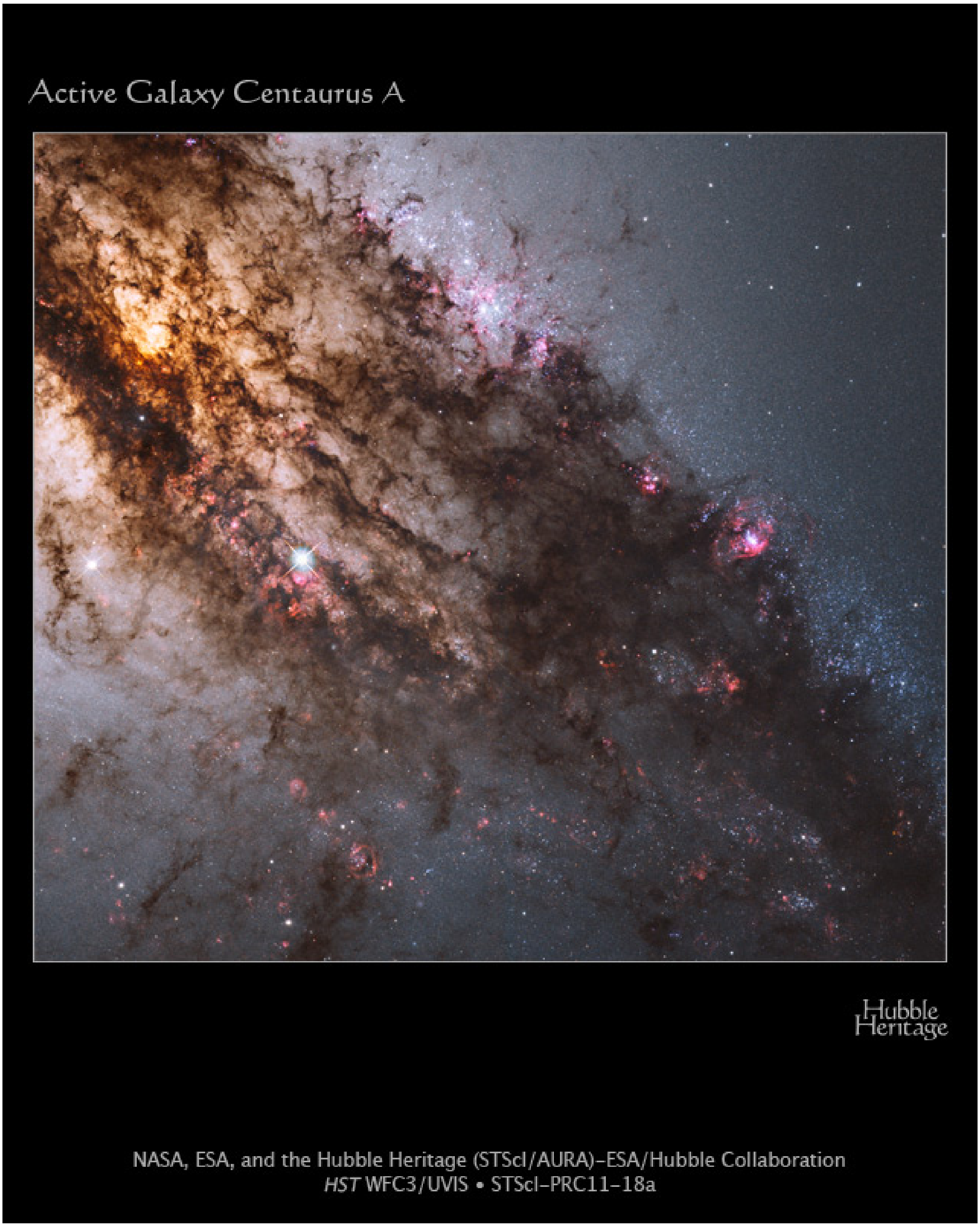
$$ds^2 = -f(r,t)\,c^2dt^2 + g(r,t)\,dr^2 + r^2\Big[d\theta^2 + \sin^2(\theta)\,d\phi^2\Big].$$

Show that the Einstein equations imply that neither f nor g depend on time and hence the only solution is still the Schwarzshild metric, *i.e.* establish Birkhoff's theorem.

11. It is often said that general relativity is a theory about the curvature of space-time; it is one of its most evocative tag-lines. A large fraction of all gravitational phenomena that we have been able to study may be understood in terms of properties of the Schwarzschild solution – this includes planetary orbits, deflection of light and lensing, gravitational redshift, Shapiro delay, and many basic properties of black holes. In that solution one solves the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

Show that $R = 0$ and hence that $R_{\mu\nu} = 0$, *i.e.* that both the Ricci scalar and all components of the Ricci curvature tensor vanish identically. Given this, what does it mean to say that the Schwarzschild space-time is curved?



The active galaxy Centaurus A taken by the Hubble Space Telescope in 2011. The centres of active galaxies, indeed all galaxies, are believed to contain supermassive black holes. Image from NASA.

Chapter 5

Gravitational Collapse and Black Holes

It seems reasonable to try to attack the problem of stellar structure by the methods of theoretical physics, i.e. *to investigate the physical nature of stellar equilibrium.*

Lev Landau (1932)

5.1 Gravitational Collapse

The Schwarzschild solution, as we have presented it, describes the space-time metric in the vacuum region outside a single massive body; it applies only to this exterior region. Most stars have a radius that far exceeds their Schwarzschild radius (6.96×10^5 km compared to 2.96 km in the case of the sun) and it is only beyond this distance that the Schwarzschild metric describes the space-time geometry. The interior part consists of the body of the star, where the stress-energy-momentum tensor is not zero, as it is for a vacuum, but may be approximated by that of a perfect fluid. The essential nature of the solution then follows directly from hydrostatic equilibrium; the pressure gradient must balance the gravitational force of attraction. In Newtonian terms this condition would be

$$-\frac{dp}{dr} = \frac{G\rho(r)M(r)}{r^2}, \tag{5.1.1}$$

where $M(r)$ is the total mass up to radius r. A full treatment of the spherically symmetric static Einstein equations leads to the *Tolman-Oppenheimer-Volkoff equation* of hydrostatic equilibrium for a spherically symmetric star

$$-\frac{dp}{dr} = \frac{G(\rho c^2 + p)}{c^2 r^2 \left(1 - \frac{2m(r)}{r}\right)} \left[\frac{m(r)c^2}{G} + \frac{4\pi r^3 p}{c^2}\right], \tag{5.1.2}$$

where $m(r) = \frac{4\pi G}{c^2} \int_0^r \rho(r') r'^2 \, dr'$. In the discussion that follows the relativistic aspects of this hydrostatic condition are not crucially important and it will be sufficient to use the Newtonian relation for the purpose of making our estimates. First, we estimate that in a star of total mass M and radius R the Newtonian expression for the gravitational pressure balance gives that the typical internal pressure the star must maintain is roughly

$$P \sim \frac{GM\rho}{R} \sim GM^{2/3}\rho^{4/3}, \tag{5.1.3}$$

where ρ is the density. What will be important in the subsequent comparison is that the 'gravitational pressure' varies with the stellar density as $\rho^{4/3}$. Now, whether or not the star can create and sustain the necessary pressure gradient, *i.e.* whether or not there is a stable stellar equilibrium, depends on its physical nature, which in our simple picture of things means the equation of state relating its pressure, temperature and density. Now, I must confess that I am not very knowledgeable about stars and their equations of state; certainly they are tremendously complicated things and I am but a simple person. Nonetheless, it is still possible to make generic statements and we will content ourselves with that.

When a star is young it burns hydrogen generating both heat and radiation that provide the pressure that supports it against gravity. But what happens when the star is old and cold? Forms of matter that we have some understanding of, such as atoms and nuclei – the things stars are made of –, have a size, an Angstrom or a Fermi. This size is set by quantum mechanics. The fundamental concept that underpins the structure and properties of simple forms of condensed matter is the exclusion principle; a large body of fermions occupy quantum states at most singly so that, at low temperatures, they fill up the available states to some Fermi surface, characterised by a Fermi wavevector k_{F}. If there are N fermions in a region of size V then

$$N = \int_0^{k_{\mathrm{F}}} \frac{V}{\pi^2}\, k^2\, dk = \frac{V}{3\pi^2}\, k_{\mathrm{F}}^3. \tag{5.1.4}$$

Assuming the fermions are non-relativistic, their energy will be roughly

$$E \simeq \int_0^{k_{\mathrm{F}}} \frac{V}{\pi^2}\, k^2\, \frac{\hbar^2 k^2}{2m}\, dk = \frac{\hbar^2 V}{10\pi^2 m}\, k_{\mathrm{F}}^5 = \frac{\hbar^2 (3\pi^2)^{5/3}}{10\pi^2 m} N^{5/3} V^{-2/3}, \tag{5.1.5}$$

and it follows that the pressure of such a degenerate Fermi gas is given by

$$P = -\frac{\partial E}{\partial V} \simeq \frac{3^{2/3}\pi^{4/3}\hbar^2}{5m} (N/V)^{5/3}. \tag{5.1.6}$$

For stars that are composed of ordinary atoms, the degeneracy we have just described is associated to electron states, so that the mass m is the electron mass and the number N is the number of electrons in the star. This is the same as the number of protons, but only a fraction of the total number of nucleons so that the total mass of the star is $M = \mu m_{\mathrm{H}} N$ where m_{H} is the mass of hydrogen and μ an average number of nucleons per electron in a typical aged star, say about 2. This given, balancing the estimate of the degeneracy pressure against the gravitational one requires that the stellar density should be roughly

$$\rho \simeq \frac{125 m_e^3 (\mu m_{\mathrm{H}})^5}{9\pi^4} \left(\frac{G}{\hbar^2}\right)^3 M^2. \tag{5.1.7}$$

Stars that are able to maintain equilibrium conditions of this kind into their old age are known as *white dwarfs*.

The density estimated here increases with the mass of the star; more massive stars are more dense, meaning that the fermions they are made of are more tightly confined. If they are confined tightly enough their typical momenta estimated from the uncertainty principle will put them into a relativistic regime. The conclusion is then different, for their typical energy is no longer the non-relativistic $\hbar^2 k^2/2m$ but rather the relativistic estimate $\sim \hbar c k$. The energy of such a relativistic degenerate Fermi gas will be roughly

$$E \simeq \int_0^{k_{\mathrm{F}}} \frac{V}{\pi^2}\, k^2\, \hbar c k\, dk = \frac{\hbar c V}{4\pi^2}\, k_{\mathrm{F}}^4 = \frac{\hbar c (3\pi^2)^{4/3}}{4\pi^2} N^{4/3} V^{-1/3}, \tag{5.1.8}$$

leading to an expression

$$P \simeq \frac{(3\pi^2)^{1/3}\hbar c}{4} (N/V)^{4/3}, \tag{5.1.9}$$

for the typical interior pressure of such stars. We may again assume electron degeneracy, however, this time a balance against the gravitational requirements apparently leads to a definite value for the stellar mass

$$M \simeq \frac{\sqrt{3}\pi}{8} \left(\frac{\hbar c}{G}\right)^{3/2} (\mu m_{\mathrm{H}})^{-2}. \tag{5.1.10}$$

Substituting relevant values one finds $\sim 0.3 M_\odot$. This is not bad, but a little smaller than the proper value. If one treats the hydrostatic equilibrium properly[1], for the degenerate Fermi gas equation of state, one obtains a value about 5 times larger

$$M \simeq 1.44 M_\odot, \tag{5.1.11}$$

known as the *Chandrasekhar limit*. Although the actual value is obviously important, the basic concept and behaviour is correctly captured by our roughshod methods. There is an upper bound to the mass of a star that can exist as a degenerate Fermi gas supported by electron degeneracy pressure. The striking prediction that comes from this is that stars more massive than the Chandrasekhar limit cannot possibly end their existence as white dwarfs; it is their fate to suffer gravitational collapse to a different state of existence. If the mass is not too great they will end up as a *neutron star*, their electronic content forcibly crushed by gravity into recombination with protons to leave an extraordinary body of nuclear matter. The equation of state of such a star may be even less well-known than that of white dwarfs, but it can still be constrained by quantum mechanics and the degeneracy pressure that comes from being a many-body state of fermions. The point is, that in the relativistic limit the degeneracy pressure will again vary with the density as $\rho^{4/3}$, by precisely the same argument, and we again arrive at an upper limit for the stellar mass, estimated this time at between 1.5 and 3 times the solar mass. This is known as the *Tolman-Oppenheimer-Volkoff limit*. More massive stars are unable to resist the force of gravity and presumably undergo continued gravitational collapse until their outer surface recedes within their Schwarzschild radius at which point a singularity is inevitable and a black hole is formed.

5.2 The Kerr Black Hole

The Schwarzschild solution was generalised almost immediately by Hans Reissner and Gunnar Nordström (independently, 1916 and 1918) to give the spherically symmetric solution of the Einstein equations for a charged black hole. Further generalisation was finally found in 1963 when Roy Kerr presented his solution for an axisymmetric rotating black hole. In 'Boyer-Lindquist' coordinates the *Kerr metric* is

$$\begin{aligned} ds^2 = -c^2dt^2 &+ \frac{2mr}{r^2+a^2\cos^2(\theta)}\big(a\sin^2(\theta)\,d\phi - cdt\big)^2 + \big(r^2+a^2\big)\sin^2(\theta)\,d\phi^2 \\ &+ \frac{r^2+a^2\cos^2(\theta)}{r^2-2mr+a^2}\,dr^2 + \big(r^2+a^2\cos^2(\theta)\big)d\theta^2, \end{aligned} \tag{5.2.1}$$

where $M = mc^2/G$ is the *mass* of the black hole and $J = aMc$ its *angular momentum*. When the angular momentum a is zero it reduces to the Schwarzschild metric in Schwarzschild coordinates. Unlike the Schwarzschild metric, the Kerr metric does not currently have a derivation that can be widely considered physically intuitive, or peripatetic, or approachable at undergraduate level. Even verifying that the metric, once given, is indeed an exact solution of the Einstein equations is a lengthy and unenlightening process[2]. I make no attempt here either to motivate it, or verify it. However, for those who are interested, Roy Kerr has given a fascinating historical account of how he arrived at the rotating black hole metric [arXiv:0706.1109v2]. There is reason to believe – the uniqueness theorems – that the charged, rotating black hole (Kerr-Newman metric) is the most general form of a stationary black hole. In other words, we do not need to search for further generalisation of (5.2.1) (aside from allowing a non-zero charge), and stationary black holes are described by just three numbers, their mass m, charge

[1]Doing so leads to a Lane-Emden equation for a polytrope with index 3, whose solution identifies the correct numerical prefactor.

[2]Or so I am told; I have to confess that I, myself, have never gone through this calculation.

q, and angular momentum a. This is a strong statement; even if I do not justify it, I encourage you to give it some thought[3].

5.2.1 Geodesics of the Kerr Metric

The motion of celestial bodies in the gravitational field of a central, massive, rotating body are given by the time-like geodesics of the Kerr metric. We look at them briefly now. The Kerr metric has a number of symmetries; two of them are explicitly evident the way that it is written in (5.2.1). These are that the metric, and hence the Kerr space-time, is unchanged by a constant shift in either of the coordinates t or ϕ. The technical terms for these two symmetries are that the metric (and Kerr space-time) is *stationary* and *axisymmetric*. Associated to these two symmetries are conserved first integrals of the motion for test particles moving along geodesics of the Kerr metric. Direct computation of the geodesic equations for these two coordinates gives these conserved first integrals as

$$\left[1-\frac{2mr}{r^2+a^2\cos^2(\theta)}\right]c\partial_\tau t+a\frac{2mr\sin^2(\theta)}{r^2+a^2\cos^2(\theta)}\,\partial_\tau\phi=\gamma, \tag{5.2.2}$$

$$-a\frac{2mr\sin^2(\theta)}{r^2+a^2\cos^2(\theta)}\,c\partial_\tau t+\left[r^2+a^2+\frac{2mra^2\sin^2(\theta)}{r^2+a^2\cos^2(\theta)}\right]\sin^2(\theta)\,\partial_\tau\phi=\ell. \tag{5.2.3}$$

The fact that the geodesic is parameterised by 'arc length' (c times proper time), $g_{\mu\nu}\partial_\tau x^\mu\,\partial_\tau x^\nu = -1$ or 0, according to whether the geodesic is time-like or null, contributes a third first integral of the motion, which can be written

$$-\gamma\,c\partial_\tau t+\ell\,\partial_\tau\phi+\frac{r^2+a^2\cos^2(\theta)}{r^2-2mr+a^2}\left(\partial_\tau r\right)^2+\Big(r^2+a^2\cos^2(\theta)\Big)\left(\partial_\tau\theta\right)^2=-\kappa, \tag{5.2.4}$$

with κ equal to 1 or 0 according to whether the geodesic is time-like or null, respectively. There remain the geodesic equations for the coordinates $r(\tau)$ and $\theta(\tau)$, which represent only one further independent equation. One may check that a solution of the θ equation is $\theta(\tau)=\pi/2$, corresponding to geodesic motion in the 'equatorial plane' of the black hole. However, unlike the situation with the Schwarzschild metric this solution does not represent the generic case, since the Kerr black hole is only axisymmetric and not fully spherically symmetric. A fourth first integral was found by Brandon Carter (1968) using the Hamilton-Jacobi method of classical mechanics, leading to a complete description of the geodesics of the Kerr metric[4]. We will not consider this here and satisfy ourselves with a brief description of the geodesics that lie in the equatorial plane ($\theta=\pi/2$).

In the equatorial plane, and provided $r^2-2mr+a^2\neq 0$, the first two integrals of motion give

$$\begin{bmatrix}c\partial_\tau t\\ \partial_\tau\phi\end{bmatrix}=\frac{1}{r^2-2mr+a^2}\begin{bmatrix}r^2+a^2(1+2m/r) & -a\,2m/r\\ a\,2m/r & 1-2m/r\end{bmatrix}\begin{bmatrix}\gamma\\ \ell\end{bmatrix}. \tag{5.2.5}$$

Using this the third first integral of the motion can be reduced to

$$\left(\partial_\tau r\right)^2-\frac{2m\kappa}{r}+\frac{\ell^2-a^2(\gamma^2-\kappa)}{r^2}-\frac{2m(\ell-\gamma a)^2}{r^3}=\gamma^2-\kappa, \tag{5.2.6}$$

so that the 'effective potential' is

$$2V_{\text{eff}}=-\frac{2m\kappa}{r}+\frac{\ell^2-a^2(\gamma^2-\kappa)}{r^2}-\frac{2m(\ell-\gamma a)^2}{r^3}. \tag{5.2.7}$$

[3] In his 1983 Nobel Prize lecture, Subrahmanyan Chandrasekhar said the following: *"I do not know if the full import of what I have said is clear. Let me explain. Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described* exactly *by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object."*

[4] The 'Carter constant' can be thought of as analogous to the conserved Laplace-Runge-Lenz vector of the ordinary Kepler problem in the sense that it does not arise from a manifest symmetry of the space (like the constants γ and ℓ do).

The rotation of the black hole does not add fundamentally new terms to the effective potential as compared to the Schwarzschild black hole, *i.e.* terms with different powers of $1/r$, all it does is alter the coefficients of the various different terms. It might seem that the effects of the rotation are relatively minor; indeed the predictions of planetary orbits on the basis of the Schwarzschild metric are already in good agreement with observations so that any modification deriving from the rotation of the sun must indeed be minor. We leave a quantitative estimate to the problems.

5.2.2 Event Horizon and Ergoregion

As in the Schwarzschild metric, there is a value of r at which the metric component g_{rr} formally diverges, corresponding to a coordinate singularity. This happens when

$$r^2 - 2mr + a^2 = 0. \tag{5.2.8}$$

Whether there is a (real) solution or not depends on the value of the angular momentum a. When $|a| < m$ there are two solutions, and so two distinct surfaces on which g_{rr} is formally divergent. The outer of these, $r_+ = m + \sqrt{m^2 - a^2}$, is the *event horizon* of the black hole; massive, or massless, particles can cross it falling inwards, but can then never cross the same surface again to get back out. As in the Schwarzschild space-time it is a null surface. When $|a| > m$ there are no real solutions and no coordinate singularities for $r > 0$. The curvature singularity at $r = 0$ is, in this case, not hidden from a far-field observer, lying concealed behind an event horizon, but sits openly in plain sight. It is a *naked singularity.* An open conjecture of Roger Penrose – cosmic censorship – claims that such naked singularities are never produced in any physical process, such as the gravitational collapse of a star. The borderline case $|a| = m$ is called an *extremal Kerr black hole.* If Penrose's conjecture is correct it represents the maximum value of the angular momentum that a black hole can acquire.

In the Schwarzschild metric two things happened at the event horizon; the metric component g_{rr} diverged and, at the same time, the metric component g_{00} vanished. This is not so in the Kerr metric. g_{00} vanishes on a different surface, defined by

$$r^2 - 2mr + a^2\cos^2(\theta) = 0, \tag{5.2.9}$$

and called the *ergosphere.* The region between the ergosphere and the event horizon is called the *ergoregion.* To give an indication of why, consider the particle trajectory $\big(ct(\tau), r(\tau), \theta(\tau), \phi(\tau)\big) = (ct_0 + \tau, r_0, \theta_0, \phi_0)$. This is not a geodesic, so it is not the (free) motion of a test particle, but it is a special trajectory. It is an integral curve of a flow, generated by the vector field $K = (1, 0, 0, 0)$, that simply changes the t-coordinate and nothing else. As the metric does not depend on t, it is unchanged under this flow, which corresponds to a symmetry of the space-time. We say that the flow generates an *isometry*; the vector field K that generates it is called a *Killing vector.* This vector, the 'velocity' of our particle trajectory, has (squared) magnitude g_{00} and so is only time-like outside the ergoregion. What this means is that this trajectory can only correspond to the motion of an actual massive particle if it is outside the ergoregion. What does the motion of a particle inside the ergoregion look like? Consider a more general trajectory $\big(ct(\tau), r(\tau), \theta(\tau), \phi(\tau)\big) = (ct_0 + \tau, r_0, \theta_0, \phi_0 + \omega\tau)$, corresponding to an integral curve of the vector field $K_\omega = (1, 0, 0, \omega)$. What is its magnitude? A short calculation gives the answer

$$\frac{-[r^2 - 2mr + a^2\cos^2(\theta)]}{r^2 + a^2\sin^2(\theta)} - \frac{4mra\sin^2(\theta)\,\omega}{r^2 + a^2\cos^2(\theta)} + \left[\frac{2mra^2\sin^4(\theta)}{r^2 + a^2\cos^2(\theta)} + (r^2 + a^2)\sin^2(\theta)\right]\omega^2, \tag{5.2.10}$$

although it is more expedient to simply write $g_{00} + 2g_{0\phi}\omega + g_{\phi\phi}\omega^2$. The vector is only time-like if $\omega_- < \omega < \omega_+$ where

$$\omega_\pm = -\frac{g_{0\phi}}{g_{\phi\phi}} \pm \left[\left(\frac{g_{0\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{00}}{g_{\phi\phi}}\right]^{1/2}. \tag{5.2.11}$$

When g_{00} is positive both of these roots are positive (assuming a is positive) so that the interval between them only includes strictly positive values of ω. Thus motion inside the ergoregion, even for light, necessarily involves co-rotation with the black hole. It is impossible to counter-rotate, or even to simply 'stand still'.

5.3 Observational Evidence for Black Holes

Do black holes really exist? The Schwarzschild solution is an *exact* solution of the Einstein equations; as such its properties can be trusted to give a faithful representation of gravity in the strongly relativistic regime opened up by general relativity. Yet, it is only one solution; and a very particular one at that, with perfect spherical symmetry and a static metric. It was easy for *'doubters and nay-sayers'* to claim that the singularity it contained was not something that would actually show up in the real world. As I am sure you all know, the main intellectual evidence against this came in the 1960s with the work of Roger Penrose and Stephen Hawking on the singularity theorems. These established firmly that the Einstein field equations contained solutions with singularities, like that of the Schwarzschild space-time, and that such solutions were not special but would arise under a generic set of initial conditions, including as the end fate of sufficiently large stars. In their celebrated text, Hawking & Ellis state this with due lack of pomp and circumstance

> *To summarize, it seems that certainly some, and probably most, bodies of mass $> M_L$ will eventually collapse within their Schwarzschild radius, and so give rise to a closed trapped surface. There are at least 10^9 stars more massive than M_L in our galaxy. Thus there are a large number of situations in which theorem 2 predicts the existence of singularities.*
>
> from The large scale structure of space-time, page 308

This utterly striking theoretical prediction is eminently worthy of serious experimental attention and observational verification.

Observational evidence for black holes is indirect by necessity, for as we have seen it is impossible to receive any signals from the event horizon, or the region of space-time within it, so long as we stay safely in the asymptotic far-field. Thus we must infer; typically what we try to infer is that a certain large amount of mass is confined to a certain small region of space and yet cannot be seen 'directly' as ordinary matter that we can otherwise identify. One then invokes that we know of nothing that it could be other than a black hole, and we do know it could be a black hole. On this basis, there is now widespread consensus that black holes are common, as Hawking & Ellis said they should be.

The strongest evidence comes from observations of galactic nuclei; active nuclei and quasars are amongst the most energetic phenomena known and supermassive black holes offer one possible source for the enormous release of energy. Not only do they produce enormous amounts of energy – their luminosity can exceed 100 times that of the entire Milky Way – but the source is compact, with a size of only about a light year across. It is believed that their luminosity derives from an accretion disc falling onto a supermassive black hole, with mass between 10^6 and $10^{10}\ M_\odot$. The efficiency of this energy generation mechanism is impressive; of order 10% of the mass is converted to light, which should be contrasted with the efficiency of nuclear fission $\sim 0.8\%$. More generally, observations of galactic nuclei and the behaviour of visible material closest to them have led to the widespread view that all galaxies contain supermassive black holes at their centres. Unsurprisingly, it is the one closest to us, in our own galaxy, that offers the best observational evidence. The centre of the Milky Way is identified with a compact radio source Sagittarius A*. Over a period of about two decades the orbits of over 100 stars in close proximity to the galactic centre have been monitored and those of at least 28 accurately determined[5]. One in particular, labelled S2, has an orbital period of 15 years, a pericentre

[5] Numbers quoted from S. Gillessen *et al.* Astrophys. J. **692**, 1075 (2009).

distance of only 1.8×10^{10} km, and has completed one full orbit during the period of observation. The motion of all stars whose orbits have been determined are consistent with a single central gravitating object with mass $4.3 \times 10^6 \, M_\odot$, localised within a distance certainly smaller than the pericentre distance of the star S2.

Observational evidence for stellar-mass black holes derives primarily from x-ray observations of compact sources in binary systems. Typically, a visible star is seen in orbit with an unseen, highly compact companion, which can be inferred to be a neutron star or a black hole. The distinction between them depends on its mass; the Oppenheimer-Volkoff limit places an upper bound on the mass of neutron stars, so that if its mass (significantly) exceeds this then we may conclude that it is unlikely to be a neutron star and the presumption becomes that it is a black hole. It is perhaps worth saying a few words about the observations that are made. Orbital properties of the visible star are deduced from measurements of the Doppler shift of its spectrum. These reflect variations in its line of sight velocity, the period of which gives the orbital period P, and the semi-amplitude K gives the average component of the orbital velocity projected onto the line of sight

$$K = \frac{2\pi a \sin(i)}{P}, \tag{5.3.1}$$

where a is the orbital radius of the visible star about the centre of mass and i is the angle of inclination of the orbital plane as we see it on Earth[6]. Using a Newtonian description of binary orbits $(2\pi/P)^2 = G(M + M_\mathrm{c})/r^3$ – Kepler's third law – and the definition of centre of mass $Ma = M_\mathrm{c} a_\mathrm{c}$, where M_c is the mass of the compact companion, one can then derive the *binary mass function*

$$\frac{PK^3}{2\pi G} = \frac{\big(M_\mathrm{c} \sin(i)\big)^3}{(M + M_\mathrm{c})^2}. \tag{5.3.2}$$

If in addition to the Doppler shift data one has estimates for the mass of the visible star M and the angle of inclination i, then the mass of the companion can be deduced.

The first black hole candidate of this kind was the x-ray source Cygnus X-1, discovered in the 1970s. Conservative estimates of its mass give a value of about $3.3 \, M_\odot$ that is only a little above the Oppenheimer-Volkoff limit, but estimates that are not overly cautious, or pessimistic, suggest a mass closer to $9 \, M_\odot$. Thus Cygnus X-1 is a good candidate for a stellar-mass black hole. Better estimates have come from so-called *soft x-ray transients*, which are strong x-ray sources only for short intervals of time; during their quiet periods the accretion disc becomes faint and accurate optical measurements of the visible companion can be made. These binaries can have visible stars very much lighter than that in the Cygnus X-1 system, which is one reason why they lead to better black hole candidates. One of the best such candidates is V404 Cygni where the compact source has a mass estimated at $12 \pm 2 \, M_\odot$.

[6] We have neglected the eccentricity of the orbit.

Problems

1. Derive the effective potential (5.2.7) characterising equatorial geodesics of the Kerr metric.

   Find an explicit expression for elliptical planetary orbits in the form $u = u(\phi)$, where $u = 1/r$, and comment on the changes associated to the non-zero rotation.

   Find also an explicit expression for the deflection of light and again comment on the changes associated to the non-zero rotation.

2. Although we do not show it, analogously to the Schwarzschild metric the Kerr metric has a singularity at $r = 0$. However, the nature of this singularity is a little different and is often referred to as a 'ring singularity'. Show that, unless one approaches the singularity in the equatorial plane $\theta = \pi/2$, the limit $r \to 0$ may be taken in the expression for the metric. Subsequently taking $\theta \to \pi/2$ show that in any constant t slice the singularity has the topology of a circle with circumference $2\pi a$.

3. Consider a crude model of a star where the density takes a constant value ρ_0 out to some radius a and is zero thereafter. For such a star, show that the Tolman-Oppenheimer-Volkoff equation (5.1.2) becomes

   $$-\frac{dp}{dr} = \frac{4\pi G r/c^4}{1 - 8\pi G\rho_0 r^2/3c^2}\left(p + \frac{1}{3}\rho_0 c^2\right)(p + \rho_0 c^2),$$

   and hence that the pressure at the centre of the star is given by

   $$p(0) = \rho_0 c^2 \frac{1-(1-2m/a)^{1/2}}{3(1-2m/a)^{1/2}-1}, \qquad \text{where} \quad m = \frac{4\pi G\rho_0 a^3}{3c^2}.$$

   Finally, show that the pressure at the centre diverges when $m = 4a/9$. [Stars that are more massive than this cannot support themselves against gravity. This analysis is due to Schwarzschild.]

4. In the notes we tried to estimate the maximum mass of a star that can support itself against gravity. The result we obtained, $\sim 0.3M_\odot$, is smaller than the value found by Chandrasekhar, $1.44M_\odot$, by enough to be considered significant. Read his own account of his calculation in his Nobel Prize lecture, linked to here, and identify where our underestimate comes from. [To say more, this is more than qualitative; the qualitative part is easy, the difficult part is to determine accurately the numerical factor, which is of some importance for observations. So the problem is a quantitative one; account for a discrepancy by a factor of 5.]

5. Solutions of the Einstein equations for the interior region of a spherically symmetric star can be obtained by writing the metric in the form

   $$ds^2 = -f(r)\,c^2dt^2 + g(r)\,dr^2 + r^2\Big[d\theta^2 + \sin^2(\theta)\,d\phi^2\Big],$$

   and taking the stress-energy-momentum tensor to be that of an ideal fluid. Show that the 00 and rr Einstein equations are then

   $$\frac{f}{r^2}\frac{d}{dr}\left[r\left(1-\frac{1}{g}\right)\right] = \frac{8\pi G}{c^4}\,\rho c^2\, f,$$
   $$\frac{1}{r^2} - \frac{g}{r^2} + \frac{1}{rf}f' = \frac{8\pi G}{c^4}\,p\,g,$$

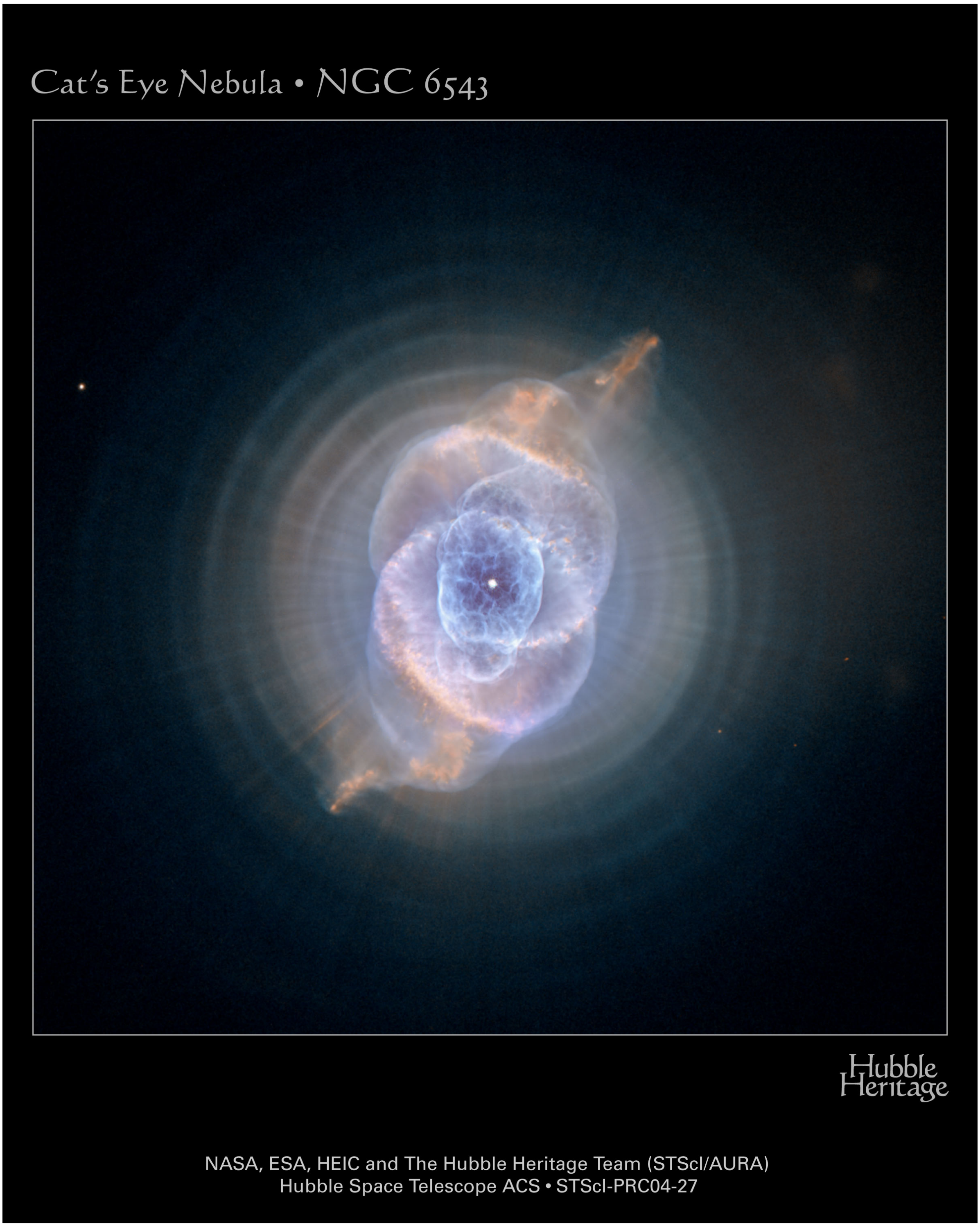
   respectively. Show that the solution of these equations can be written in the form

   $$\frac{1}{g} = 1 - \frac{2m(r)}{r}, \qquad m(r) = \frac{4\pi G}{c^2}\int_0^r \rho(r')\,r'^2\,dr',$$
   $$\partial_r \ln f = \frac{2G}{c^2r^2}\frac{1}{1-\frac{2m(r)}{r}}\left[\frac{m(r)c^2}{G} + \frac{4\pi r^3 p(r)}{c^2}\right].$$

The final linearly independent equation can be taken to be the continuity equation $\nabla_\alpha T^\alpha_r = 0$ for the stress-energy-momentum tensor. Show that it takes the form

$$\partial_r p + \frac{1}{2f} f' \big(\rho c^2 + p\big) = 0,$$

and hence obtain the Tolman-Oppenheimer-Volkoff equation (5.1.2) given in the notes.



The Cat's Eye Nebula taken by the Hubble Space Telescope. The nebula is the result of a series of large ejections of material from a massive star. Image from NASA.

Chapter 6

Gravitational Waves

And behold ... the waves be upon you at last.[1]

Keith Moffatt (1961)

In Einstein's theory of general relativity the geometry of space-time is a dynamic physical observable that supports wave-like excitations, propagating at the speed of light. These are known as gravitational waves; their elementary excitations, or normal modes, have the properties of massless, spin 2 particles with two linearly independent polarisation states. They are called gravitons. Initial experimental evidence for gravitational waves was only indirect, coming primarily from the changing orbital parameters of binary pulsars, such as the Hulse-Taylor binary. We observe the changes but not the gravitational radiation presumed responsible for them. However, from September 2015 the LIGO collaboration has made direct observations using laser interferometry. The first four signals have been identified as the coalescence of binary black holes at distances of over a billion light years; the fifth detection was a neutron star coalescence 140 million light years away.

6.1 Linearised General Relativity

It is frequently said that the gravitational interaction is weak. If that is so, then it seems reasonable to make use of the weakness and give a linearised treatment of the effect. In doing so we will obtain a linear version of the Einstein equations and since we know many methods for solving linear equations it can be anticipated that much will be learnt. We will linearise only about the vacuum, Minkowski space-time, leaving the general case to the sufficiently interested reader.

The linearisation begins by declaring a form for the components of the metric in a local coordinate system

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \big(\eta_{\mu\nu} + h_{\mu\nu}\big)\, dx^\mu dx^\nu, \tag{6.1.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is small. Two things are important here already. First, we are thinking of $\eta_{\mu\nu}$ as the *standard Minkowski metric* with $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and all other components zero. This means the coordinates x^μ are, to zeroth order approximation, standard 'Cartesian' coordinates for Minkowski space-time[2]. Second, what follows from this, but could be overlooked if we rushed, is that the particular form of the perturbation $h_{\mu\nu}$ as we have written it in (6.1.1) depends on the choice of coordinates. If we (or one of our friends) used a different coordinate system the perturbation $h_{\mu\nu}$ would assume a different functional form. But of course the space-time is exactly the same; it is not a

[1]This is taken from 'Hymn to Proserpine' (1866) by the poet Swinburne.

[2]For instance, they are not a polar system as was adopted in the derivation of the Schwarzschild solution.

different perturbation of Minkowski but only a different way of writing the same perturbation of Minkowski.

Let us address this formally; then you should take time to let it properly sink in. Suppose we make a small change in our choice of coordinate system, replacing the coordinates x^μ with a new choice y^μ obtained by the near-identity transformation

$$x^\mu = y^\mu + \epsilon^\mu, \tag{6.1.2}$$

for functions ϵ^μ that are small and slowly varying but otherwise arbitrary. In terms of our new choice of coordinates the metric takes the form

$$\begin{aligned} ds^2 &= g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\,dy^\alpha dy^\beta, &(6.1.3)\\ &= \big(\eta_{\mu\nu}+h_{\mu\nu}\big)\big(\delta^\mu_\alpha+\partial_\alpha\epsilon^\mu\big)\big(\delta^\nu_\beta+\partial_\beta\epsilon^\nu\big)dy^\alpha dy^\beta, &(6.1.4)\\ &= \big(\eta_{\alpha\beta}+h_{\alpha\beta}+\partial_\alpha\epsilon_\beta+\partial_\beta\epsilon_\alpha+O(2)\big)dy^\alpha dy^\beta, &(6.1.5)\end{aligned}$$

suppressing terms of second order, or higher, in small quantities. In other words, a near-identity coordinate transformation, which has no effect whatsoever on the space-time (only on the way we view it), transforms the perturbation of the metric according to

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu. \tag{6.1.6}$$

By analogy with electromagetism this is called a *gauge transformation*. Considerable simplification arises from making the 'right' choice of gauge when doing our calculations.

Remark: Another way of expressing this, which some of you may find preferable, is that, if it happens that the perturbation is equal to the symmetric derivative of a function, $h_{\mu\nu} = \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$, then it is no perturbation at all but only a view of Minkowski using coordinates that are close to inertial but not precisely so. A simple coordinate transformation then puts things straight. One should say this more strongly; any part of the perturbation that is equal to the symmetric derivative of a function can be eliminated by a coordinate transformation and so carries no physical significance.

This preamble being said, let us proceed to determine the linearised form of the Ricci tensor. To linear order, the Christoffel symbols are given by

$$\begin{aligned} \Gamma^\alpha_{\mu\nu} &= \frac{1}{2}g^{\alpha\beta}\big(\partial_\mu g_{\beta\nu}+\partial_\nu g_{\mu\beta}-\partial_\beta g_{\mu\nu}\big), &(6.1.7)\\ &= \frac{1}{2}\eta^{\alpha\beta}\big(\partial_\mu h_{\beta\nu}+\partial_\nu h_{\mu\beta}-\partial_\beta h_{\mu\nu}\big)+O(2), &(6.1.8)\end{aligned}$$

and it then follows that the components of the Ricci tensor, in our coordinate basis, are

$$\begin{aligned} R_{\mu\nu} &= \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\alpha\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}, &(6.1.9)\\ &= \frac{1}{2}\eta^{\alpha\beta}\big(\partial_\alpha\partial_\mu h_{\beta\nu}+\partial_\alpha\partial_\nu h_{\mu\beta}-\partial_\alpha\partial_\beta h_{\mu\nu}\big)\\ &\quad - \frac{1}{2}\eta^{\alpha\beta}\big(\partial_\mu\partial_\alpha h_{\beta\nu}+\partial_\mu\partial_\nu h_{\alpha\beta}-\partial_\mu\partial_\beta h_{\alpha\nu}\big)+O(2), &(6.1.10)\\ &= \frac{1}{2}\eta^{\alpha\beta}\big(-\partial_\alpha\partial_\beta h_{\mu\nu}+\partial_\mu\partial_\beta h_{\alpha\nu}+\partial_\alpha\partial_\nu h_{\mu\beta}-\partial_\mu\partial_\nu h_{\alpha\beta}\big)+O(2). &(6.1.11)\end{aligned}$$

This is the linearised Ricci tensor. It depends on second derivatives of the perturbation $h_{\mu\nu}$, reflecting the fact that curvature is associated to second derivatives. It also means that constant and linear terms in $h_{\mu\nu}$ carry no physical significance; they can be removed by coordinate transformations.

One may verify easily that the components of the Ricci tensor are unchanged, to linear order, under a gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$, as they should be. Of course, this reflects the fact that curvature is a directly observable physical quantity independent of any choice of coordinate system. In particular, if the perturbation is *pure gauge* then the space-time is Minkowski and is flat; the components of the Ricci tensor must then all be zero regardless of the coordinate system.

6.2 The Graviton is a Spin 2 Particle

In electromagnetism, the gauge field A_μ is a 1-form that allows for a manifestly Lorentz covariant description of electromagnetic phenomena. Naively it conveys four independent components at any point in space-time – one for each value of the index μ. However, it is well-known that the photon has only two linearly independent polarisation states. The description, therefore, necessarily contains redundancy. Part of this is gauge freedom – the transformation $A_\mu \to A_\mu + \partial_\mu \chi$ changes the gauge field but has no effect whatsoever on any physically observable quantities – and part of it is more significant; the photon is a spin 1 particle but does not have three distinct spin states because it is massless[3]. In calculations, the redundancy is often removed by imposing an additional condition on the gauge field; one that can always be satisfied by freedom under gauge transformations. Any particular choice of additional condition to impose is called *gauge fixing*. One of the most common choices is *Lorenz gauge*[4]

$$\partial_\mu A^\mu = 0. \tag{6.2.1}$$

This entire situation is repeated in the linear theory of gravity with perhaps the only difference being that is it not so obvious, *a priori*, what the analogue of Lorenz gauge should be. We study this now, in a presentation strongly inspired by Feynman's treatment.

Consider the form of the Ricci tensor for a plane wave $h_{\mu\nu} = e_{\mu\nu}\, \mathrm{e}^{iq_\alpha x^\alpha}$ for some constants $e_{\mu\nu}$ and a wavevector q_α. Away from masses the Ricci tensor vanishes and this leads to the condition

$$\frac{1}{2}q^2 e_{\mu\nu} - \frac{1}{2}\Big[q_\mu q^\alpha e_{\alpha\nu} + q_\nu q^\alpha e_{\mu\alpha} - q_\mu q_\nu \eta^{\alpha\beta} e_{\alpha\beta}\Big] = 0, \tag{6.2.2}$$

where, of course, $q^2 = \eta^{\mu\nu} q_\mu q_\nu$ is the magnitude squared of the wavevector. Let us suppose that $q^2 \neq 0$. Then this equation can be rearranged into the form

$$e_{\mu\nu} = \frac{1}{q^2}\Big[q_\mu\Big(q^\alpha e_{\alpha\nu} - \frac{1}{2}q_\nu \eta^{\alpha\beta} e_{\alpha\beta}\Big) + q_\nu\Big(q^\alpha e_{\mu\alpha} - \frac{1}{2}q_\mu \eta^{\alpha\beta} e_{\alpha\beta}\Big)\Big]. \tag{6.2.3}$$

But this is pure gauge. If $h_{\mu\nu}$ is pure gauge then we will have

$$\begin{aligned} & h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu, \\ \Rightarrow \qquad & e_{\mu\nu} = \big[q_\mu f_\nu + q_\nu f_\mu\big], \end{aligned} \tag{6.2.4}$$

for some choice of f_μ. The choice $f_\mu = \frac{1}{q^2}(q^\alpha e_{\mu\alpha} - \frac{1}{2}q_\mu \eta^{\alpha\beta} e_{\alpha\beta})$ shows that $h_{\mu\nu}$ is pure gauge. Thus plane wave modes for which $q^2 \neq 0$ are always pure gauge and hence do not correspond to anything physical. More positively, the only physical modes are those with $q^2 = 0$. Let's add the fanfare that this result deserves; we have just shown that the graviton is a massless particle.

When $q^2 = 0$ the vanishing of the Ricci tensor provides the constraint

$$q_\mu q^\sigma\Big[e_{\sigma\nu} - \frac{1}{2}\eta_{\sigma\nu}\eta^{\alpha\beta}e_{\alpha\beta}\Big] + q_\nu q^\sigma\Big[e_{\mu\sigma} - \frac{1}{2}\eta_{\mu\sigma}\eta^{\alpha\beta}e_{\alpha\beta}\Big] = 0, \tag{6.2.5}$$

which implies

$$q^\sigma\Big[e_{\sigma\nu} - \frac{1}{2}\eta_{\sigma\nu}\eta^{\alpha\beta}e_{\alpha\beta}\Big] = 0. \tag{6.2.6}$$

Again, this deserves proper fanfare; we have just shown that the graviton is a transverse mode, the same as the photon. Written in real-space this condition has the form of a divergence

$$\eta^{\sigma\nu}\partial_\sigma\Big[h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}\Big] = 0, \tag{6.2.7}$$

[3]The best description I know of this crucial part of (particle) physics is given in chapter 2 of Steven Weinberg's book *Quantum Theory of Fields, volume 1* (Cambridge University Press, Cambridge, 1996).

[4]Lorenz gauge is named after the Danish physicist Ludvig Lorenz, not the Dutch physicist Hendrik Lorentz, who is another person entirely. It would appear that confusion persists.

which is the analogue, for linearised gravity, of the Lorenz gauge condition in electromagnetism. The discussion we have given of it led to the significant findings that gravitational waves (gravitons) are massless and transverse, but it was also intended to bring out where this gauge choice comes from, rather than simply plucking it from thin air.

The choice of Lorenz gauge does not completely fix the gauge freedom in the choice of the metric perturbation $h_{\mu\nu}$. Perhaps the easiest way to see this is through the Ricci tensor, since it is an observable physical quantity. In Lorenz gauge the Ricci tensor reduces to

$$R_{\mu\nu} = -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}, \tag{6.2.8}$$

and is preserved by gauge transformations for which

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\epsilon_\mu = 0, \tag{6.2.9}$$

in other words coordinate transformations that satisfy the wave equation. One may use this additional freedom to impose an extra constraint on $h_{\mu\nu}$ and one that is frequently chosen is that it is traceless,

$$\eta^{\mu\nu}h_{\mu\nu} = 0. \tag{6.2.10}$$

The choice of Lorenz gauge in which the metric perturbation is also traceless is called *transverse traceless gauge.*

Perhaps the easiest way to see the final structure is to spell everything out in a specific case. So let's take the wavevector q_μ to be $(-k, k, 0, 0)$. The Lorenz gauge condition then reads

$$q^\nu e_{\mu\nu} = 0, \qquad \Rightarrow \qquad e_{\mu 0} + e_{\mu 1} = 0. \tag{6.2.11}$$

The tensor is pure gauge if it is of the form $e_{\mu\nu} = q_\mu f_\nu + q_\nu f_\mu$ for some f_μ. One may check that each of the three tensors

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \tag{6.2.12}$$

is pure gauge. Finally, accounting for $h_{\mu\nu}$ being traceless we can write the gauge fixed form of $e_{\mu\nu}$ for a plane wave perturbation as

$$e_{\mu\nu} = h_+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + h_\times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{6.2.13}$$

and see that there are only two distinct polarisation states, both of which are 'transverse traceless'.

A polarisation state of the electromagnetic field is a direction in space (orthogonal to the direction of propagation) along which the electric field points, and oscillates. A polarisation state of a gravitational wave is a structure for the deformation of the metric in directions orthogonal to the direction of propagation of the wave. These two deformations are

$$dy^2 - dz^2, \qquad \text{and} \qquad 2dydz. \tag{6.2.14}$$

The former represents an extension of the arc length along y and equal compression along z. The latter is the same distortion, but rotated by $\pi/4$. A rotation of the coordinate system[5] will 'mix' the two polarisation states, in the same way that such a rotation 'mixes' the two

[5]In doing this it is important that we *do not* simultaneously rotate the gravitational wave since otherwise we will have only succeeded in doing nothing at all.

linear polarisation states of the electromagnetic field. We compute this as follows. Under a rotation through angle $-\phi$ [6] about the x-axis the coordinates y and z transform according to

$$y \to y\cos(\phi) + z\sin(\phi), \qquad \text{and} \qquad z \to -y\sin(\phi) + z\cos(\phi). \tag{6.2.15}$$

It then follows that

$$dy^2 - dz^2 \to \cos(2\phi)\big[dy^2 - dz^2\big] + \sin(2\phi)\, 2dydz, \tag{6.2.16}$$

and we see that the rotation between the two polarisation states is at twice the rate of the rotation. This was surely evident from the picture of the two polarisation states; we show it algebraically purely for completeness. This behaviour is what is meant when one says that gravitons are spin 2 particles.

6.3 Gravitational Radiation

We turn now to the issue of how gravitational waves are created and the nature of the astrophysical sources that produce them. We start with the linearised Einstein equations in Lorenz gauge[7]

$$-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\Big[h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}h_{\sigma\tau}\Big] = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{6.3.1}$$

We will find it convenient to define $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}h_{\sigma\tau}$, called the 'trace-reversed metric perturbation', and then write the linearised equations in the form

$$\left(\frac{-1}{c^2}\partial_t^2 + \partial_i\partial_i\right)\overline{h}_{\mu\nu} = \frac{-16\pi G}{c^4}T_{\mu\nu}. \tag{6.3.2}$$

This is an inhomogeneous (tensor) wave equation with the stress-energy-momentum tensor playing the role of the source. Because Minkowski space has full ($\mathbb{R}^4$) translational symmetry, the solution can be conveniently developed using Fourier transforms

$$\overline{h}_{\mu\nu} = \frac{-16\pi G}{c^4}\int_{\mathbb{R}^{1,3}} \frac{d\omega d^3k}{(2\pi)^4}\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\, \frac{\tilde{T}_{\mu\nu}}{(\omega/c)^2 - k^2}, \tag{6.3.3}$$

$$= \frac{-16\pi G}{c^4}\int_{\mathbb{R}^{1,3}} dt'd^3x'\, G^{\mathrm{ret}}(\mathbf{x}-\mathbf{x}', t-t')T_{\mu\nu}(\mathbf{x}', t'), \tag{6.3.4}$$

where $\tilde{T}_{\mu\nu}$ is the Fourier transform of the stress-energy-momentum tensor, $k = |\mathbf{k}|$, $G^{\mathrm{ret}}(\mathbf{x}, t)$ is the *retarded Green function* for the wave operator, and we have made use of the convolution theorem. The appearance of the retarded Green function comes from physical considerations of the relevant boundary conditions; the matter content is a source for the gravitational radiation and a compact source should produce outgoing waves rather than ingoing waves. The gravitational disturbance produced by a compact source can therefore be given as

$$\overline{h}_{\mu\nu} = \frac{-16\pi G}{c^4}\int_{\mathbb{R}^{1,3}} dt'd^3x'\, \frac{-1}{4\pi|\mathbf{x}-\mathbf{x}'|}\, \delta\big((t-t') - |\mathbf{x}-\mathbf{x}'|/c\big)\, T_{\mu\nu}(\mathbf{x}', t'), \tag{6.3.5}$$

$$= \frac{16\pi G}{c^4}\int_{\mathbb{R}^3} d^3x'\, \frac{T_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{4\pi|\mathbf{x}-\mathbf{x}'|}. \tag{6.3.6}$$

Let us focus on the production of gravitational waves. Now, as we have seen, gravitational waves have a transverse character and are associated with the spatial components h_{ij} of the

[6] This is equivalent to a rotation of the gravitational wave by $+\phi$

[7] In writing this we should note that the stress-energy-momentum tensor appearing on the right-hand-side is also a leading order approximation, consistent with that taken for the curvature terms. We do not describe this in any detail.

perturbed metric. If we focus on these components and look at distances far from the source, $|\mathbf{x}| \gg |\mathbf{x}'|$, then

$$\bar{h}_{ij} = \frac{4G}{c^4|\mathbf{x}|} \int_{\text{source}} T_{ij}(\mathbf{x}', t - |\mathbf{x}|/c)\, d^3x'. \tag{6.3.7}$$

The integral over the source can be manipulated with the help of the continuity equation for the stress-energy-momentum tensor, $\eta^{\sigma\nu}\partial_\nu T_{\mu\nu} = 0$ or

$$\frac{1}{c}\partial_t T_{\mu 0} = \partial_k T_{\mu k}, \tag{6.3.8}$$

to show that the source of gravitational radiation, in conformance with the nature of gravitational waves, is characteristic of a *mass quadrupole.* To see this, first note that

$$\int T_{ij}\, d^3x = \int (\partial_k x^j) T_{ik}\, d^3x, \tag{6.3.9}$$

$$= -\int x^j\, \partial_k T_{ik}\, d^3x, \tag{6.3.10}$$

$$= \frac{-1}{c}\partial_t \int x^j T_{i0}\, d^3x. \tag{6.3.11}$$

Since the stress-energy-momentum-tensor is symmetric, $T_{\mu\nu} = T_{\nu\mu}$, the same expression holds with indices i and j interchanged. This observation allows us to proceed as follows

$$\int T_{ij}\, d^3x = \frac{-1}{2c}\,\partial_t \int \Big[x^j T_{i0} + x^i T_{j0}\Big] d^3x, \tag{6.3.12}$$

$$= \frac{-1}{2c}\,\partial_t \int \Big[x^j (\partial_k x^i) T_{0k} + x^i (\partial_k x^j) T_{0k}\Big] d^3x, \tag{6.3.13}$$

$$= \frac{1}{2c}\,\partial_t \int x^i x^j\, \partial_k T_{0k}\, d^3x, \tag{6.3.14}$$

$$= \frac{1}{2c^2}\,\partial_t^2 \int x^i x^j\, T_{00}\, d^3x. \tag{6.3.15}$$

The 00-component of the stress-energy-momentum tensor is the energy density, equal to ρc^2 for ordinary matter, where ρ is the mass density. So the integral $\int x^i x^j T_{00}\, d^3x$ is the second moment of the mass density, or the mass quadrupole. In non-relativistic mechanics it is called the moment of inertia tensor. In any case, we arrive at the *mass quadrupole formula* for the gravitational radiation produced by a compact massive source

$$\bar{h}_{ij}(\mathbf{x}, t) = \frac{2G}{c^6|\mathbf{x}|}\,\partial_t^2 \int_{\text{source}} x'^i x'^j\, T_{00}(\mathbf{x}', t - |\mathbf{x}|/c)\, d^3x'. \tag{6.3.16}$$

Remark: In electromagnetism, the source of electromagnetic waves (*e.g.* radio waves) is an oscillating electric dipole and the radiation is correspondingly of a characteristic 'dipole' form. Gravitational waves are not characteristic of a 'dipole' since the mass dipole moment is a constant by virtue of conservation of momentum. This difference in character of gravitational waves as compared to their electromagnetic cousins contributes to them being rather more feeble and consequently difficult to detect.

To effectively generate gravitational waves a source should have a large moment of inertia that is rapidly changing. Individual spinning stars, or other objects that are axisymmetric, are therefore not good sources. Binary systems are rather better. For the simplest situation, consider a binary system of two equal mass stars with a circular orbit about their common centre of mass of radius $a/2$ and period $2\pi/\Omega$. The 00-component of the stress-energy-momentum tensor can then be taken to be

$$T_{00} = Mc^2\delta(x)\Big[\delta\big(y - \frac{a}{2}\cos(\Omega t)\big)\delta\big(z - \frac{a}{2}\sin(\Omega t)\big) + \delta\big(y + \frac{a}{2}\cos(\Omega t)\big)\delta\big(z + \frac{a}{2}\sin(\Omega t)\big)\Big], \tag{6.3.17}$$

choosing the orbital plane (yz) to orient the spatial coordinate system x, y, z. We then find

$$\overline{h}_{yy}(\mathbf{x},t) = -\overline{h}_{zz}(\mathbf{x},t) = \frac{-2GM\Omega^2 a^2}{c^4|\mathbf{x}|}\cos\big(2\Omega(t-|\mathbf{x}|/c)\big), \tag{6.3.18}$$

$$\overline{h}_{yz}(\mathbf{x},t) = \frac{-2GM\Omega^2 a^2}{c^4|\mathbf{x}|}\sin\big(2\Omega(t-|\mathbf{x}|/c)\big). \tag{6.3.19}$$

Using Kepler's third law – $\Omega^2 a^3 = G(M_1+M_2)$ – the amplitude can be simplified to

$$\frac{2GM\Omega^2 a^2}{c^4|\mathbf{x}|} = \frac{(2GM/c^2)^2}{a|\mathbf{x}|} = \frac{(2m)^2}{a|\mathbf{x}|}. \tag{6.3.20}$$

We might make three general remarks. First, the amplitude is tiny; it is the product of the Schwarzschild radii for the two stars divided by their separation and the distance they are from the point of observation. Not only is the Schwarzschild radius small on astronomical scales, but the distance to the source is truly astronomical. Second, the frequency of the radiation is twice that of the source; one may think of this as another vestige of the fact that gravitational waves are spin 2 excitations. And for our third remark, we should say something about the polarisation. This depends on the viewing direction relative to the orbital plane of the binary system[8] but in the particular case where we are viewing the binary 'face on' (specifically at large distances along the positive x-axis) the waves are right-circulary polarised.

6.4 Inspiral – the fate of binary systems

Like all types of waves, gravitational waves carry energy and indeed transport it away from the source. In this process, since energy is conserved, the binary system is losing energy. In a Newtonian description the energy of our binary system is given by

$$E = \frac{1}{2}M\left(\frac{a}{2}\Omega\right)^2 \times 2 - \frac{GM^2}{a} = -\frac{GM^2}{2a}, \tag{6.4.1}$$

where the last form follows from Kepler's third law, $\Omega^2 a^3 = G(M_1+M_2)$. The loss of energy from the binary leads to a reduction in the separation a, or inspiral. It is clear that this is a run-away process: the closer they get together, the more they radiate; and the more they radiate, the closer they get together. In general relativity, binary systems are inherently unstable and undergo a slow inspiral and eventual merger to a single object. Not only is this a remarkable theoretical prediction, it is also now a process that we have observed directly, multiple times.

We should like to describe how this happens, for which we need an understanding of how much energy the gravitational waves are carrying away from the binary source. It turns out that the definition of energy in general relativity is difficult (see any advanced text, *e.g.* Landau & Lifshitz) but we will describe what we can. In any linear wave theory, the energy of the wave is a quadratic expression in the linear wave amplitude. The guide for what this expression should be comes from the Einstein equations

$$T_{\mu\nu} = \frac{c^4}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right). \tag{6.4.2}$$

Now, the idea is that we can evaluate the right-hand-side for our linear gravitational wave metric, to second order in the wave amplitude, and will obtain an expression that has all the properties of a stress-energy-momentum tensor. This turns out to be the correct object.

Let us focus on a specific case: the energy flux along the x-axis (rotation axis of the binary). This flux is given by the component T_{0x} of the stress-energy-momentum tensor, so

$$\frac{1}{c}\Big(\text{energy flux along } x\Big) = \frac{c^4}{8\pi G}R_{0x}, \tag{6.4.3}$$

[8]For details of the directional dependence see, as usual, Landau & Lifshitz.

since in the linear gravitational wave metric $g_{0x} = 0$. The calculation of R_{0x} to quadratic order in the gravitational wave amplitude h is lengthy and tedious; the final result is $2(\Omega h/c)^2$. Recalling that $h = (2m)^2/a|\mathbf{x}|$ and (again) using Kepler's third law we finally obtain

$$\frac{1}{c}\Big(\text{energy flux along } x\Big) = \frac{c^2\Omega^2}{4\pi G}\frac{(2m)^4}{a^2|\mathbf{x}|^2} = \frac{Mc^2(2m)^4}{2\pi a^5}\frac{1}{|\mathbf{x}|^2}. \tag{6.4.4}$$

Since the flux decays like $1/\text{distance}^2$ the total flux, going in all directions, through any spherical surface is a constant and energy is truly being carried away to infinite distance. This total flux is equal to

$$\frac{1}{c}\Big(\text{total energy flux}\Big) = \mathcal{A}\frac{Mc^2(2m)^4}{a^5}, \tag{6.4.5}$$

where $\mathcal{A}$ is a numerical prefactor coming from integrating over all directions and the variation in the amplitude and polarisation of the gravitational waves that there is with that. Detailed calculation gives the value $\mathcal{A} = \frac{4}{5}$ [9]. Equating this to the energy loss from the binary we find

$$\frac{-1}{c}\frac{dE}{dt} = \frac{4Mc^2(2m)^4}{5a^5} \qquad \Rightarrow \qquad \frac{-1}{c}\frac{da}{dt} = \frac{16(2m)^3}{5a^3}. \tag{6.4.6}$$

Finally, integrating we obtain the time-dependence of the separation

$$a_0^4 - a^4 = \frac{64(2m)^3}{5}c(t - t_0), \tag{6.4.7}$$

where t_0 is a reference time, at which $a = a_0$. Importantly, we see from this that the separation vanishes in finite time, *i.e.* binary systems inspiral completely, ending with the masses merging.

At least at early times, when our linear calculations should be good, the separation varies with the time to merger as $|t_{\text{merger}} - t|^{1/4}$, which leads to the following prediction for the variation of the amplitude and frequency of the emitted gravitational waves

$$h \sim |t_{\text{merger}} - t|^{-1/4}, \qquad \omega \sim |t_{\text{merger}} - t|^{-3/8}. \tag{6.4.8}$$

The signal, rising in both amplitude and frequency as the merger approaches, is known as a 'chirp' waveform. Detailed comparison with the LIGO observations should not be expected since the signals we have detected so far are certainly in the strong-field regime, however, the qualitative behaviour is amply accounted for.

6.5 Observations of Gravitational Waves

There are two principal observations that involve gravitational wave phenomena; radio pulses coming from binary neutron stars; and direct detection of the gravitational wave amplitude using laser interferometry. The first binary pulsar was discovered in 1974 by Russell Hulse and Joseph Taylor. They received the Nobel Prize in 1993 for this discovery and the confirmation of inspiral from gravitational wave emission that it brought. Direct detection of gravitational waves was first achieved on 14th September 2015 by the LIGO collaboration and received the Nobel Prize in 2017. At the time of writing the LIGO and VIRGO collaborations have released a total of five confirmed detections, although this number is now expected to increase rapidly.

6.5.1 Hulse-Taylor Binary

The Hulse-Taylor binary, known formally as PSR $1913 + 16$, is a binary neutron star system from which we receive regular radio pulses from one of the stars. Since its discovery in 1974 these pulses have been recorded and studied extensively so that we have a very accurate picture of the system and its evolution. The main star has mass $1.44M_\odot$ and the companion $1.39M_\odot$.

[9] Again, see Landau & Lifshitz for details.

They form a close binary orbit with semi-major axis 1.95×10^6km (about 3 times the radius of the sun) and eccentricity $e = 0.62$. The orbital period is only 0.32 days, or about 7.7 hours. Its distance from the sun is estimated at about 21,000 light years, or 2×10^{17}km. It follows that the typical amplitude of gravitational waves produced by this source, when they arrive at the Earth, can be estimated at

$$h \sim \frac{(2m)^2}{a|\mathbf{x}|} = 4.7 \times 10^{-23}.$$

This is *tiny!!* and underscores the enormous challenge presented to experimental groups to manufacture any sort of instrument that could detect such miniscule ripples in space-time.

Over the 43 year period during which the Hulse-Taylor binary has been observed its orbital parameters have changed. If we assume that the changes are due to gravitational wave emission, as per the previous section, then we can use (6.4.7) to estimate them. This gives that

$$a_0^4 - a^4 = 4.03 \times 10^{17}\,\text{km}^4,$$

or a decrease of the semi-major axis of about 13 metres. This estimate turns out to be too small by almost an order of magnitude, the reason being that the formula (6.4.7) assumes a circular orbit while the eccentricity of the Hulse-Taylor binary is quite considerable. A more detailed calculation (by Peters and Matthew) gives the dependence on eccentricity through the formulae[10]

$$\begin{aligned} \left\langle \frac{da}{dt} \right\rangle &= \frac{-64}{5} \frac{G^3 M_1 M_2 (M_1 + M_2)}{c^5 a^3 (1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right), \\ \left\langle \frac{de}{dt} \right\rangle &= \frac{-304}{15} \frac{G^3 M_1 M_2 (M_1 + M_2)}{c^5 a^4 (1 - e^2)^{5/2}} \left(e + \frac{121}{304} e^3 \right), \end{aligned} \tag{6.5.1}$$

and using the value 0.62 for the eccentricity of the Hulse-Taylor binary we find that the rate of decrease of the semi-major axis is increased (relative to the circular case) by a factor of about 12.4 – translating to a total decrease of the semi-major axis of about 160 metres. The ratio of observed to predicted values is 0.997 ± 0.002.

6.5.2 LIGO and VIRGO Collaborations

Despite their almost unfamothably tiny amplitudes, gravitational waves can be detected directly using laser interferometry. This was first achieved by the LIGO (Laser Interferometer Gravitational-Wave Observatory) collaboration, on 14th September 2015 and identified with the merger of two $\sim 30 M_\odot$ Kerr black holes at a distance of over a billion light years. Since then, there have been a further four confirmed detections; three black hole mergers like the original event, and the most recent a binary neutron star merger.

The LIGO experiment, based in the United States, consists of two instruments, one located in Louisiana (Livingstone) and another in Washington State (Hanford), each a Michelson interferometer ('L-shaped') with 4 km long arms. They are sensitive in the frequency range 40-7000 Hz and can detect strain amplitudes down to about 10^{-23} with peak sensitivity of about 10^{-22}. Cross-correlation of the signal between the different interferometers allows for local noise sources to be largely eliminated, and partial triangulation of the location of the source in the sky. Accurate triangulation requires at least three interferometers; the VIRGO instrument is located in Cascina, near Pisa in Italy, and began taking scientific data in 2017. This led almost immediately to an accurate triangulation of a binary neutron star merger on the 17th of August, which was then subsequently also observed as an optical signal, sparking a flurry of excitment and activity.

The observed waveform detected by the interferometer is a 'chirp' signal with an amplitude and frequency that increase sharply towards merger of the binary sources, followed by a rapid

[10] These are copied from P.C. Peters, Phys. Rev. **136**, B1224 (1964).

'ring-down' to a quiescent state. Although this fits qualitatively with the linear theory predictions of the previous section we should not expect close quantitative agreement from what is evidently a very strong (nonlinear) gravitational event. The observed waveform is compared against templates provided by numerical simulations of the full Einstein equations to infer the nature of the source, and also of the final state. This identifies the masses, and in the case of black hole mergers values for the spin parameter of the Kerr metric, as well as the distance to the source. For further details the original detection papers are most strongly recommended:

Observation of Gravitational Waves from a Binary Black Hole Merger, Physical Review Letters **116**, 061102 (2016).

GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Physical Review Letters **116**, 241103 (2016).

GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Physical Review Letters **118**, 221101 (2017).

GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence, Physical Review Letters **119**, 141101 (2017).

GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Physical Review Letters **119**, 161101 (2017).

Problems

1. One factor contributing to the feebleness of gravitational waves is the distance to the source, which can often be thousands of light years (21,000 for the Hulse-Taylor binary). The closest source is clearly the sun. Why do we not detect gravitational waves produced by the sun?

2. For circular orbits of two equal mass stars (mass M), Peters' formula (6.5.1) for the rate of change of separation reduces to
$$\frac{da}{dt} = \frac{-128G^3M^3}{5c^5a^3}.$$
If the initial separation is a_0, show that the time taken for complete inspiral of the two stars is
$$t = \frac{5a_0^4c^5}{512G^3M^3}.$$
Calculate this time for two neutron stars, each with $1.4M_\odot$, starting from an orbital period of 2 hours.

   The LIGO experiment is expected to detect the final inspiral of binary neutron stars after the gravitational waves they emit exceed a frequency of about 50 Hz (seismic noise makes detection difficult at lower frequencies). Estimate how long a neutron star merger event detected by LIGO will last for. Would a pair of merging black holes with equal masses of $10M_\odot$ produce a longer or shorter event?

   Qualitatively describe the nature of the gravitational wave signal that will characterise such mergers, at least while the two stars can be treated as point masses.

3. The only currently known double pulsar (binary system in which both stars are neutron stars from which we receive radio pulses) was discovered in 2003 and is called PSR J0737-3039. The masses of the two stars are $1.35M_\odot$ and $1.24M_\odot$ and the period of their orbit is 2.4 hours. The system is estimated to be between 3200 and 4500 light years from Earth.

   Estimate the amplitude of the gravitational waves produced by this system when they arrive at the Earth.

   Use Peters' formula for the rate of change of the semi-major axis (setting the eccentricity to zero)
$$\frac{da}{dt} = \frac{-64G^3M_1M_2(M_1+M_2)}{5c^5a^3},$$
to determine how long it will be before the orbit completely shrinks and the neutron stars merge. By how much does the orbit shrink per day?

   Estimate the general relativistic contribution to the precession of the perihelion in PSR J0737-3039.

4. If the sun was separated into two equal parts in circular orbit around each other with radius twice that of the sun ($R_\odot = 6.96\times10^5$ km), what would be the amplitude of the gravitational waves at the location of the Earth? What would the frequency of the gravitational waves be?

5. In the notes we obtained the expression
$$\overline{h}_{\mu\nu} = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x',$$
for the perturbation to the metric created by some matter distribution, here considered to be stationary, *i.e.* independent of time. Taking for the stress-energy-momentum tensor

the expression for a perfect, pressure-less fluid, $T_{\mu\nu} = \rho c^2 u_\mu u_\nu$, and with a non-relativistic velocity $u^\mu = (1, v^i/c)$, show that

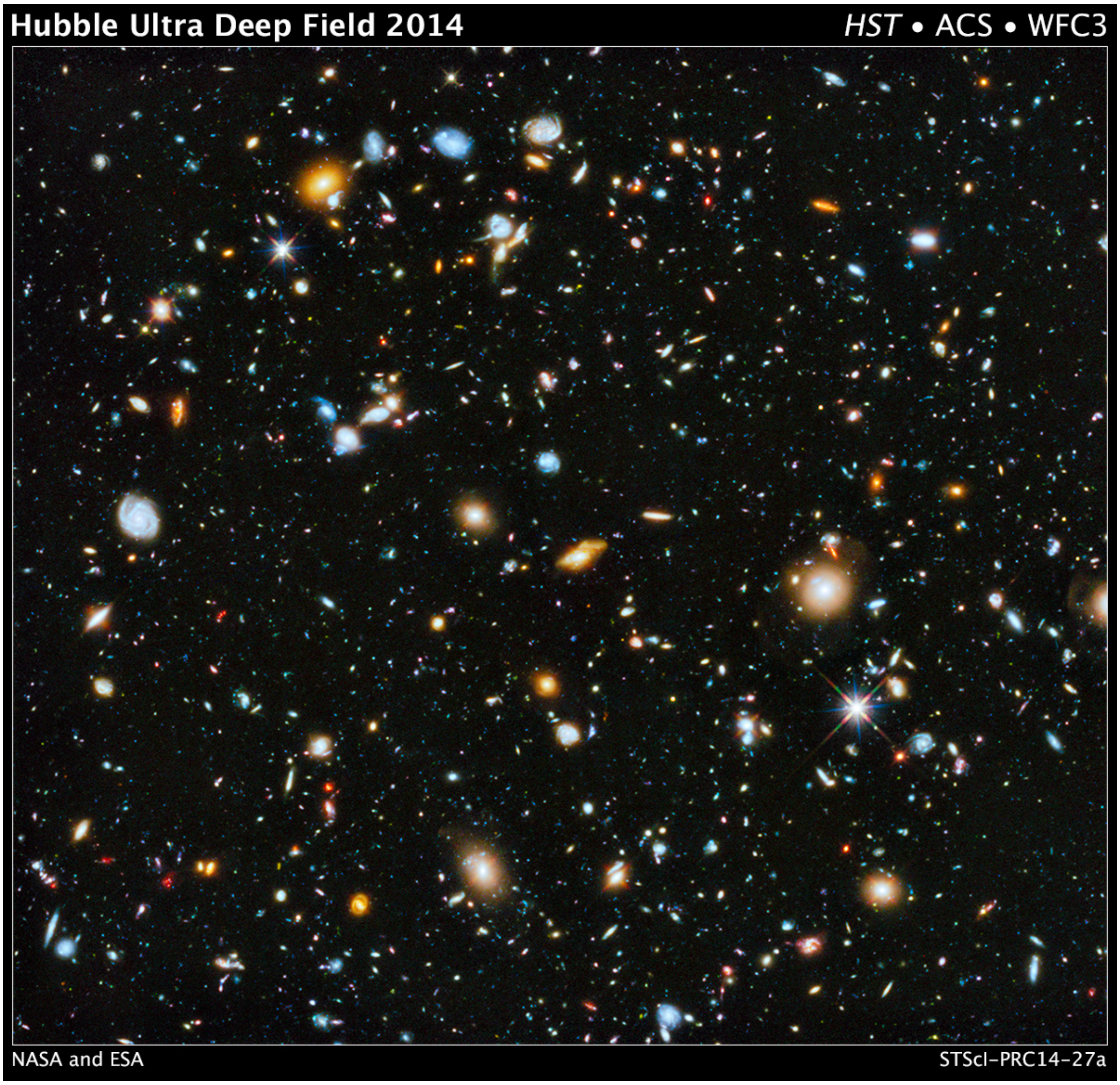
$$\begin{aligned}
\overline{h}_{00} &= \frac{4G}{c^2}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \equiv -\frac{4\phi}{c^2}, \\
\overline{h}_{0i} &= -\frac{4G}{c^3}\int \frac{\rho(\mathbf{x}')v^i}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \equiv -\Omega_i, \\
\overline{h}_{ij} &= \frac{4G}{c^4}\int \frac{\rho(\mathbf{x}')v^i v^j}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' = O\big(|v/c|^2\big),
\end{aligned}$$

to lowest order in $|v/c|$.

Hence show that the metric is given by

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - \Omega_i \Big[cdt\, dx^i + dx^i\, cdt\Big] + \left(1 - \frac{2\phi}{c^2}\right) dx^i dx^i.$$

ϕ has an interpretation as the Newtonian potential (it is defined by the same formula as Newton gave). What is the physical interpretation of the vector Ω_i?



The Ultra Deep Field image was taken by the Hubble Space Telescope in 2014. Almost everything you see in this image is a galaxy, some of them the most distant ever imaged. The few stars are easy to identify from their characteristic diffraction spikes. Image from NASA.

Chapter 7

Cosmology

This singularity is the most striking feature of the Robertson-Walker solutions. It occurs in all models in which $\mu + 3p$ is positive and Λ negative, zero, or with not too large a positive value. It would imply that the universe (or at least that part of which we can have any physical knowledge) had a beginning a finite time ago.

Hawking & Ellis (1973)

The solutions of the Einstein equations we have looked at so far have all been interpreted in terms of local gravitational problems; planetary orbits, lensing by a massive object, stellar collapse, individual black holes, and so on. The metric that arises as a particular solution of the Einstein equations is then taken to represent a decent approximation of some local region of the universe, for instance our own solar system. One can also conceive of an interpretation of suitable solutions of the Einstein equations as providing a metric that describes the entire universe, at some level. The study of the universe on its largest scales is called cosmology, and solutions of the Einstein equations used to interpret it are known as cosmological models. Their study originated with Einstein, in 1917.

7.1 de Sitter Universe and Cosmological Constant

The first cosmological models to be deduced from Einstein's theory of general relativity were Einstein's own *static universe* and the space-times of constant scalar curvature – the Lorentzian analogues of spheres – introduced by Willem de Sitter in 1917. There are two de Sitter universes: the *de Sitter space* dS_4 is the subset of $\mathbb{R}^{1,4}$ a space-like distance a_0 from the origin

$$dS_4 : \qquad -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a_0^2, \tag{7.1.1}$$

whereas the *anti-de Sitter space* AdS_4 is the subset of $\mathbb{R}^{2,3}$ a time-like distance a_0 from the origin

$$AdS_4 : \qquad -(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = -a_0^2. \tag{7.1.2}$$

Both are isotropic, homogeneous, four-dimensional Lorentzian manifolds with constant scalar curvature. They are also exact solutions of the Einstein equations, provided a certain interpretation is adopted. To see this, introduce coordinates (ct, χ, θ, ϕ) for dS_4 such that

$$\begin{gathered} x^0 = a_0 \sinh(ct/a_0), \qquad x^1 = a_0 \cosh(ct/a_0)\cos(\chi), \\ \left(x^2, x^3, x^4\right) = a_0 \cosh(ct/a_0)\sin(\chi)\Big(\cos(\theta), \sin(\theta)\cos(\phi), \sin(\theta)\sin(\theta)\Big), \end{gathered} \tag{7.1.3}$$

in terms of which the metric is given by

$$ds^2 = -c^2 dt^2 + a_0^2 \cosh^2(ct/a_0)\Big[d\chi^2 + \sin^2(\chi)\big(d\theta^2 + \sin^2(\theta)\, d\phi^2\big)\Big]. \tag{7.1.4}$$

After a wearisome calculation one arrives at expressions for the components of the Ricci tensor in this coordinate basis. They may be conveniently summarised as $R_{\mu\nu} = (3/a_0^2)g_{\mu\nu}$, from which it follows that the Ricci scalar is $R = 12/a_0^2$; de Sitter space is a space-time of constant positive scalar curvature. Now, for the Einstein equations to be satisfied we must have

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu}, \qquad \Rightarrow \qquad T_{\mu\nu} = \frac{-3c^4}{8\pi G a_0^2}\,g_{\mu\nu}. \tag{7.1.5}$$

In other words, the stress-energy-momentum tensor is a constant negative multiple of the metric. This corresponds to a perfect fluid, $T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu + p g_{\mu\nu}$, with constant *negative* pressure, equal in magnitude to the constant *positive* energy density ($p = -\rho c^2$). Ordinary matter does not have this property, so this is not the stress-energy-momentum tensor of any ordinary form of matter.

Historically, a term of this form – a constant multiple of the metric – was introduced into the field equations by Einstein as part of his static universe cosmology; the constant prefactor was referred to as the *cosmological constant* and denoted Λ. It was not originally introduced as a contribution to the stress-energy-momentum tensor, as it is now viewed, but as a modification of the Einstein equations to

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu}. \tag{7.1.6}$$

Nowadays, one moves it to the right-hand-side and interprets a contribution of the form $T_{\mu\nu} = -(c^4/8\pi G)\Lambda g_{\mu\nu}$ to the stress-energy-momentum tensor as arising from 'non-ordinary' matter and calls it *dark energy*. Modern observations are compatible with simple cosmological models based on the Einstein equations only if one assumes that a substantial fraction of the total stress-energy-momentum tensor comes from dark matter (26.8%) and dark energy (68.3%)[1]. Such cosmologies, dominated by the cosmological constant, suggest that our universe will evolve to become increasingly like de Sitter space in the future. Despite the fact that modern observations suggest that only 4.9% of the stress-energy-momentum tensor comes from ordinary forms of matter that we know and understand, it is evident that one must study, and understand, cosmological models in which the stress-energy-momentum tensor corresponds to ordinary forms of matter. This is what we turn to now.

7.2 The Friedmann-Lemaître-Robertson-Walker Metric

The de Sitter universes are isotropic and homogeneous. A natural generalisation is to associate these two properties – isotropy and homogeneity – only to space, and its matter content. Such a situation is captured by a very simple form for the metric

$$ds^2 = -c^2dt^2 + a^2(t)\,d\ell^2, \tag{7.2.1}$$

where $d\ell^2$ is the metric for an isotropic, homogeneous, three-dimensional Riemannian manifold of constant scalar curvature. There are precisely three of these, corresponding to the three possibilities of positive scalar curvature, zero scalar curvature and negative scalar curvature. These are, respectively, the 3-sphere, three-dimensional Euclidean space and three-dimensional hyperbolic space $\mathbb{H}^3$. The metric is commonly expressed in either of the forms

$$d\ell^2 = \begin{cases} d\chi^2 + \sin^2(\chi)\big(d\theta^2 + \sin^2(\theta)\,d\phi^2\big), \\ d\chi^2 + \chi^2\big(d\theta^2 + \sin^2(\theta)\,d\phi^2\big), \\ d\chi^2 + \sinh^2(\chi)\big(d\theta^2 + \sin^2(\theta)\,d\phi^2\big), \end{cases} \tag{7.2.2}$$

$$d\ell^2 = \frac{dr^2}{1-kr^2} + r^2\big(d\theta^2 + \sin^2(\theta)\,d\phi^2\big), \qquad k = +1, 0, -1. \tag{7.2.3}$$

[1]These figures are derived from observations of the cosmic microwave background by the ESA Planck satellite, released in 2013.

We will use the former, together with the symbol $k = +1, 0, -1$ to distinguish the three different cases for the scalar curvature. Any metric of the form (7.2.1) is called a *Friedmann-Lemaître-Robertson-Walker space-time.*

Metrics of this form, and the cosmology they give rise to, were first studied in 1922 by the Russian mathematical physicist Alexander Friedmann. The same results were obtained, independently, by the Belgian priest and scientist Georges Lemaître in 1927, who was also the first to point out that the expansion factor $a(t)$ could account for the observed redshifts of distant galaxies, obtain the expression that subsequently became known as 'Hubble's law' and determine the present value of the 'Hubble constant' from experimental data. The American mathematical physicist Howard Robertson and British mathematician Arthur Walker made their (independent) contributions in the mid 1930s. Their work established that the two conditions of spatial isotropy and homogeneity were sufficient to imply that the metric could always be written in the form (7.2.1), *i.e.* there is no loss of generality.

The assumption of spatial isotropy and homogeneity appears to be in reasonable accord with observations of the universe on the largest scales. Every direction we look in we see galaxies and clusters of galaxies; at least this is the impression one immediately gets from looking at the Hubble Deep Field images. Of course we do not see galaxies in *every* direction, nor do we see precisely the *same* galaxy, or cluster of galaxies, in *every* direction. There is variation; some galaxies are spiral, some elliptical; some parts of any image are perfectly dark, others are bright with the light of a star or galaxy. But 'on average' the night sky 'looks the same' in all directions. Moreover, this is true at all wavelengths. In addition to optical measurements we have been surveying the night sky in infrared, microwave, radio and x-ray parts of the electromagnetic spectrum. In each case one can say the same; on average the sky looks the same in every direction. Thus the FLRW cosmology has come to be the standard cosmological model, used for the interpretation of the majority of observations. At the same time, it is important to bear in mind and acknowledge that the universe is *neither* isotropic *nor* homogeneous in a strict sense; it is patchy; there are galaxies and clusters of galaxies, and within galaxies there are stars and planets and so forth. The point is that this patchiness can be studied and one can try to learn new things from it. A major topic, of high current interest, is the inhomogeneity, or fluctuations, in the cosmic microwave background. Nonetheless, here we will content ourselves with a study of the isotropic, homogeneous FLRW cosmologies.

The cosmology envisaged by Friedmann is a solution of the Einstein equations for the metric (7.2.1), with the stress-energy-momentum tensor of a perfect fluid. One finds by a tedious, but straightforward, calculation that

$$R_{00} - \frac{1}{2} R\, g_{00} = \frac{3}{a^2}\Big[\big(c^{-1}\partial_t a\big)^2 + k\Big], \tag{7.2.4}$$

$$R_{ij} - \frac{1}{2} R\, g_{ij} = \frac{-1}{a^2}\Big[2ac^{-2}\partial_t^2 a + \big(c^{-1}\partial_t a\big)^2 + k\Big] g_{ij}. \tag{7.2.5}$$

Equating this to the stress-energy-momentum tensor for a perfect fluid, $T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu + p g_{\mu\nu}$, in its rest frame, yields

$$\left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{8\pi G}{3c^2}\,\rho c^2 = 0, \tag{7.2.6}$$

$$\frac{\ddot a}{a} + \frac{4\pi G}{3c^2}\big(\rho c^2 + 3p\big) = 0, \tag{7.2.7}$$

where $\dot a = \partial_t a$. These are known as the *Friedmann equations.* Sometimes the second is called separately the *acceleration equation.* Either from these, by taking the time derivative of the first equation, or from the continuity of the stress-energy-momentum tensor, $\nabla_\alpha T^\alpha_\mu = 0$, one can derive the *fluid equation*

$$\partial_t \rho c^2 + \frac{3\dot a}{a}\big(\rho c^2 + p\big) = 0. \tag{7.2.8}$$

The Friedmann equations do not yield a closed system; one needs to give also an equation of state for the fluid[2]. We will consider a couple of approximations, or idealised equations of state, corresponding to a dust dominated universe, where the pressure is negligible compared to the energy density (p negligible compared to ρc^2), and a radiation dominated universe, where the pressure is taken to be one-third the energy density[3] ($p = \frac{1}{3}\rho c^2$). But before this, some general comments can be made.

Regardless of the equation of state for the fluid, the main feature of the Friedmann equations is that the scale factor a is not a constant[4], *i.e.* the space-time is not static and 'space' is either expanding or contracting. This is an awe-inspiring prediction. It can, of course, be tested against observation. Observations of distant galaxies reveal that they exhibit a generic redshift in their spectral lines. Moreover, the value of the redshift depends only on the distance of the galaxy from us, matching with the assumption that the universe appears isotropic to us on the largest length scales. If the spectral redshift is attributed to a Doppler shift, then we would say that all[5] other galaxies are moving away from us, and those that are more distant are receding faster. It was Georges Lemaître who first recognised that these observations are reproduced by the time-dependence of the scale factor a in the FLRW metric (7.2.1); the universe is expanding isotropically. He recognised more than this; if the universe is expanding, $\dot{a} > 0$, and $\rho c^2 + 3p$ is non-negative, then (7.2.7) shows that $\ddot{a} \leq 0$, and the universe must have been expanding at its present rate, or faster, in the past. So in the past the universe was smaller, and smaller still. Extrapolating backwards its size must have been vanishingly small at a time of no more than $(a/\dot{a})|_{\text{now}} \equiv H^{-1}$ long ago. This is the concept of a 'primevial atom' from which the universe originated, first introduced by Lemaître in 1931; the name 'Big Bang' was coined by Fred Hoyle in 1949. It is perhaps the most striking feature of simple isotropic homogeneous cosmologies.

To say more, consider a galaxy that at some instant in time, $t = 0$ say, is at the spatial point $(\chi_0, 0, 0)$ and is travelling along a time-like geodesic[6]. It is not difficult to verify, from the geodesic equation, that its trajectory is

$$\Big(ct(\tau), \chi(\tau), \theta(\tau), \phi(\tau)\Big) = \big(\tau, \chi_0, 0, 0\big). \tag{7.2.9}$$

What is important here is that the spatial point where the galaxy is located is not changing; it is 'at rest'[7]. Now consider two such galaxies, one at the spatial point $(\chi_0, 0, 0)$ – the galaxy we are observing, M31 for the sake of giving it a name – and one at the spatial point $(0, 0, 0)$ – our own galaxy, where we are observing from. When we look at M31 we see light that left it and travelled to us along a 'radial' null geodesic. Its trajectory therefore satisfies

$$cdt = -a(t)d\chi, \tag{7.2.10}$$

the minus sign coming because it is travelling inwards from $\chi = \chi_0$ to $\chi = 0$. The total time

[2]For instance specifying the pressure as a function of density. As you will surely know, the equation of state for a simple fluid or ideal gas is typically a relation between three thermodynamic state variables, not just two. The third is the temperature or entropy. In such circumstances one then needs in addition an entropy balance equation.

[3]The reason for this is that then the stress-energy-momentum tensor is traceless as it is for the electromagnetic field, however, one should say that this is only a proxy and certainly in the radiation dominated model we do not employ the correct stress-energy-momentum tensor for the electromagnetic field, nor self-consistently solve Maxwell's equations.

[4]Indeed, there are no solutions for constant a without a cosmological constant; this was Einstein's motivation for introducing it.

[5]Not quite all, some of those closest to us – in the local cluster – are moving towards us. But if we average over length scales large compared to intergalactic distances then the observations are 'the same for all galaxies'. Remarks of this kind are implicit in any statement that the universe appears isotropic and homogeneous on large scales.

[6]It is a 'test particle' for the cosmological metric.

[7]You should recall that this condition was also used in deriving the Friedmann equations; the stress-energy-momentum tensor for the fluid was given in its rest frame.

taken is given implicitly by

$$\chi_0 = \int_{t_{\rm emit}}^{t_{\rm obs}} \frac{cdt}{a(t)}, \tag{7.2.11}$$

and should be thought of as an expression for the observation time $t_{\rm obs}$ as a function of the emission time $t_{\rm emit}$. Differentiating with respect to the emission time we obtain

$$0 = \frac{c}{a(t)}\bigg|_{t=t_{\rm obs}} \frac{\partial t_{\rm obs}}{\partial t_{\rm emit}} - \frac{c}{a(t)}\bigg|_{t=t_{\rm emit}}, \tag{7.2.12}$$

or, since the derivative $\partial t_{\rm obs}/\partial t_{\rm emit}$ is the ratio $\omega_{\rm emit}/\omega_{\rm obs}$ of the frequencies of the emitted and observed light,

$$\frac{\omega_{\rm emit}}{\omega_{\rm obs}} = \frac{a(t_{\rm obs})}{a(t_{\rm emit})} = \frac{\lambda_{\rm obs}}{\lambda_{\rm emit}} = 1+z, \tag{7.2.13}$$

where z is the *redshift*. Thus the redshift is given by the difference between the value of the scale factor a at the time the light is observed and the time the light is emitted. The expansion of the universe accounts for the observed redshift of galactic spectra.

If the galaxies are close to each other, so that $t_{\rm emit} = t_{\rm obs} - \delta t$ with δt small, then we may approximate

$$\frac{a(t_{\rm obs})}{a(t_{\rm emit})} \approx \frac{a(t_{\rm obs})}{a(t_{\rm obs}) - \partial_t a(t_{\rm obs}).\delta t} \approx 1 + \frac{\dot a}{a}\bigg|_{t_{\rm obs}} \delta t \approx 1 + H(t_{\rm obs})\,\frac{a(t_{\rm obs})\chi_0}{c}, \tag{7.2.14}$$

where $H(t_{\rm obs}) = \dot a/a$ is the current value of the *Hubble constant*. For small values of the redshit, the interpretation as a Doppler shift due to the apparent recession velocity gives $z = v/c$ and we obtain *Hubble's law*

$$v = H(t_{\rm obs})\,\ell_0, \tag{7.2.15}$$

where $\ell_0 = a(t_{\rm obs})\chi_0$ is the current spatial distance between the galaxies as measured using the current spatial metric $a^2(t_{\rm obs})d\ell^2$ [8]. Observations of the redshifts of nearby galaxies therefore provide a means of measuring the current value of the Hubble constant, providing the current distances to the galaxies are known accurately. This gives a directly observable measure of the current rate of expansion of the universe and connects it to cosmological models. The main issue is in determining the distances to nearby galaxies with sufficient accuracy. Measurements by the Hubble Space Telescope give a value for the Hubble constant of 74.2 ± 3.6 km s^{-1} Mpc^{-1}, while those from the ESA Planck satellite yield a slightly smaller value of 67.8 ± 0.8 km s^{-1} Mpc^{-1}. Given that $\ddot a \le 0$ these values for the Hubble constant imply that the universe is no older than 13.5 or 14.7 billion years, respectively.

Let us return now to the Friedmann equations and consider how the nature of the universe reflects its spatial curvature, *i.e.* how the scale factor $a(t)$ depends on the parameter $k \in \{+1, 0, -1\}$ that encodes whether the spatial metric corresponds to that of a sphere, Euclidean space, or hyperbolic space. As an equation of state, let us write $p = w\rho c^2$ so that the dust filled universe corresponds to $w = 0$ and the radiation dominated universe is $w = 1/3$. One finds immediately from the fluid equation (7.2.8) that

$$a^{3(1+w)}\rho c^2 = \text{const.} \tag{7.2.16}$$

It follows that as the universe expands the energy density decreases, the energy density associated with radiation decreasing more rapidly than that associated with dust, or the matter content. Of course, going backwards in time to when the universe was very much smaller, the converse is true and radiation would have dominated the energy content. A picture that emerges is that at early times the universe was small and the main contribution to the energy density was from (electromagnetic) radiation, but as it expanded that contribution died out more rapidly than the energy density of the dust so that at some point there was a cross-over

[8] Note that this is not the same as geodesic distance, but the difference is small under the assumptions made.

to a matter dominated regime in which the galaxies could form. The discovery of the cosmic microwave background radiation by Penzias and Wilson in 1964 is interpreted as a remnant of this transition from the early universe and rightly considered one of the most significant observations of the evolutionary history of our universe.

Substituting the expression for the energy density into the Friedmann equation gives for the scale factor

$$\dot{a}^2 = Ka^{-(1+3w)} - kc^2, \tag{7.2.17}$$

where K is a constant that we do not give explicitly. In the dust filled regime, $w = 0$, the solution is

$$a(t) = \begin{cases} \frac{K}{c^2}\sin^2(\eta(t)),\ \eta - \frac{1}{2}\sin(2\eta) = \frac{c^3}{K}t & k = +1, \\ \left(\frac{3}{2}K^{1/2}t\right)^{2/3} & k = 0, \\ \frac{K}{c^2}\sinh^2(\eta(t)),\ \frac{1}{2}\sinh(2\eta) - \eta = \frac{c^3}{K}t & k = -1. \end{cases} \tag{7.2.18}$$

The most striking feature of the solution is that the scale factor increases without bound if the universe is spatially flat ($k = 0$) or hyperbolic ($k = -1$) but not if there is positive curvature ($k = +1$). If the spatial sections are 3-spheres then the universe expands to a maximum size $a_{\text{max}} = K/c^2$ at a time $t_{\text{max}} = \frac{\pi K}{2c^3}$ after which it begins to contract until it eventually shrinks back to nothing after a time $t_{\text{end}} = \frac{\pi K}{c^3}$. This fate for the universe is sometimes called the 'Big Crunch'. Naturally, it is interesting to know the spatial curvature of our universe; current observations suggest that it is flat, $k = 0$, but it is understandably very difficult to tell.

7.3 Dark Energy and Λ-CDM

In the late 1990s new observational results indicated that the expansion of the universe is accelerating, meaning $\ddot{a} > 0$. This is inconsistent with the Friedmann equation (7.2.7) for any ordinary form of matter. The observations were of Type Ia supernovae. The brightness – intrinsic luminosity – of such supernovae provides a measure for their distance, which can be correlated against the value of the scale factor $a(t_{\text{sn}})$ at that time as derived from the redshift of the galaxy they belong to. From the present value of the Hubble constant (rate of expansion) and the redshift one can estimate how far away the supernovae would be if the universe had always been expanding at its present rate and consequently how bright you expect it to be. It is not hard to see that if the expansion of the universe is slowing down, $\ddot{a} < 0$, then the time required for the observed change in the scale factor ($a(t_{\text{sn}})$ to $a(t_{\text{now}})$) would be less, so too the distance to the supernova, and consequently it would be brighter than expected on the basis of a constant rate of expansion. The observation is precisely the converse; the Type Ia supernovae are dimmer than expected, consistent with a positive value for the acceleration, $\ddot{a} > 0$.

The general consensus on how to interpret these observations has been to reinstate Einstein's cosmological constant, Λ, albeit under the new (but no less mysterious) moniker of *dark energy*. Current estimates give a value for the cosmological constant of 10^{-52} m^{-2}; tiny, but supposedly non-zero and positive. One may think of it as a contribution to the stress-energy-momentum tensor of the form

$$-\frac{c^4}{8\pi G}\Lambda g_{\mu\nu}, \tag{7.3.1}$$

and, as mentioned previously, view it as like a fluid with negative pressure, equal in magnitude to the energy density. For this reason, one sometimes views dark energy as having an equation of state $p = w\rho c^2$ with $w = -1$. Accounting for dark energy, the Friedmann equations are modified to read

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{8\pi G}{3c^2}\,\rho c^2 - \frac{1}{3}\Lambda c^2 = 0, \tag{7.3.2}$$

$$\frac{\ddot{a}}{a} + \frac{4\pi G}{3c^2}\big(\rho c^2 + 3p\big) - \frac{1}{3}\Lambda c^2 = 0, \tag{7.3.3}$$

while the fluid equation remains unchanged[9]. A large enough value of the cosmological constant allows for positive acceleration, $\ddot{a} > 0$. If one neglects the matter density ρ compared to the cosmological constant, then, for a spatially flat universe $k = 0$, the expansion is exponential

$$a(t) = a(0)\,\mathrm{e}^{\sqrt{\Lambda/3}\,ct}. \tag{7.3.4}$$

This solution should not be that surprising, given how we started this chapter with a discussion of the de Sitter universe. Indeed, setting $k = +1$, one obtains the de Sitter metric (7.1.4) with $a_0 = (3/\Lambda)^{1/2}$. Since in this case $(3/\Lambda)^{1/2}$ is the radius of curvature, a tiny value for the cosmological constant makes it exceedingly difficult to tell if spatial sections of the universe are flat ($k = 0$) or curved ($k = \pm 1$).

The Λ-CDM cosmology in which the universe started with the Big Bang, was initially radiation dominated until the recombination transition that left the cosmic microwave background radiation, was then dominated by the matter content (dust) for a period and then more recently by the cosmological constant, or dark energy, summarises the current view of the history and nature of our universe.

[9] So long as the cosmological constant Λ does not depend on time.

Problems

1. The fluid equation in the FLRW cosmology can be described in terms of classical thermodynamics. As the universe is homogeneous and isotropic there can be no temperature gradients and therefore no flow of heat, so that the first law of thermodynamics is simply

$$dU = -p\,dV.$$

By dimensional analysis show that

$$\frac{dV}{V} = 3\frac{da}{a} = 3H\,dt,$$

where H is the Hubble constant. Finally, taking the energy of an ideal fluid to be $U = \rho V c^2$, show that the density satisfies the fluid equation

$$\partial_t \rho + 3H\left(\rho + p/c^2\right) = 0.$$

2. The universe at the current time is dominated by matter and the cosmological constant, with the contribution of radiation negligible. Show that Friedmann's equation for the current epoch can be written as

$$H(z) = H_0\left[\Omega_M(1+z)^3 + \Omega_\Lambda - \left(\Omega_M + \Omega_\Lambda - 1\right)(1+z)^2\right]^{1/2},$$

where $H(z)$ is the Hubble constant at redshift z, H_0 is its present day value, and Ω_M, Ω_Λ are the ratios of the current matter density and cosmological constant density to the *critical density*, defined by $\rho_{\text{crit}} = 3H_0^2/8\pi G$.

Which term in the relation above represents the spatial curvature of the universe?

In the lectures we made use of the relation

$$\chi = \int \frac{c\,dt}{a(t)},$$

to find that the galactic redshift z was related to the scale factor $a(t)$ by

$$1 + z = \frac{a_0}{a(t)},$$

where a_0 is the current value of the scale factor. By differentiating the latter relation, show that

$$a_0\chi = \int_0^z \frac{c\,dz'}{H(z')}.$$

Hence show that $a_0\chi$ is a monotonically increasing function of Ω_Λ. How does this relate to the use of supernovae in cosmology?

3. Use the fluid equation to show that a cosmological fluid containing only dark energy (no ordinary matter) has constant energy density. What does this imply about the homogeneity of the space-time? Does it conflict with the scale factor $a(t)$ having time-dependence?

4. For a universe with only cosmological constant (dark energy, no ordinary matter) solve the Friedmann equations to find the possible forms that the universe may take and describe their evolution. You should distinguish the cases $\Lambda > 0$ and $\Lambda < 0$, and consider all possible spatial geometries.

[Hint: do not try to find the most general solution, but rather a (representative) solution for each possibility, if such exists.]

5. Einstein's static universe is the cylinder

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a_0^2,$$

of the Minkowski space $\mathbb{R}^{1,4}$ with its usual metric. Show that it is an exact solution of the Einstein equations with cosmological constant and perfect fluid form for the stress-energy-momentum tensor (dust, no pressure).

[Hint: parameterise the space and determine the metric in your local coordinates; then compute from it the components of the Ricci tensor; finally solve the Einstein equations. This is easier than it sounds as if you make a sensible choice for the coordinates you will find that the metric is of FLRW form, for which you have been given the calculation of the Ricci tensor and solution of the Einstein equations in the notes.]

6. Show that the anti-de Sitter space is an exact solution of the Einstein equations with cosmological constant and no ordinary matter content.

Show that there are closed time-like curves in the anti-de Sitter space.

Solutions to selected problems

I hope over time to include sketched solutions to as many of the end of chapter problems as I can. This is a laborious process but I will try to get enough solutions written up quickly, so as to be useful. As always, comments, suggestions, corrections, or help with producing solutions are all most welcome. It is worth reiterating that the end of chapter problems are deliberately designed to be distinct, both in content and style, from past exam problems, so as to minimise overlap and retain the latter as a useful repository for exam preparation.

Chapter 1 – Gravity

PROBLEM 1

The Kepler problem considers two masses m and M moving under their mutual gravitational interaction. There is an implicit assumption that m is the mass of a planet, and may be considered small, while M is the mass of the sun (or star), and may be considered large, although the analysis is certainly not restricted to this scenario. If $\mathbf{x}_1$, $\mathbf{x}_2$ denote the positions of the two masses in some inertial frame, then Newton's law of motion reads

$$m\frac{d^2\mathbf{x}_1}{dt^2} = -\frac{GMm}{|\mathbf{x}_1-\mathbf{x}_2|^2}\frac{\mathbf{x}_1-\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|}, \qquad M\frac{d^2\mathbf{x}_2}{dt^2} = -\frac{GMm}{|\mathbf{x}_1-\mathbf{x}_2|^2}\frac{\mathbf{x}_2-\mathbf{x}_1}{|\mathbf{x}_1-\mathbf{x}_2|}.$$

Adding, we find

$$\frac{d}{dt}\left[m\frac{d\mathbf{x}_1}{dt} + M\frac{d\mathbf{x}_2}{dt}\right] = \mathbf{0},$$

with the interpretation that the total linear momentum is conserved. Dividing by m or M, as appropriate, and subtracting we find

$$\frac{d^2(\mathbf{x}_1-\mathbf{x}_2)}{dt^2} = -\frac{G(M+m)}{|\mathbf{x}_1-\mathbf{x}_2|^2}\frac{\mathbf{x}_1-\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|},$$

or, writing $\mathbf{x}_1-\mathbf{x}_2 = r\mathbf{n}$, with $\mathbf{n}$ a unit vector,

$$\frac{d}{dt}\left[\frac{dr}{dt}\mathbf{n} + r\frac{d\mathbf{n}}{dt}\right] = -\frac{G(M+m)}{r^2}\mathbf{n}.$$

The term in square brackets is the instantaneous relative velocity, which lies in a plane spanned by the orthogonal vectors $\mathbf{n}$ and $d\mathbf{n}/dt$ (don't forget $\mathbf{n}$ is a unit vector). Its change is purely in the direction $\mathbf{n}$, which lies in this plane, and so the motion continues to lie in this same plane. Introducing a standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for this plane, and polar coordinates (r, ϕ), we may write

$$\mathbf{x}_1 - \mathbf{x}_2 = r\mathbf{n} = r\big[\mathbf{e}_1\cos(\phi) + \mathbf{e}_2\sin(\phi)\big],$$

and the equation of motion for the relative separation becomes

$$\frac{d^2r}{dt^2}\mathbf{n} + 2\frac{dr}{dt}\frac{d\phi}{dt}\mathbf{n}^{\perp} + r\frac{d^2\phi}{dt^2}\mathbf{n}^{\perp} - r\left(\frac{d\phi}{dt}\right)^2\mathbf{n} = -\frac{G(M+m)}{r^2}\mathbf{n},$$

where $\mathbf{n}^{\perp} = -\sin(\phi)\,\mathbf{e}_1 + \cos(\phi)\,\mathbf{e}_2$. By linear independence, the coefficients of $\mathbf{n}$ and $\mathbf{n}^{\perp}$ must vanish separately. Taking the latter first we find

$$0 = 2\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\phi}{dt^2} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right),$$

from which we conclude that

$$r^2\frac{d\phi}{dt} = \ell,$$

with ℓ a constant. This is Kepler's second law of planetary motion; a line segment between a planet and the sun sweeps out equal areas in equal times. It is a statement of conservation of orbital angular momentum. The remaining equation reads

$$0 = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 + \frac{G(M+m)}{r^2} = \frac{d^2r}{dt^2} - \frac{\ell^2}{r^3} + \frac{G(M+m)}{r^2}.$$

Finally, describe the motion by $u = 1/r$ and think of u as parameterised by angle ϕ rather than time t. We have

$$\begin{aligned} \frac{dr}{dt} &= \frac{d\phi}{dt}\frac{du^{-1}}{d\phi} = -\ell\frac{du}{d\phi}, \\ \frac{d^2r}{dt^2} &= \frac{d\phi}{dt}\frac{d}{d\phi}\left(-\ell\frac{du}{d\phi}\right) = -\ell^2u^2\frac{d^2u}{d\phi^2}, \end{aligned}$$

and hence the equation of motion becomes

$$0 = \frac{d^2r}{dt^2} - \frac{\ell^2}{r^3} + \frac{G(M+m)}{r^2} = -\ell^2u^2\left[\frac{d^2u}{d\phi^2} + u - \frac{G(M+m)}{\ell^2}\right].$$

The solution is

$$u = \frac{G(M+m)}{\ell^2}\Big[1 + e\cos(\phi)\Big],$$

where e is the eccentricity. This is the equation of an ellipse with the centre of our coordinate system (r,ϕ) at one of the foci; Kepler's first law. With this solution one can compute the orbital period from the equation $r^2\,d\phi/dt = \ell$

$$T = \int_0^{2\pi} \frac{\ell^3}{G^2(M+m)^2}\frac{d\phi}{(1+e\cos(\phi))^2} = \frac{2\pi\ell^3}{G^2(M+m)^2(1-e^2)^{3/2}}.$$

The integral can presumably be done a number of ways; I used contour methods. The semi-major axis of the orbit is $\ell^2/G(M+m)(1-e^2)$ so that the square of the orbital period is proportional to the cube of the semi-major axis, Kepler's third law of planetary motion.

A Lagrangian approach is also insightful, especially given the method used in general relativity where the planetary orbits are described using time-like geodesics of the Schwarzschild metric. The Lagrangian is

$$\begin{aligned} L &= \frac{m}{2}\left\|\frac{d\mathbf{x}_1}{dt}\right\|^2 + \frac{M}{2}\left\|\frac{d\mathbf{x}_2}{dt}\right\|^2 + \frac{GMm}{|\mathbf{x}_1 - \mathbf{x}_2|}, \\ &= \frac{M+m}{2}\left\|\frac{d}{dt}\frac{m\mathbf{x}_1 + M\mathbf{x}_2}{M+m}\right\|^2 + \frac{Mm}{2(M+m)}\left\|\frac{d(\mathbf{x}_1 - \mathbf{x}_2)}{dt}\right\|^2 + \frac{GMm}{|\mathbf{x}_1 - \mathbf{x}_2|}, \\ &= \frac{M+m}{2}\left\|\frac{d}{dt}\frac{m\mathbf{x}_1 + M\mathbf{x}_2}{M+m}\right\|^2 + \frac{Mm}{2(M+m)}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right] + \frac{GMm}{r}, \end{aligned}$$

writing the relative position vector as $\mathbf{x}_1 - \mathbf{x}_2 = r\mathbf{n}$ as before. It follows that $\frac{d}{dt}(m\mathbf{x}_1 + M\mathbf{x}_2)$ is a constant, as is the magnitude of the orbital angular momentum $\frac{Mm}{M+m}r^2\frac{d\phi}{dt}$. [We implicitly used the constancy of its direction in writing the Lagrangian in the form given above.] Calling the latter ℓ we can the total energy of the relative motion as

$$\frac{Mm}{2(M+m)}\left[\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{r^2}\right] - \frac{GMm}{r} = E,$$

which is a constant as the Lagrangian has no explicit time dependence. This energy equation can be used to characterise the nature of planetary orbits – circular, bound elliptical, unbound hyperbolic – without having to find the explicit form of the orbits. It should be contrasted with the analogous expression obtained for the time-like geodesics of the Schwarzschild space-time.

PROBLEM 6

The expressions $A^\alpha + B_\alpha$, $R_{\mu\nu} = S_\gamma$, and $g_{\mu\nu}g^{\mu\nu}R_{\mu\nu}$ are each meaningless. The first because it tries to add a vector to a 1-form and these are simply different things, and the second because it tries to equate a type $\binom{0}{2}$ tensor to a 1-form; they too are simply different objects. The third has both the index μ and ν repeated three times, which does not correspond to anything in the index notation; in that notation an index either appears once only, and is not summed, or twice exactly, and is summed. [Exceptions can be made only if the expression is explicilty accompanied by the words *"not summed"*, although they are fairly rare and do not appear at all in this course.]

PROBLEM 7

The equation $\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$ is the usual wave equation, since

$$\begin{aligned}\eta^{\mu\nu}\partial_\mu\partial_\nu &= \eta^{00}\partial_0\partial_0 + \eta^{11}\partial_1\partial_1 + \eta^{22}\partial_2\partial_2 + \eta^{33}\partial_3\partial_3, \\ &= \frac{-1}{c^2}\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2.\end{aligned}$$

Let the symbol ∂'_μ denote the partial derivative $\partial/\partial x'^\mu$. Then, by the chain rule we have

$$\begin{aligned}\eta^{\mu\nu}\partial_\mu\partial_\nu &= \eta^{\mu\nu}\,\frac{\partial x^\alpha}{\partial x^\mu}\partial'_\alpha\,\frac{\partial x^\beta}{\partial x^\nu}\partial'_\beta, \\ &= \eta^{\mu\nu}\Lambda^\alpha_\mu\Lambda^\beta_\nu\,\partial'_\alpha\partial'_\beta, \\ &= \eta^{\alpha\beta}\partial'_\alpha\partial'_\beta,\end{aligned}$$

using the properties of the inverse metric and of the Lorentz transformation. It follows that under a Lorentz transformation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0 \quad\mapsto\quad \eta^{\alpha\beta}\partial'_\alpha\partial'_\beta\phi = 0,$$

the wave equation takes the same form (assuming that the field ϕ transforms as a scalar).

From the foregoing we have that the wave operator $\eta^{\mu\nu}\partial_\mu\partial_\nu$ acts on the plane wave $\phi \sim \mathrm{e}^{i(k_jx_j-\omega t)}$ to give $[(\omega/c)^2 - k_jk_j]\phi$. Thus the action of the Klein-Gordon operator gives

$$\left[\hbar^2c^2\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2c^4\right]\phi = \left[\hbar^2\omega^2 - \hbar^2c^2k_jk_j - m^2c^4\right]\phi$$

and the plane wave satisfies the Klein-Gordon equation provided the relativistic dispersion

$$\hbar^2\omega^2 = \hbar^2c^2k_jk_j + m^2c^4.$$

is satisfied.

PROBLEM 8

δ^α_α is the sum of the 'diagonal elements' of δ^μ_ν, all of which are equal to $+1$. Thus $\delta^\alpha_\alpha = 4$, or n in general dimension.

For the full contraction of the Levi-Civita symbol we should note that $\epsilon_{0123} = +1$ but $\epsilon^{0123} = -1$. The indices $(\alpha\beta\mu\nu)$ all need to be distinct, but other than that we are required to sum over each of the 24 permutations of (0123); for each permutation the value of the product of the two symbols is -1. Thus $\epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\mu\nu} = -24$.

Let $F_{\mu\nu} = -F_{\nu\mu}$ be antisymmetric and $T^{\mu\nu} = T^{\nu\mu}$ symmetric. Then

$$F_{\mu\nu}T^{\mu\nu} = -F_{\nu\mu}T^{\mu\nu} = -F_{\nu\mu}T^{\nu\mu} = -F_{\mu\nu}T^{\mu\nu}.$$

Here the first equality comes from the antisymmetry of F, the second from the symmetry of T and the third from a simple relabelling of indices $(\mu\nu) \mapsto (\nu\mu)$. It follows that $F_{\mu\nu}T^{\mu\nu}$ is equal to its negative and so must be zero.

In the case where no symmetry conditions are assumed for $T^{\mu\nu}$ we need only note the identity

$$T^{\mu\nu} = \frac{1}{2}\Big(T^{\mu\nu} + T^{\nu\mu}\Big) + \frac{1}{2}\Big(T^{\mu\nu} - T^{\nu\mu}\Big),$$

which writes T as the sum of a symmetric part and an antisymmetric part. The previous result then gives the current one.

The symbol $\Gamma^\alpha_{\mu\nu}$ has three indices each of which can take any of n different values. There are therefore n^3 different components. If it is symmetric $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$ then only $\frac{1}{2}n(n+1)$ of the combinations of possibilities for the lower two indices give independent values. So there are $\frac{1}{2}n^2(n+1)$ independent components in this case. [For instance, if $n = 4$ there are 40 independent components; this is the general situation for the Christoffel symbols.]

PROBLEM 11

The field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where the components of the electromagnetic gauge field are $A_\mu = (-\phi, c\mathbf{A})$. By direct calculation we have

$$\begin{aligned} F_{i0} &= \partial_i\big(-\phi\big) - \frac{1}{c}\partial_t\, cA_i = E_i, \\ \epsilon_{ijk}F_{jk} &= \epsilon_{ijk}\big[\partial_j cA_k - \partial_k cA_j\big] = 2cB_i, \end{aligned}$$

and both results follow.

Note that $F^{\mu\nu}$ is antisymmetric, since

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} = -\eta^{\mu\alpha}\eta^{\nu\beta}F_{\beta\alpha} = -\eta^{\mu\beta}\eta^{\nu\alpha}F_{\alpha\beta} = -F^{\nu\mu}.$$

Then, accounting for the values of the components of the inverse metric, we have at once

$$\begin{aligned} F^{0i} &= \eta^{0\alpha}\eta^{i\beta}F_{\alpha\beta} = (-1)(1)\,F_{0i} = -(-E_i) = E_i, \\ F^{ij} &= \eta^{i\alpha}\eta^{j\beta}F_{\alpha\beta} = (1)(1)\,F_{ij} = \epsilon_{ijk}cB_k. \end{aligned}$$

It follows that the Lorentz scalar $F_{\mu\nu}F^{\mu\nu}$ can be expressed as

$$F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} = -E_iE_i + E_i\big(-E_i\big) + \epsilon_{ijk}cB_k\,\epsilon_{ijl}cB_l = -2\mathbf{E}\cdot\mathbf{E} + 2c^2\mathbf{B}\cdot\mathbf{B},$$

in terms of the electric and magnetic fields.

By direct calculation we have the components

$$\begin{aligned} \star F_{0i} &= \frac{1}{2}\epsilon_{0ijk}F^{jk} = \frac{1}{2}\epsilon_{ijk}\epsilon_{jkl}cB_l = cB_i, \\ \star F_{ij} &= \frac{1}{2}\epsilon_{ij0k}F^{0k} + \frac{1}{2}\epsilon_{ijk0}F^{k0} = \epsilon_{ijk}E_k. \end{aligned}$$

The others follow by antisymmetry. The dual field strength tensor can therefore be thought of as obtained from the field strength tensor by the replacements $(\mathbf{E}, c\mathbf{B}) \mapsto (-c\mathbf{B}, \mathbf{E})$.

The Lorentz pseudoscalar $\star F_{\mu\nu}F^{\mu\nu}$ is expressed in terms of the electric and magnetic fields as

$$\star F_{\mu\nu}F^{\mu\nu} = \star F_{0i}F^{0i} + \star F_{i0}F^{i0} + \star F_{ij}F^{ij} = 4c\mathbf{E}\cdot\mathbf{B}.$$

PROBLEM 12

For reference, we have $F_{i0} = E_i$, $F_{ij} = \epsilon_{ijk}cB_k$ and the components of the 4-current are $J^\mu = (\rho c, \mathbf{J})$.

Consider first the Bianchi identities

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0.$$

Take the particular values $\alpha = 0, \mu = 1, \nu = 2$. Then we have

$$\begin{aligned} \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} &= \frac{1}{c}\partial_t\, cB_z + \partial_x E_y + \partial_y\big(-E_x\big), \\ &= \partial_t B_z + \big(\nabla\times\mathbf{E}\big)_z, \end{aligned}$$

and so recover the z-component of the Maxwell equation $\nabla\times\mathbf{E} + \partial_t\mathbf{B} = \mathbf{0}$. The particular values $\alpha = 0, \mu = 2, \nu = 3$ and $\alpha = 0, \mu = 3, \nu = 1$ produce the x- and y-components of the same equation. Now consider the particular values $\alpha = 1, \mu = 2, \nu = 3$. Then we have

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_x\, cB_x + \partial_y\, cB_y + \partial_z\, cB_z = c\nabla\cdot\mathbf{B},$$

and so recover the Maxwell equation $\nabla\cdot\mathbf{B} = 0$.

Next consider the field equations

$$\partial_\nu F^{\mu\nu} = \mu_0 c\, J^\mu.$$

First, set $\mu = 0$. Then

$$\begin{aligned} \partial_\nu F^{0\nu} &= \partial_0 F^{00} + \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03}, \\ &= 0 + \partial_x E_x + \partial_y E_y + \partial_z E_z, \end{aligned}$$

and it follows that $\nabla\cdot\mathbf{E} = \mu_0 c^2\rho$. Recalling that $c^2 = 1/\mu_0\epsilon_0$ we recover the first Maxwell equation. Now set $\mu = i$. Then

$$\begin{aligned} \partial_\nu F^{i\nu} = \partial_0 F^{i0} + \partial_j F^{ij} &= \frac{1}{c}\partial_t\big(-E_i\big) + \partial_j\epsilon_{ijk}cB_k, \\ &= c\Big[\big(\nabla\times\mathbf{B}\big)_i - \frac{1}{c^2}\partial_t E_i\Big], \end{aligned}$$

and we recover Maxwell's fourth equation $\nabla\times\mathbf{B} - \mu_0\epsilon_0\partial_t\mathbf{E} = \mu_0\mathbf{J}$.

PROBLEM 13

The motion of the particle is given by the solution of the Lorentz force equation

$$mc\eta_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} = qF_{\mu\nu}\frac{dx^\nu}{d\tau},$$

where the field strength tensor is that corresponding to a uniform electric field, say along the x-direction, and no magnetic field. What this means is that $F_{10} = -F_{01} = E$, while all other components are zero. The Lorentz force equation then gives the motion of the charged particle as the solution of the simultaneous equations

$$\frac{d^2x^0}{d\tau^2} = \frac{qE}{mc}\frac{dx^1}{d\tau}, \qquad\qquad \frac{d^2x^1}{d\tau^2} = \frac{qE}{mc}\frac{dx^0}{d\tau}.$$

Viewed as a pair of first order equations for the velocities $dx^0/d\tau$ and $dx^1/d\tau$, the solution is

$$\frac{dx^0}{d\tau} = \cosh\left(\frac{qE}{mc}\tau\right), \qquad\qquad \frac{dx^1}{d\tau} = \sinh\left(\frac{qE}{mc}\tau\right),$$

since the particle starts from rest, so $dx^1/d\tau|_{\tau=0} = 0$, and the trajectory is parameterised by c times proper time, so that $\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$. Finally, the trajectory of the particle is

$$x^0(\tau) = \frac{mc}{qE}\sinh\left(\frac{qE}{mc}\tau\right), \qquad x^1(\tau) = \frac{mc}{qE}\cosh\left(\frac{qE}{mc}\tau\right) + \left(x^1(0) - \frac{mc}{qE}\right).$$

It is already clear from the structure of the solution that the trajectory is time-like; the tangent vector to the particle trajectory is a time-like vector, in fact with magnitude squared -1, since it is parameterised by c times proper time.

PROBLEM 14

The stress-energy-momentum tensor is obtained from the fundamental translational symmetry of Minkowski. It can be derived from an action for the physical field. The action for the Klein-Gordon field is

$$S[\phi] = \int \frac{1}{2}\left[\hbar^2c^2\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2c^4\phi^2\right]d^4x.$$

Let $\phi + \psi$ be a field configuration close to ϕ, which is a critical point of the Klein-Gordon action. Then

$$\begin{aligned} S[\phi+\psi] &= S[\phi] + \int\left[\hbar^2c^2\eta^{\mu\nu}\partial_\mu\psi\,\partial_\nu\phi + m^2c^4\psi\phi\right]d^4x + O(2), \\ &= S[\phi] + \int\left[\partial_\mu\left(\psi\,\hbar^2c^2\eta^{\mu\nu}\partial_\nu\phi\right) - \psi\left(\hbar^2c^2\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2c^4\phi\right)\right]d^4x + O(2). \end{aligned}$$

Now let $\phi'(x) = \phi(x+\epsilon)$ be a field configuration obtained from ϕ by translation by a constant amount ϵ. The difference between the action for the two field configurations can only come from the flux through the boundary and so must be of the form

$$S[\phi'] - S[\phi] = \int \partial_\alpha\left[\epsilon^\mu\delta^\alpha_\mu\frac{1}{2}\left(\hbar^2c^2\eta^{\gamma\beta}\partial_\gamma\phi\,\partial_\beta\phi + m^2c^4\phi^2\right)\right]d^4x + O(2).$$

Since we also have $\phi'(x) = \phi(x+\epsilon) = \phi(x) + \epsilon^\mu\partial_\mu\phi + O(2)$ we may identify $\psi = \epsilon^\mu\partial_\mu\phi$ and it follows that, when ϕ is a critical point of the Klein-Gordon action so that the field equations are satisfied,

$$0 = \int \partial_\alpha\left(\epsilon^\mu\left[\hbar^2c^2\eta^{\alpha\beta}\partial_\mu\phi\,\partial_\beta\phi - \frac{1}{2}\delta^\alpha_\mu\left(\hbar^2c^2\eta^{\gamma\beta}\partial_\gamma\phi\,\partial_\beta\phi + m^2c^4\phi^2\right)\right]\right)d^4x.$$

Writing the term in square brackets as $-T^\alpha_\mu$ we can identify the stress-energy-momentum tensor for the Klein-Gordon field as

$$T_{\mu\nu} = -\hbar^2c^2\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{2}\eta_{\mu\nu}\left[\hbar^2c^2\eta^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + m^2c^4\phi^2\right].$$

Chapter 2 – Differential Geometry

PROBLEM 1

An explicit parameterisation of the torus is given by

$$x^1 = \Big(R - \rho\cos(u^2)\Big)\cos(u^1), \qquad x^2 = \Big(R - \rho\cos(u^2)\Big)\sin(u^1), \qquad x^3 = \rho\sin(u^2),$$

where $0 \le u^1, u^2 < 2\pi$.

To compute the metric in these local coordinates, write the displacement between nearby points on the surface as

$$\begin{aligned} d\mathbf{X} = \Big(R - \rho\cos(u^2)\Big)&\big[-\sin(u^1)\,\mathbf{e}_1 + \cos(u^1)\,\mathbf{e}_2\big]du^1 \\ &+ \rho\sin(u^2)\big[\cos(u^1)\,\mathbf{e}_1 + \sin(u^1)\,\mathbf{e}_2\big]du^2 + \rho\cos(u^2)\,du^2\,\mathbf{e}_3. \end{aligned}$$

One then finds easily that

$$ds^2 = d\mathbf{X}\cdot d\mathbf{X} = \Big(R - \rho\cos(u^2)\Big)^2\big(du^1\big)^2 + \rho^2\big(du^2\big)^2.$$

In the local coordinates (u^1, u^2) the area element is given by

$$d\text{area} = \sqrt{\det g}\,du^1du^2 = \big(R - \rho\cos(u^2)\big)\rho\,du^1du^2,$$

and the total surface area is

$$\text{area} = \int_{u^1=0}^{2\pi}\int_{u^2=0}^{2\pi}\big(R - \rho\cos(u^2)\big)\rho\,du^1du^2 = 2\pi R\,.\,2\pi\rho.$$

PROBLEM 2

Note that the displacement between nearby points of the surface is

$$d\mathbf{X} = \frac{1}{\sqrt{2}}\big[-\sin(u^1)\,\mathbf{e}_1 + \cos(u^1)\,\mathbf{e}_2\big]du^1 + \frac{1}{\sqrt{2}}\big[-\sin(u^2)\,\mathbf{e}_3 + \cos(u^2)\,\mathbf{e}_4\big]du^2.$$

It follows that the metric in these coordinates is given by

$$ds^2 = d\mathbf{X}\cdot d\mathbf{X} = \frac{1}{2}\big(du^1\big)^2 + \frac{1}{2}\big(du^2\big)^2.$$

This is the *flat metric* on the torus.

The total surface area is

$$\begin{aligned} \text{area} = \int_{T^2} d\text{area} &= \int_{u^1=0}^{2\pi}\int_{u^2=0}^{2\pi}\sqrt{\det g}\,du^1du^2, \\ &= \int_{u^1=0}^{2\pi}\int_{u^2=0}^{2\pi}\frac{1}{2}\,du^1du^2 = 2\pi^2. \end{aligned}$$

PROBLEM 6

We are asked to consider the subset

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -1, \qquad x^0 > 0,$$

of Minkowski $\mathbb{R}^{1,3}$. A parameterisation of it in terms of local coordinates (χ, θ, ϕ) can be given by

$$\begin{aligned}
x^0 &= \cosh(\chi), \\
x^1 &= \sinh(\chi)\cos(\theta), \\
x^2 &= \sinh(\chi)\sin(\theta)\cos(\phi), \\
x^3 &= \sinh(\chi)\sin(\theta)\sin(\phi).
\end{aligned}$$

To determine the metric in these coordinates note that the displacement between nearby points can be written as

$$\begin{aligned}
d\mathbf{X} = &\Big[\sinh(\chi)\,\mathbf{e}_0 + \cosh(\chi)\Big(\cos(\theta)\,\mathbf{e}_1 + \sin(\theta)\big[\cos(\phi)\,\mathbf{e}_2 + \sin(\phi)\,\mathbf{e}_3\big]\Big)\Big]d\chi \\
&+ \sinh(\chi)\Big(-\sin(\theta)\,\mathbf{e}_1 + \cos(\theta)\big[\cos(\phi)\,\mathbf{e}_2 + \sin(\phi)\,\mathbf{e}_3\big]\Big)d\theta \\
&+ \sinh(\chi)\sin(\theta)\big[-\sin(\phi)\,\mathbf{e_2} + \cos(\phi)\,\mathbf{e}_3\big]d\phi,
\end{aligned}$$

One then finds that the metric is given by (the 'dot' product is with respect to the Minkowski metric of the embedding space)

$$ds^2 = d\mathbf{X}\cdot d\mathbf{X} = d\chi^2 + \sinh^2(\chi)\big[d\theta^2 + \sin^2(\theta)\,d\phi^2\big].$$

This is the standard *round metric* on the hyperbolic three-space.

The volume element in these coordinates is given by the usual formula

$$d\text{vol} = \sqrt{\det g}\,d\chi d\theta d\phi = \sinh^2(\chi)\sin(\theta)\,d\chi d\theta d\phi.$$

PROBLEM 8

We are given a cylinder of radius a with explicit embedding in $\mathbb{R}^3$ $(a\cos(u1), a\sin(u^1), u^2)$. The displacement between nearby points on the surface is

$$d\mathbf{X} = a\big[-\sin(u^1)\,\mathbf{e}_1 + \cos(u^1)\,\mathbf{e}_2\big]du^1 + du^2\,\mathbf{e}_3,$$

from which it follows that the metric is

$$ds^2 = a^2(du^1)^2 + (du^2)^2.$$

The metric is flat; it is the same as the standard Cartesian metric on $\mathbb{R}^2$ with Cartesian coordinates (au^1, u^2). This is enough to determine the geodesics, but we may also note that since the components of the metric are all constants the Christoffel symbols are all zero. The geodesic equations are therefore

$$\frac{d^2u^i}{d\tau^2} = 0 \qquad i = 1, 2,$$

and so are linear functions, $u^1 = c_1\tau + d_1$, $u^2 = c_2\tau + d_2$, with c_1, d_1 and c_2, d_2 all constants.

PROBLEM 10

Let us define the de Sitter space dS_4 as the subset

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2,$$

of $\mathbb{R}^{1,4}$ and introduce local coordinates (ct, χ, θ, ϕ) according to

$$\begin{gathered}
x^0 = a\sinh(ct/a), \quad x^1 = a\cosh(ct/a)\cos(\chi), \quad x^2 = a\cosh(ct/a)\sin(\chi)\cos(\theta), \\
x^3 = a\cosh(ct/a)\sin(\chi)\sin(\theta)\cos(\phi), \quad x^4 = a\cosh(ct/a)\sin(\chi)\sin(\theta)\sin(\phi),
\end{gathered}$$

in terms of which the metric is given by

$$ds^2 = -c^2 dt^2 + a^2 \cosh^2(ct/a)\Big[d\chi^2 + \sin^2(\chi)\big(d\theta^2 + \sin^2(\theta)\, d\phi^2\big)\Big].$$

Now, let γ be a geodesic parameterised by c times proper time (τ) and γ' any nearby curve, also parameterised by τ. By direct calculation the metric on γ' is found to be

$$\begin{aligned} ds^2 = &-d\tau^2 - 2c\, dt\, d\epsilon^0 + 2a\epsilon^0 \sinh(ct/a)\cosh(ct/a)\Big[d\chi^2 + \sin^2(\chi)\big(d\theta^2 + \sin^2(\theta)\, d\phi^2\big)\Big] \\ &+ a^2\cosh^2(ct/a)\Big\{2d\epsilon^\chi d\chi + 2\epsilon^\chi\ \sin(\chi)\cos(\chi)\big(d\theta^2 + \sin^2(\theta)\, d\phi^2\big) \\ &+ \sin^2(\chi)\Big[2d\epsilon^\theta d\theta + 2\epsilon^\theta\ \sin(\theta)\cos(\theta)\, d\phi^2 + 2\sin^2(\theta)\, d\epsilon^\phi d\phi\Big]\Big\} + O(2). \end{aligned}$$

It follows that the length of γ' is

$$\begin{aligned} \int_{\gamma'} = &\int_{\tau_i}^{\tau_f}\Big\{1 + \frac{d\epsilon^0}{d\tau}\frac{d(ct)}{d\tau} - \epsilon^0\, a\sinh(ct/a)\cosh(ct/a)\Big[\frac{d\chi}{d\tau}\frac{d\chi}{d\tau} + \sin^2(\chi)\Big(\frac{d\theta}{d\tau}\frac{d\theta}{d\tau} + \sin^2(\theta)\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\Big)\Big] \\ &- a^2\cosh^2(ct/a)\Big[\frac{d\epsilon^\chi}{d\tau}\frac{d\chi}{d\tau} + \epsilon^\chi\ \sin(\chi)\cos(\chi)\Big(\frac{d\theta}{d\tau}\frac{d\theta}{d\tau} + \sin^2(\theta)\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\Big)\Big] \\ &- a^2\cosh^2(ct/a)\sin^2(\chi)\Big[\frac{d\epsilon^\theta}{d\tau}\frac{d\theta}{d\tau} + \epsilon^\theta\ \sin(\theta)\cos(\theta)\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\Big] \\ &- a^2\cosh^2(ct/a)\sin^2(\chi)\sin^2(\theta)\frac{d\epsilon^\phi}{d\tau}\frac{d\phi}{d\tau}\Big\}d\tau + O(2), \\ = &\int_\gamma ds - \int_{\tau_i}^{\tau_f}\Big\{\epsilon^0\Big[\frac{d^2(ct)}{d\tau^2} + a\sinh(ct/a)\cosh(ct/a)\Big[\frac{d\chi}{d\tau}\frac{d\chi}{d\tau} + \sin^2(\chi)\Big(\frac{d\theta}{d\tau}\frac{d\theta}{d\tau} + \sin^2(\theta)\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\Big)\Big]\Big] \\ &- \epsilon^\chi\Big[\frac{d}{d\tau}\Big(a^2\cosh^2(ct/a)\frac{d\chi}{d\tau}\Big) - a^2\cosh^2(ct/a)\sin(\chi)\cos(\chi)\Big(\frac{d\theta}{d\tau}\frac{d\theta}{d\tau} + \sin^2(\theta)\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\Big)\Big] \\ &- \epsilon^\theta\Big[\frac{d}{d\tau}\Big(a^2\cosh^2(ct/a)\sin^2(\chi)\frac{d\theta}{d\tau}\Big) - a^2\cosh^2(ct/a)\sin^2(\chi)\sin(\theta)\cos(\theta)\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\Big] \\ &- \epsilon^\phi\frac{d}{d\tau}\Big(a^2\cosh^2(ct/a)\sin^2(\chi)\sin^2(\theta)\frac{d\phi}{d\tau}\Big)\Big\}d\tau + O(2). \end{aligned}$$

From this one can read off the geodesic equations, which we summarise by giving the complete list of non-zero Christoffel symbols

$$\begin{aligned} &\Gamma^0_{\chi\chi} = a\sinh(ct/a)\cosh(ct/a), \quad \Gamma^0_{\theta\theta} = a\sinh(ct/a)\cosh(ct/a)\sin^2(\theta), \\ &\Gamma^0_{\phi\phi} = a\sinh(ct/a)\cosh(ct/a)\sin^2(\theta)\sin^2(\phi), \\ &\Gamma^\chi_{0\chi} = \Gamma^\chi_{\chi 0} = \frac{1}{a}\frac{\sinh(ct/a)}{\cosh(ct/a)}, \quad \Gamma^\chi_{\theta\theta} = -\sin(\chi)\cos(\chi), \quad \Gamma^\chi_{\phi\phi} = -\sin(\chi)\cos(\chi)\sin^2(\theta), \\ &\Gamma^\theta_{0\theta} = \Gamma^\theta_{\theta 0} = \frac{1}{a}\frac{\sinh(ct/a)}{\cosh(ct/a)}, \quad \Gamma^\theta_{\chi\theta} = \Gamma^\theta_{\theta\chi} = \frac{\cos(\chi)}{\sin(\chi)}, \quad \Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta), \\ &\Gamma^\phi_{0\phi} = \Gamma^\phi_{\phi 0} = \frac{1}{a}\frac{\sinh(ct/a)}{\cosh(ct/a)}, \quad \Gamma^\phi_{\chi\phi} = \Gamma^\phi_{\phi\chi} = \frac{\cos(\chi)}{\sin(\chi)}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)}. \end{aligned}$$

The hint recommends that we do not try to solve the geodesic equation but instead use what we know about geodesics on spheres in the Euclidean setting. What we know there is that the geodesics on $S^n \subset \mathbb{R}^{n+1}$ are given by the intersection of the sphere with a 2-plane through the origin of $\mathbb{R}^{n+1}$. In fact more is true; it suffices to consider the particular 2-plane given by $(x^1, x^2, 0, \ldots, 0)$, or $x^k = 0$ for $k = 3, \ldots, n$. This intersection is the curve

$$\mathbf{c}(\tau) = a\cos(\tau/a)\,\mathbf{e}_1 + a\sin(\tau/a)\,\mathbf{e}_2.$$

The reason this is sufficient is because all other geodesics can be obtained from this one by the action of the symmetry group of the sphere: the rotation group $SO(n+1)$ acts on $\mathbb{R}^{n+1}$ preserving the sphere and also acts transitively on the 2-planes passing through the origin (there is some rotation taking the particular 2-plane we have considered so far to any other). [We do not prove this statement, but you may like to do so.]

The same is true in the Lorentzian setting; the geodesics are given by the intersection of the de Sitter space with 2-planes through the origin of $\mathbb{R}^{1,4}$. The only difference is the 2-plane may be space-like, time-like or null. A typical space-like 2-plane is $x^0 = x^3 = x^4 = 0$ and a typical time-like 2-plane is $x^2 = x^3 = x^4 = 0$. Their intersections with dS_4 are a space-like and a time-like geodesic, respectively. All others can be obtained from these two by the action of the Lorentz group $SO(1,4)$, which is the symmetry group of dS_4 and acts transitively on the 2-planes through the origin of $\mathbb{R}^{1,4}$, preserving their type (space-like, time-like or null). [We do not prove this statement, but you may like to do so.] Now a typical null 2-plane can be given as $x^0 = x^1$, $x^3 = x^4 = 0$. Its intersection with the de Sitter space can be expressed as the light ray $(\tau, \tau, a, 0, 0)$. All other light rays can be obtained from this one by the action of the subgroup of $SO(1,4)$ isomorphic to $SO(4)$ that corresponds to rotations of the spatial coordinates x^1, x^2, x^3, x^4 of $\mathbb{R}^{1,4}$. [The action is transitive but of course not free; the isotropy subgroup is isomorphic to $SO(2)$ and corresponds to rotations acting only on the coordinates x^3, x^4.]

PROBLEM 11

Anti-de Sitter space AdS_4 is the subset of $\mathbb{R}^{2,3}$ given by

$$-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = -a^2.$$

A parameterisation that covers the entire space is

$$x^0 = a\cosh(\chi)\cos(\tau/a), \qquad x^1 = a\cosh(\chi)\sin(\tau/a),$$
$$\left(x^2, x^3, x^4\right) = a\sinh(\chi)\Big(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\Big).$$

The displacement between nearby points of anti-de Sitter can then be written as

$$\begin{aligned} d\mathbf{X} = {} & a\sinh(\chi)\big[\cos(\tau/a)\,\mathbf{e}_0 + \sin(\tau/a)\,\mathbf{e}_1\big]d\chi + a\cosh(\chi)\Big(\sin(\theta)\big[\cos(\phi)\,\mathbf{e}_2 + \sin(\phi)\,\mathbf{e}_3\big] + \cos(\theta)\,\mathbf{e}_4\Big)d\chi \\ & + \cosh(\chi)\big[-\sin(\tau/a)\,\mathbf{e}_0 + \cos(\tau/a)\,\mathbf{e}_1\big]d\tau + a\sinh(\chi)\Big(\cos(\theta)\big[\cos(\phi)\,\mathbf{e}_2 + \sin(\phi)\,\mathbf{e}_3\big] - \sin(\theta)\,\mathbf{e}_4\Big)d\theta \\ & + a\sinh(\chi)\sin(\theta)\big[-\sin(\phi)\,\mathbf{e}_2 + \cos(\phi)\,\mathbf{e}_3\big]d\phi, \end{aligned}$$

and it follows that the metric on anti-de Sitter in these coordinates is

$$\begin{aligned} ds^2 &= -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2, \\ &= -\cosh^2(\chi)d\tau^2 + a^2\Big[d\chi^2 + \sinh^2(\chi)\big(d\theta^2 + \sin^2(\theta)\,d\phi^2\big)\Big]. \end{aligned}$$

To show that the curve $x^0 = a\cos(\tau/a)$, $x^1 = a\sin(\tau/a)$, $x^2 = x^3 = x^4 = 0$ is a geodesic we may either show that it satisfies the geodesic equation or, equivalently, that its tangent vector is parallel transported along itself. In this case the latter is easier. The tangent to the curve is

$$\mathbf{T} = -\sin(\tau/a)\,\mathbf{e}_0 + \cos(\tau/a)\,\mathbf{e}_1,$$

and its derivative along itself is (τ is c times proper time along the curve, since along the curve $\chi = 0$)

$$\nabla_{\mathbf{T}}\mathbf{T} = \frac{d}{d\tau}\mathbf{T} = \frac{-1}{a}\big[\cos(\tau/a)\,\mathbf{e}_0 + \sin(\tau/a)\,\mathbf{e}_1\big].$$

Now the expression given above for the displacement $d\mathbf{X}$ between nearby points in anti-de Sitter allows one to identify a basis for the tangent space at each point and one can see directly that the vector giving the direction that **T** is changing into is orthogonal to all of the tangent vectors. [Don't forget that $\chi = 0$ along the curve.]

PROBLEM 14

We are asked to consider the three metrics

$$ds^2 = \begin{cases} dr^2 + a^2 \sin^2(r/a)\, d\phi^2, \\ dr^2 + r^2\, d\phi^2, \\ dr^2 + a^2 \sinh^2(r/a)\, d\phi^2, \end{cases}$$

corresponding to standard 'round' expressions for the metric on the three isotropic homogeneous two-dimensional manifolds of constant scalar curvature; the 2-sphere, the plane, and the hyperbolic plane. I give the calculation for the 2-sphere only; the others are identical in structure.

First we determine the Christoffel symbols from the geodesics. Let $(r(\tau), \phi(\tau))$ be a geodesic parameterised by arc length and let $(r + \epsilon^r, \phi + \epsilon^\phi)$ be a variation. The metric on this latter curve is

$$\begin{aligned} ds^2 &= d(r+\epsilon^r)\, d(r+\epsilon^r) + a^2 \sin^2\bigl((r+\epsilon)/a\bigr)\, d(\phi+\epsilon^\phi)\, d(\phi+\epsilon^\phi), \\ &= d\tau^2 + 2\frac{d\epsilon^r}{d\tau}\frac{dr}{d\tau}\, d\tau^2 + 2a\sin(r/a)\cos(r/a)\,\epsilon^r\,\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\, d\tau^2 + 2a^2\sin^2(r/a)\,\frac{d\epsilon^\phi}{d\tau}\frac{d\phi}{d\tau}\, d\tau^2 + O(2), \end{aligned}$$

from which it follows that

$$\begin{aligned} \int_{\gamma'} ds &= \int_{\tau_i}^{\tau_f} \Bigl[1 + \frac{d\epsilon^r}{d\tau}\frac{dr}{d\tau} + a\sin(r/a)\cos(r/a)\,\epsilon^r\,\frac{d\phi}{d\tau}\frac{d\phi}{d\tau} + a^2\sin^2(r/a)\,\frac{d\epsilon^\phi}{d\tau}\frac{d\phi}{d\tau}\Bigr] d\tau + O(2), \\ &= \int_\gamma ds - \int_{\tau_i}^{\tau_f} \Bigl\{\epsilon^r\Bigl[\frac{d^2r}{d\tau^2} - a\sin(r/a)\cos(r/a)\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\Bigr] + \epsilon^\phi\frac{d}{d\tau}\Bigl(a^2\sin^2(r/a)\frac{d\phi}{d\tau}\Bigr)\Bigr\} d\tau + O(2). \end{aligned}$$

The vanishing of the first order term yields the geodesic equations

$$\begin{aligned} \frac{d^2r}{d\tau^2} - a\sin(r/a)\cos(r/a)\frac{d\phi}{d\tau}\frac{d\phi}{d\tau} &= 0, \\ \frac{d^2\phi}{d\tau^2} + \frac{2}{a}\frac{\cos(r/a)}{\sin(r/a)}\frac{dr}{d\tau}\frac{d\phi}{d\tau} &= 0, \end{aligned}$$

from which we can read off the non-zero Christoffel symbols

$$\Gamma^r_{\phi\phi} = -a\sin(r/a)\cos(r/a), \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{a}\frac{\cos(r/a)}{\sin(r/a)}.$$

For the components of the Ricci tensor we have by direct calculation

$$\begin{aligned} R_{rr} &= \partial_i\Gamma^i_{rr} - \partial_r\Gamma^i_{ir} + \Gamma^i_{rr}\Gamma^j_{ji} - \Gamma^i_{jr}\Gamma^j_{ir}, \\ &= -\partial_r\Gamma^\phi_{\phi r} - \Gamma^\phi_{\phi r}\Gamma^\phi_{\phi r}, \\ &= \frac{1}{a^2}, \\ R_{r\phi} = R_{\phi r} &= \partial_i\Gamma^i_{r\phi} - \partial_r\Gamma^i_{i\phi} + \Gamma^i_{r\phi}\Gamma^j_{ji} - \Gamma^i_{jr}\Gamma^j_{i\phi}, \\ &= 0, \\ R_{\phi\phi} &= \partial_i\Gamma^i_{\phi\phi} - \partial_\phi\Gamma^i_{i\phi} + \Gamma^i_{\phi\phi}\Gamma^j_{ji} - \Gamma^i_{j\phi}\Gamma^j_{i\phi}, \\ &= \partial_r\Gamma^r_{\phi\phi} + \Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r} - \Gamma^r_{\phi\phi}\Gamma^\phi_{r\phi} - \Gamma^\phi_{r\phi}\Gamma^r_{\phi\phi}, \\ &= \sin^2(r/a). \end{aligned}$$

Finally, the Ricci scalar is

$$R = g^{ij} R_{ij} = \frac{1}{a^2} + \frac{1}{a^2 \sin^2(r/a)} \sin^2(r/a) = \frac{2}{a^2}.$$

PROBLEM 16

The idea is to use the fact that the covariant derivative of a function is the ordinary derivative of a function, $\nabla f = df$, or in local coordinates

$$\nabla_\mu f = \partial_\mu f.$$

Now choose for f the action of a 1-form A on the vector Z, which in terms of components is given by $A(Z) = A_\nu Z^\nu$. Then by the Leibniz formula we have

$$\begin{aligned} \nabla_\mu \big(A_\nu Z^\nu\big) &= \big(\nabla_\mu A_\nu\big) Z^\nu + A_\nu \nabla_\mu Z^\nu, \\ &= \big(\nabla_\mu A_\nu\big) Z^\nu + A_\nu \big[\partial_\mu Z^\nu + \Gamma^\nu_{\mu\alpha} Z^\alpha\big]. \end{aligned}$$

But also we have

$$\nabla_\mu \big(A_\nu Z^\nu\big) = \partial_\mu \big(A_\nu Z^\nu\big) = \big(\partial_\mu A_\nu\big) Z^\nu + A_\nu \partial_\mu Z^\nu,$$

from which it follows that

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha.$$

Applying this result to the Maxwell field strength tensor we find

$$\begin{aligned} F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu &= \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha - \partial_\nu A_\mu + \Gamma^\alpha_{\nu\mu} A_\alpha, \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned}$$

since the Christoffel symbols are symmetric $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$.

One gets at the next part in exactly the same way by considering the covariant derivative of the function $F_{\mu\nu} Y^\mu Z^\nu$ where Y, Z are vectors. An identical calculation then shows that

$$\nabla_\alpha F_{\mu\nu} = \partial_\alpha F_{\mu\nu} - \Gamma^\beta_{\alpha\mu} F_{\beta\nu} - \Gamma^\beta_{\alpha\nu} F_{\mu\beta}.$$

Consider the function $f = T^\alpha_\mu A_\alpha Z^\mu$ for a 1-form A and vector Z. Then we have

$$\begin{aligned} \nabla_\nu \big(T^\alpha_\mu A_\alpha Z^\mu\big) &= \big(\nabla_\nu T^\alpha_\mu\big) A_\alpha Z^\mu + T^\alpha_\mu \big[\big(\nabla_\nu A_\alpha\big) Z^\mu + A_\alpha \nabla_\nu Z^\mu\big], \\ \partial_\nu \big(T^\alpha_\mu A_\alpha Z^\mu\big) &= \big(\partial_\nu T^\alpha_\mu\big) A_\alpha Z^\mu + T^\alpha_\mu \big[\big(\partial_\nu A_\alpha\big) Z^\mu + A_\alpha \partial_\nu Z^\mu\big]. \end{aligned}$$

Equating these, and setting $\nu = \alpha$, we arrive at the expression

$$\nabla_\alpha T^\alpha_\mu = \partial_\alpha T^\alpha_\mu + \Gamma^\alpha_{\alpha\beta} T^\beta_\mu - \Gamma^\beta_{\alpha\mu} T^\alpha_\beta.$$

Chapter 4 – The Schwarzschild Solution

PROBLEM 3

The effective potential (up to unimportant scale factor) is

$$V_{\text{eff}}(r) = -\frac{2m}{r} + \frac{\ell^2}{r^2} - \frac{2m\ell^2}{r^3}.$$

The critical points are given by

$$\begin{aligned} 0 = \frac{dV_{\text{eff}}}{dr} &= \frac{2m}{r^2} - \frac{2\ell^2}{r^3} + \frac{6m\ell^2}{r^4}, \\ &= \frac{2m}{r^4}\Big[r^2 - \frac{\ell^2}{m}r + 3\ell^2\Big], \end{aligned}$$

whose solutions are evidently

$$r_\pm = \frac{\ell^2}{2m}\left(1 \pm \left[1 - \frac{12m^2}{\ell^2}\right]^{1/2}\right).$$

These solutions only exist provided $12m^2/\ell^2 < 1$. (Equality is still a critical point, but it has degenerate Hessian.)

Write the effective potential as follows

$$V_{\text{eff}}(r) = -\frac{2m}{r^3}\Big[r^2 - \frac{\ell^2}{2m}r + \ell^2\Big] = -\frac{2m}{r^3}\left[\left(r - \frac{\ell^2}{4m}\right)^2 + \ell^2\left(1 - \frac{\ell^2}{16m^2}\right)\right].$$

This shows that the potential is non-positive if $\ell^2/16m^2 \leq 1$ but positive for some range of values of r if $\ell^2/16m^2 > 1$. When $\ell^2 = 16m^2$ it takes its maximum value of 0 at $r = \ell^2/4m$ (ignore the point $r \to \infty$). One readily verifies that when $\ell^2 = 16m^2$ the previous calculation yields

$$r_\pm = \frac{\ell^2}{2m}\left(1 \pm \frac{1}{2}\right) = \frac{(2 \pm 1)\ell^2}{4m},$$

so that there is complete consistency. [The easiest way to see the answer to this question is to sketch the form of the effective potential. Certainly this is how I obtained the solution outlined above.]

PROBLEM 4

We are asked to consider the change of variables $r \mapsto \rho$ defined by

$$r = \left(1 + \frac{m}{2\rho}\right)^2 \rho,$$

in the Schwarzschild metric. One finds by direct calculation that

$$\begin{aligned} \frac{1}{1 - \frac{2m}{r}} = \frac{r}{r - 2m} &= \frac{(1 + m/2\rho)^2}{(1 + m/2\rho)^2 - 2m/\rho} = \frac{(1 + m/2\rho)^2}{(1 - m/2\rho)^2}, \\ dr &= \left(1 + \frac{m}{2\rho}\right)^2 d\rho - \frac{m}{\rho}\left(1 + \frac{m}{2\rho}\right)d\rho = \left(1 + \frac{m}{2\rho}\right)\left(1 - \frac{m}{2\rho}\right)d\rho, \end{aligned}$$

and it then follows swiftly that

$$\begin{aligned} \frac{dr^2}{1 - \frac{2m}{r}} + r^2\Big[d\theta^2 + \sin^2(\theta)\,d\phi^2\Big] &= \left(1 + \frac{m}{2\rho}\right)^4 d\rho^2 + \left(1 + \frac{m}{2\rho}\right)^4 \rho^2\Big[d\theta^2 + \sin^2(\theta)\,d\phi^2\Big], \\ &= \left(1 + \frac{m}{2\rho}\right)^4 \big[dx^2 + dy^2 + dz^2\big]. \end{aligned}$$

The form of the Minkowski metric given then follows at once.

To see the coordinate range, note that

$$r = \frac{m}{2}\left(\left[\frac{2\rho}{m}\right]^{1/2} + \left[\frac{m}{2\rho}\right]^{1/2}\right)^2.$$

This makes it clear that there is a symmetry under the inversion

$$\frac{2\rho}{m} \mapsto \frac{m}{2\rho}.$$

The fixed point of this inversion, $2\rho/m = 1$ corresponds to the minimum value of r covered by the new coordinate chart. One finds by direct substitution that this minimum value is $r = 2m$.

PROBLEM 5

In the 'Cartesian' coordinates of the previous problem the Schwarzschild metric is

$$ds^2 = -\left(\frac{1-\frac{m}{2\rho}}{1+\frac{m}{2\rho}}\right) c^2 dt^2 + \left(1+\frac{m}{2\rho}\right)^4 d\ell^2,$$

where $d\ell^2 = dx^2 + dy^2 + dz^2$ is the usual Euclidean metric for Cartesian coordinates on $\mathbb{R}^3$ and $\rho = \sqrt{x^2+y^2+z^2}$. Along a null geodesic $ds^2 = 0$ and hence

$$cdt = \left(1-\frac{m}{2\rho}\right)^{-1}\left(1+\frac{m}{2\rho}\right)^3 d\ell \approx \left(1+\frac{4m}{\rho}\right) d\ell.$$

We describe the trajectory just using the Cartesian coordinates x, y, z and standard Euclidean geometry. If the distance of closest approach to the Sun is b then the distance travelled from that point to a point radial distance ρ from the origin is $\ell = \sqrt{\rho^2 - b^2}$. It follows that the total time taken along the photon trajectory is

$$\begin{aligned}
ct &= 2\left\{\int_0^{\sqrt{R_{\rm E}^2-b^2}} d\ell + \int_b^{R_{\rm E}} \frac{2m}{\sqrt{\rho^2-b^2}}\, d\rho + \int_0^{\sqrt{R_{\rm P}^2-b^2}} d\ell + \int_b^{R_{\rm P}} \frac{2m}{\sqrt{\rho^2-b^2}}\, d\rho\right\},\\
&= 2\left\{\sqrt{R_{\rm E}^2-b^2} + \sqrt{R_{\rm P}^2-b^2} + 2m\left[\operatorname{arccosh}\frac{R_{\rm E}}{b} + \operatorname{arccosh}\frac{R_{\rm P}}{b}\right]\right\},\\
&= 2\left\{\sqrt{R_{\rm E}^2-b^2} + \sqrt{R_{\rm P}^2-b^2} + 2m\left[\ln\left(\frac{R_{\rm E}}{b} + \sqrt{\frac{R_{\rm E}^2}{b^2}-1}\right) + \ln\left(\frac{R_{\rm P}}{b} + \sqrt{\frac{R_{\rm P}^2}{b^2}-1}\right)\right]\right\},\\
&\approx 2\left\{\sqrt{R_{\rm E}^2-b^2} + \sqrt{R_{\rm P}^2-b^2} + 2m\ln\frac{4R_{\rm E}R_{\rm P}}{b^2}\right\}.
\end{aligned}$$

Calculation in Schwarzschild coordinates to be added.

PROBLEM 6

From the figure one has immediately that the deflection angle is

$$\begin{aligned}
\Delta &= \arctan\frac{b}{d} + \arctan\frac{b-x}{D},\\
&\approx \frac{b}{d} + \frac{b-x}{D}.
\end{aligned}$$

The calculation using the Schwarzschild metric gives the deflection angle as $\Delta = 4m/b$. It then follows that

$$4m = \frac{b^2}{d} + \frac{b^2}{D} - \frac{xb}{D} = \theta^2\left(d + \frac{d^2}{D}\right) - \frac{xd}{D}\,\theta,$$

or

$$\theta^2 - \frac{x}{d+D}\,\theta - \frac{4mD}{d(d+D)} = 0.$$

The solutions are evidently

$$\theta_\pm = \frac{x}{2(d+D)} \pm \left[\frac{4mD}{d(d+D)} + \frac{x^2}{4(d+D)^2}\right]^{1/2}.$$

There are two values because there are two images, one either side of the lens. This matches with Fermat's principle of least time; one of the paths is a minimum of the time taken, the other a local maximum (given the constraint that the deflection angle is given by general relativity). As x increases one image tends to the direct line-of-sight location while the other disappears behind the lens. As $x \to 0$ all directions going around the lens become equivalent and the image becomes an Einstein ring of angular radius $\sqrt{4mD/d(d+D)}$.

PROBLEM 7

The velocity of the observer, in the Schwarzschild coordinate basis, is

$$\frac{dx^\mu}{d\tau} = \big(\gamma, 0, 0, 0\big),$$

and hence its magnitude squared is

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\gamma^2\left(1 - \frac{2m}{r_0}\right).$$

τ is c times proper time if this magnitude squared is equal to -1.

The observer's velocity is $u = (dx^\mu/d\tau)\,\partial_\mu = \gamma\,\partial_0$. Their acceleration is

$$a = \frac{du}{d\tau} = \gamma\frac{d}{d\tau}\,\partial_0 = \gamma\frac{dx^\mu}{d\tau}\,\Gamma^\alpha_{\mu 0}\,\partial_\alpha = \gamma^2\Gamma^\alpha_{00}\,\partial_\alpha.$$

Looking up the Christoffel symbols for the Schwarzschild metric we find that only Γ^r_{00} is non-zero and is given by

$$\Gamma^r_{00} = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right).$$

It then follows that the acceleration of the observer is

$$a = \frac{m}{r_0^2}\partial_r,$$

and since the magnitude squared of the basis vector ∂_r is the metric component g_{rr} we can say that the magnitude of the acceleration is

$$\|a\| = \frac{m}{r_0^2}\left(1 - \frac{2m}{r_0}\right)^{-1/2}.$$

Its direction is everywhere tangent to the coordinate r curves, *i.e.* to the curves $(ct_0, r, \theta_0, \phi_0)$ or, more plainly, 'radial' – indeed radially outwards.

Chapter 5 – Gravitational Collapse and Black Holes

PROBLEM 2

Fix a value of the coordinate t. The Kerr metric becomes

$$ds^2 = \frac{2mra^2\sin^4(\theta)}{r^2 + a^2\cos^2(\theta)}\,d\phi^2 + \left(r^2 + a^2\right)\sin^2(\theta)\,d\phi^2 + \frac{r^2 + a^2\cos^2(\theta)}{r^2 - 2mr + a^2}\,dr^2 + \left(r^2 + a^2\cos^2(\theta)\right)d\theta^2.$$

We wish to take both $r \to 0$ and $\theta \to \pi/2$. We need to do the former first if the limit $r \to 0$ is to be non-singular in these coordinates. So, setting $r = 0$ we have

$$ds^2 = a^2\sin^2(\theta)\,d\phi^2 + a^2\cos^2(\theta)\,d\theta^2.$$

Then setting $\theta = \pi/2$ we find the metric is

$$ds^2 = a^2\,d\phi^2.$$

What this tells us is that the coordinate $(ct_0, 0, \pi/2, \phi)$, which we might be tempted to associate with the 'origin' of a polar system since $r = 0$, describes a circle of radius a. Although we do not show it, one finds by computing the Riemann curvature that the coordinate location $r = 0$ is a curvature singularity. Thus in the Kerr metric the singularity is not a single point, as it is in the Schwarzschild space-time, but rather a ring of circumference $2\pi a$.

PROBLEM 3

For such a star the 'mass function' $m(r)$ is given by

$$m(r) = \frac{4\pi G}{c^2}\int_0^r \rho(r')r'^2\,dr' = \frac{4\pi G}{3c^2}\rho_0 r^3,$$

for $r \leq a$ and takes the value $m(a)$ for $r \geq a$. We will denote this latter simply by m. The Tolman-Oppenheimer-Volkoff equation then becomes

$$-\frac{dp}{dr} = \frac{4\pi G}{c^4}\frac{r}{1 - \frac{8\pi G}{3c^2}\,\rho_0 r^2}\left(p + \rho_0 c^2\right)\left(p + \frac{1}{3}\rho_0 c^2\right).$$

Separating variables this is the same as

$$\frac{dp}{(p + \rho_0 c^2)(p + \rho c^2/3)} = \frac{3}{4\rho_0 c^2}\,\frac{-\frac{16\pi G\rho_0}{3c^2}\,r\,dr}{1 - \frac{8\pi G\rho_0}{3c^2}\,r^2}.$$

Next, use the partial fraction expansion

$$\frac{1}{(p + \rho_0 c^2)(p + \rho c^2/3)} = \frac{3}{2\rho_0 c^2}\left[\frac{1}{p + \frac{1}{3}\rho_0 c^2} - \frac{1}{p + \rho_0 c^2}\right],$$

to arrive at

$$\frac{dp}{p + \frac{1}{3}\rho_0 c^2} - \frac{dp}{p + \rho_0 c^2} = \frac{1}{2}\,d\ln\left(1 - \frac{8\pi G\rho_0}{3c^2}\,r^2\right),$$

Integrating and using that $p(a) = 0$ we obtain

$$\ln\frac{\frac{1}{3}\rho_0 c^2}{p(0) + \frac{1}{3}\rho_0 c^2} - \ln\frac{\rho_0 c^2}{p(0) + \rho_0 c^2} = \frac{1}{2}\ln\left(1 - \frac{8\pi G\rho_0 a^2}{3c^2}\right),$$

or

$$\ln\frac{p(0) + \rho_0 c^2}{3p(0) + \rho_0 c^2} = \ln\left(1 - \frac{2m}{a}\right)^{1/2}.$$

Exponentiating and rearranging we find the desired expression for the pressure at the centre of the star

$$p(0) = \rho_0 c^2 \frac{1 - \bigl(1 - 2m/a\bigr)^{1/2}}{3\bigl(1 - 2m/a\bigr)^{1/2} - 1}.$$

Evidently the pressure diverges when $3(1 - 2m/a)^{1/2} = 1$, which is the same as $m = 4a/9$.

It is instructive to estimate this limiting mass for a star composed of ordinary matter, like the Sun; *i.e.* take for ρ_0 the average density of the Sun. Using that the total mass is $M = \frac{4\pi}{3}\rho_0 a^3$ and that $m = GM/c^2$ we obtain for the mass limit

$$\frac{GM}{c^2} = \frac{4}{9}\left(\frac{3M}{4\pi\rho_0}\right)^{1/3} \qquad \Rightarrow \quad M^{2/3} = \frac{4c^2 R_\odot}{9G} M_\odot^{-1/3},$$

which is the same as

$$\bigl(M/M_\odot\bigr)^{2/3} = \frac{4R_\odot}{9GM_\odot/c^2}.$$

Substituting in appropriate values, $R_\odot = 6.96 \times 10^5\,\mathrm{km}$ and $GM_\odot/c^2 = 1.48\,\mathrm{km}$, gives the limiting mass as $\sim 9.6 \times 10^7 M_\odot$. This enormous value helps to appreciate the importance of Chandrasekhar's estimate for the mass limit. [You might like to repeat the estimate for a star with the density of a neutron star.]

Chapter 6 – Gravitational Waves

PROBLEM 2

For two equal mass stars, mass M, in circular orbits ($e = 0$) with semi-major radius a, the rate of decrease of the semi-major radius due to emission of gravitational waves is given by

$$\frac{da}{dt} = -\frac{128G^3M^3}{5c^5a^3}.$$

It follows by direct integration that

$$\frac{1}{4}\left[a^4(t) - a^4(0)\right] = -\frac{128G^3M^3}{5c^5}\, t$$

Writing $a(0) = a_0$, the time for complete inspiral – $a(t) = 0$ – is given by

$$t = \frac{5c^5a_0^4}{512G^3M^3}.$$

Recall Kepler's third law; $\Omega^2a^3 = G(M_1 + M_2)$, where M_1, M_2 are the two masses and $\Omega = 2\pi/P$ with P the orbital period. It follows that $a_0 = (2GM/\Omega^2)^{1/3}$ and hence

$$t = \frac{5c^5(2GM/\Omega^2)^{4/3}}{64(2GM)^3} = \frac{5c^{5/3}}{64}\left(\frac{P}{2\pi}\right)^{8/3}\left(\frac{2GM}{c^2}\right)^{-5/3} = \frac{5P}{128\pi}\left(\frac{c}{\Omega.2GM/c^2}\right)^{5/3}.$$

Substituting in the numbers given we find that the time to merger is

$$t_{\text{merge}} = \frac{5}{64\pi}\left(8.2\times10^7\right)^{5/3} \quad \text{hours} = 4.4\times10^7 \quad \text{years}.$$

If the frequency of the gravitational waves is 50 Hz, then the frequency of the orbit is 25 Hz. Continuing to assume stars of mass $1.4M_\odot$ this gives a time to merger of

$$t_{\text{merge}} = 13.4 \text{ s}.$$

The qualitative features of the gravitational waves follow from the time dependence of the orbital parameters of the binary star system and of the amplitude of the gravitational waves they produce. From Peters' formula we have that the semi-major axis of the orbit evolves with the time to merger as

$$a \sim \left(t_{\text{merge}} - t\right)^{1/4}.$$

It then follows from Kepler's third law, $\Omega^2a^3 = G(M_1 + M_2)$ that the frequency of the orbital motion scales as

$$\Omega \sim \left(t_{\text{merge}} - t\right)^{-3/8}.$$

The gravitational waves have twice the frequency of the source, but this is simply a multiplicative factor and they display the same scaling with time to merger. Finally, the amplitude of the gravitational waves produced by such a binary source is proportional to Ω^2a^2 and hence scales as

$$h \sim \left(t_{\text{merge}} - t\right)^{-1/4},$$

with the time to merger. These scalings are enough to give a rough sketch of the anticipated chirp signal, at least in the early part of the merger.

PROBLEM 5

For a perfect pressureless fluid $T_{\mu\nu} = \rho c^2\, u_\mu u_\nu$ and in a non-relativistic limit

$$T_{00} \approx \rho c^2, \qquad T_{0i} \approx -\rho c v^i, \qquad T_{ij} \approx \rho v^i v^j,$$

and the three formulae asked for follow immediately.

To find the metric, recall that $h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(\eta^{\alpha\beta}\overline{h}_{\alpha\beta})$. Since $\eta^{\alpha\beta}\overline{h}_{\alpha\beta} = 4\phi/c^2$ we have

$$\begin{aligned} h_{00} &= -\frac{4\phi}{c^2} - \frac{1}{2}(-1)\frac{4\phi}{c^2} = -\frac{2\phi}{c^2}, \\ h_{0i} &= -\Omega_i - \frac{1}{2}(0)\frac{4\phi}{c^2} = -\Omega_i, \\ h_{ij} &= 0 - \frac{1}{2}(\delta_{ij})\frac{4\phi}{c^2} = -\frac{2\phi}{c^2}\,\delta_{ij}. \end{aligned}$$

It then follows that the metric is given by

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = -\left(1 + \frac{2\phi}{c^2}\right)c^2dt^2 - \Omega_i\left[cdt\,dx^i + dx^i\,cdt\right] + \left(1 - \frac{2\phi}{c^2}\right)dx^i dx^i.$$

The vector Ω_i represents the angular momentum of the mass distribution.

Chapter 7 – Cosmology

PROBLEM 2

To be added.

PROBLEM 5

Einstein's static universe is the subset of $\mathbb{R}^{1,4}$ defined by

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a_0^2.$$

We can introduce local coordinates parameterising the entire space as follows

$$x^0 = ct, \quad x^1 = a_0 \cos(\chi), \quad x^2 = a_0 \sin(\chi)\cos(\theta),$$
$$x^3 = a_0 \sin(\chi)\sin(\theta)\cos(\phi), \quad x^4 = a_0 \sin(\chi)\sin(\theta)\sin(\phi),$$

where (χ, θ, ϕ) are the usual polar angles on the 3-sphere.

It is obvious that the constant t slices are 3-spheres and that in the usual coordinates (χ, θ, ϕ) the metric is the usual round one. However, to spell things out more fully, tangent vectors to the surface are given by

$$\begin{aligned}
\partial_0 \mathbf{X} &= \big(1, 0, 0, 0, 0\big), \\
\partial_\chi \mathbf{X} &= a_0 \Big(0, -\sin(\chi), \cos(\chi)\cos(\theta), \cos(\chi)\sin(\theta)\cos(\phi), \cos(\chi)\sin(\theta)\sin(\phi)\Big), \\
\partial_\theta \mathbf{X} &= a_0 \sin(\chi) \Big(0, 0, -\sin(\theta), \cos(\theta)\cos(\phi), \cos(\theta)\sin(\theta)\Big), \\
\partial_\phi \mathbf{X} &= a_0 \sin(\chi)\sin(\theta) \Big(0, 0, 0, -\sin(\phi), \cos(\phi)\Big),
\end{aligned}$$

and it follows in the usual way that

$$\begin{aligned}
ds^2 = d\mathbf{X} \cdot d\mathbf{X} &= \partial_0 \mathbf{X} \cdot \partial_0 \mathbf{X} \, c^2 dt^2 + \partial_\chi \mathbf{X} \cdot \partial_\chi \mathbf{X} \, d\chi^2 + \partial_\theta \mathbf{X} \cdot \partial_\theta \mathbf{X} \, d\theta^2 + \partial_\phi \mathbf{X} \cdot \partial_\phi \mathbf{X} \, d\phi^2, \\
&= -c^2 dt^2 + a_0^2 \, d\chi^2 + a_0^2 \sin^2(\chi) \, d\theta^2 + a_0^2 \sin^2(\chi)\sin^2(\theta) \, d\phi^2, \\
&= -c^2 dt^2 + a_0^2 \Big[d\chi^2 + \sin^2(\chi) \Big(d\theta^2 + \sin^2(\theta) \, d\phi^2 \Big) \Big],
\end{aligned}$$

where by the dot " $\cdot$ " we mean the usual inner product on Minkowski space.

The metric for the Einstein static universe, in these coordinates, is of FLRW form with scale factor $a(t)$ equal to the constant a_0 and for spatial metric of positive scalar curvature, $k = 1$. We can therefore look up the components of the Ricci tensor

$$\begin{aligned}
R_{00} - \frac{1}{2} R g_{00} &= \frac{3}{a_0^2}, \\
R_{ij} - \frac{1}{2} R g_{ij} &= \frac{-1}{a_0^2} \, g_{ij}.
\end{aligned}$$

Including the cosmological constant and a pressure-free fluid (in its rest frame) the Einstein equations read

$$\begin{aligned}
\frac{3}{a_0^2} - \Lambda &= \frac{8\pi G}{c^4} \, \rho c^2, \\
\frac{-1}{a_0^2} \, g_{ij} + \Lambda g_{ij} &= 0.
\end{aligned}$$

It follows that there is a solution with

$$a_0 = \Lambda^{-1/2}, \qquad \rho = \frac{c^2}{4\pi G} \, \Lambda.$$