LECTURE NOTES ON
INTERMEDIATE FLUID MECHANICS

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updated
26 July 2019, 9:10pm

Contents

Preface

These are lecture notes for AME 60635, Intermediate Fluid Mechanics, taught in the Department of Aerospace and Mechanical Engineering of the University of Notre Dame. Most of the students in this course are beginning graduate students and advanced undergraduates in engineering. The objective of the course is to provide a survey of a wide variety of topics in fluid mechanics, including a rigorous derivation of the compressible Navier-Stokes equations, vorticity dynamics, compressible flow, potential flow, and viscous laminar flow.

While there is a good deal of rigor in the development here, it is not absolute. It is not hard to find gaps in some of the developments; consequently, the student should call on textbooks and other reference materials for a full description. A great deal of the development and notation for the governing equations closely follows Panton[1], who I find gives an especially clear presentation. The material in the remaining chapters is drawn from a wide variety of sources. A full list is given in the bibliography, though few specific citations are given in the text. The notes, along with much information on the course itself, can be found on the world wide web at `https://www3.nd.edu/~powers/ame.60635`. At this stage, anyone is free to duplicate the notes.

The notes have been transposed from written notes I developed in teaching this in 1992 and a related course in 1991. Many enhancements have been added, and thanks go to many students and faculty who have pointed out errors. It is likely that there are more waiting to be discovered; I would be happy to hear from you regarding these or suggestions for improvement.

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[1] R. L. Panton, *Incompressible Flow*, 4th edition, John Wiley, New York, 2013.

Chapter 1

Governing equations

see Panton, Chapters 1-6,
see Yih, Chapters 1-3, Appendix 1-2,
see Aris.

1.1 Philosophy of rational continuum mechanics

1.1.1 Approaches to fluid mechanics

We seek here to present an approach to fluid mechanics founded on the principles of rational continuum mechanics. There are many paths to understanding fluid mechanics, and good arguments can be made for each. A typical first undergraduate class will combine a mix of basic equations, coupled with strong physical motivations, and allows the student to develop a knowledge which is of great practical value, often driven strongly by intuition. Such an approach works well within the confines of the intuition we develop in everyday life. It often fails when the engineer moves into unfamiliar territory. For example, lack of fundamental understanding of high Mach number flows led to many aircraft and rocket failures in the 1950's. In such cases, a return to the formalism of a careful theory, one which clearly exposes the strengths and weaknesses of all assumptions, is invaluable in both understanding the true fluid physics, and applying that knowledge to engineering design.

Probably the most formal of approaches is that of the school of thought advocated most clearly by Truesdell,[1] sometimes known as Rational Continuum Mechanics. Truesdell developed a broadly based theory which encompassed all materials which could be regarded as continua, including solids, liquids, and gases, in the limit when averaging volumes were sufficiently large so that the micro- and nanoscopic structure of these materials was unimportant. For fluids (both liquid and gas), such length scales are often on the order of microns, while for solids, it may be somewhat smaller, depending on the type of crystalline structure. The difficulty of the Truesdellian approach is that it is burdened with a difficult notation

[1] Clifford Ambrose Truesdell, III, 1919-2000, American continuum mechanician and natural philosopher. Taught at Indiana and Johns Hopkins Universities.

and tends to become embroiled in proofs and philosophy, which while ultimately useful, can preclude learning basic fluid mechanics in the time scale of the human lifetime.

In this course, we will attempt to steer between the fallible pragmatism of undergraduate fluid mechanics and the harsh formalism of the Truesdellian school. The material will pay some due homage to rational continuum mechanics and will be geared towards a basic understanding of fluid behavior. We shall first spend some time carefully developing the governing equations for a compressible viscous fluid. We shall then study representative solutions of these equations in a wide variety of physically motivated limits in order to understand how the basic conservation principles of mass, momenta, and energy, coupled with constitutive relations, influence the behavior of fluids.

1.1.2 Mechanics

Mechanics is the broad superset of the topic matter of this course. Mechanics is the science which seeks an explanation for the motion of bodies based upon models grounded in well defined axioms. Axioms, as in geometry, are statements which cannot be proved; they are useful insofar as they give rise to results which are consistent with our empirical observations. A hallmark of science has been the struggle to identify the smallest set of axioms which are sufficient to describe our universe. When we find an axiom to be inconsistent with observation, it must be modified or eliminated. A familiar example of this is the Michelson-[2] Morley[3] experiment, which motivated Einstein[4] to modify the Newtonian[5] axioms of conservation of mass and energy into a conservation of mass-energy.

In Truesdell's exposition on mechanics, he suggests the following hierarchy:

- *bodies* exist,
- bodies are assigned to *place*,
- *geometry* is the theory of place,
- change of place in *time* is the *motion* of the body,
- a description of the motion of a body is *kinematics*,

[2]Albert Abraham Michelson, 1852-1931, Prussian born American physicist, graduate of the U.S. Naval Academy and faculty member at Case School of Applied Science, Clark University, and University of Chicago.

[3]Edward Williams Morley, 1838-1923, New Jersey-born American physical chemist, graduate of Williams College, professor of chemistry at Western Reserve College.

[4]Albert Einstein, 1879-1955, German physicist who developed the theory of relativity and made fundamental contributions to quantum mechanics and Brownian motion in fluid mechanics; spent later life in the United States.

[5]Sir Isaac Newton, 1642-1727, English physicist and mathematician and chief figure of the scientific revolution of the seventeenth and eighteenth centuries. Developed calculus, theories of gravitation and motion of bodies, and optics. Educated at Cambridge University and holder of the Lucasian chair at Cambridge. In civil service as Warden of the Mint, he became the terror of counterfeiters, sending many to the gallows.

- motion is the consequence of *forces*,
- study of forces on a body is *dynamics*.

There are many subsets of mechanics, e.g. statistical mechanics, quantum mechanics, continuum mechanics, fluid mechanics, or solid mechanics. Auto mechanics, while a legitimate topic for study, does not generally fall into the class of mechanics we consider here, though the intersection of the two sets is not the empty set.

1.1.3 Continuum mechanics

Early mechanicians, such as Newton, dealt primarily with point masses and finite collections of particles. In one sense this is because such systems are the easiest to study, and it makes more sense to grasp the simple before the complex. External motivation was also present in the 18th century, which had a martial need to understand the motion of cannonballs and a theological need to understand the motion of planets. The discipline which considers systems of this type is often referred to as classical mechanics. Mathematically, such systems are generally characterized by a finite number of ordinary differential equations, and the properties of each particle (e.g. position, velocity) are taken to be functions of time only.

Continuum mechanics, generally attributed to Euler,[6] considers instead an infinite number of particles. In continuum mechanics every physical property (e.g. velocity, density, pressure) is taken to be a function of both time and space. There is an infinitesimal property variation from point to point in space. While variations are generally continuous, finite numbers of surfaces of discontinuous property variation are allowed. This models, for example, the contact between one continuous body and another. Point discontinuities are not allowed, however. Finite valued material properties are required. Mathematically, such systems are characterized by a finite number of partial differential equations in which the properties of the continuum material are functions of both space and time. It is possible to show that a partial differential equation can be thought of as an infinite number of ordinary differential equations, so this is consistent with our model of a continuum as an infinite number of particles.

1.1.4 Rational continuum mechanics

The modifier "rational" was first applied by Truesdell to continuum mechanics to distinguish the formal approach advocated by his school, from less formal, though mainly not irrational, approaches to continuum mechanics. Rational continuum mechanics is developed with tools similar to those which Euclid[7] used for his geometry: formal definitions, axioms, and theorems, all accompanied by careful language and proofs. This course will generally follow the

[6]Leonhard Euler, 1707-1783, Swiss-born mathematician and physicist who served in the court of Catherine I of Russia in St. Petersburg, regarded by many as one of the greatest mechanicians.

[7]Euclid, Greek geometer of profound influence who taught in Alexandria, Egypt, during the reign of Ptolemy I Soter, who ruled 323-283 BC.

less formal, albeit still rigorous, approach of Panton's text, including the adoption of much of Panton's notation.

1.1.5 Notions from Newtonian continuum mechanics

The following are useful notions from Newtonian continuum mechanics. Here we use Newtonian to distinguish our mechanics from Einsteinian or relativistic mechanics.

- *Space* is three-dimensional and independent of time.
- An *inertial frame* is a reference frame in which the laws of physics are invariant; further, a body in an inertial frame with zero net force acting upon it does not accelerate.
- A *Galilean transformation* specifies how to transform from one inertial frame to another inertial frame moving at constant velocity relative to the original frame. If a second inertial frame has constant velocity $\mathbf{v}_o = u_o\mathbf{i} + v_o\mathbf{j} + w_o\mathbf{k}$ relative to the original inertial frame, the Galilean transformation $(x, y, z, t) \to (x', y', z', t')$ is as follows

$$x' = x - u_o t, \tag{1.1}$$
$$y' = y - v_o t, \tag{1.2}$$
$$z' = z - w_o t, \tag{1.3}$$
$$t' = t. \tag{1.4}$$

- *Control volumes* are useful; we will study three varieties:
  - Fixed: constant in space,
  - Material: no flux of mass through boundaries, can deform,
  - Arbitrary: can move, can deform, can have different fluid contained within.
- *Control surfaces* enclose control volumes; they have the same three varieties:
  - Fixed,
  - Material,
  - Arbitrary.
- *Density* is a material property, not used in classical mechanics, which only considers point masses. We can define density ρ as

$$\rho = \lim_{V \to 0} \frac{\sum_{i=1}^{N} m_i}{V}. \tag{1.5}$$

  Here V is the volume of the space considered, N is the number of particles contained within the volume, and m_i is the mass of the ith particle. We can define a length scale

L associated with the volume V to be $L = V^{1/3}$. In commonly encountered physical scenarios, we expect the density to vary with distance on a macroscale, approach a limiting value at the microscale, and become ill-defined below a cutoff scale below which molecular effects are important. That is to say, when V becomes too small, such that only a few molecules are contained within it, we expect wild oscillations in ρ.

We will in fact assume that matter can be modeled as a *continuum*: the limit in which discrete changes from molecule to molecule can be ignored and distances and times over which we are concerned are much larger than those of the molecular scale. This will enable the use of calculus in our continuum thermodynamics.

Continuum mechanics will treat macroscopic effects only and ignore individual molecular effects. For example molecules bouncing off a wall exchange momentum with the wall and induce pressure. We could use Newtonian mechanics for each particle collision to calculate the net force on the wall. Instead our approach amounts to considering the *average* over space and time of the net effect of millions of collisions on a wall.

The continuum theory can break down in important applications where the length and time scales are of comparable magnitude to molecular time scales. Important applications where the continuum assumption breaks down include

- rarefied gas dynamics of the outer atmosphere (relevant for low orbit space vehicles), and
- nano-scale heat transfer (relevant in cooling of computer chips).

To get some idea of the scales involved, we note that for air at atmospheric pressure and temperature that the time and distance between molecular collisions provides the limits of the continuum. Under these conditions, we observe for air that

- length $> 0.1\ \mu$m, and
- time > 0.1 ns,

will be sufficient to admit the continuum assumption. For denser gases, these cutoff scales are smaller. For lighter gases, these cutoff scales are larger. A sketch of a possible density variation in a gas near atmospheric pressure is given in Fig. 1.1.

Details of collision theory can be found in advanced texts such as that of Vincenti and Kruger, pp. 12-26. They show for air that the mean free path λ is well modeled by the equation:

$$\lambda = \frac{M}{\sqrt{2}\pi \mathcal{N} \rho d^2}. \tag{1.6}$$

Here M is the molecular mass, $\mathcal{N}$ is Avogadro's number, and d is the molecular diameter.

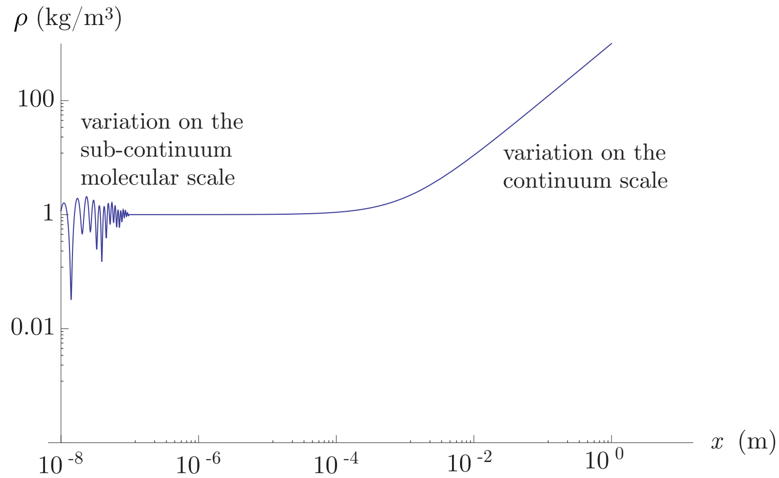

Figure 1.1: Sketch of possible density variation of a gas near atmospheric pressure.

Example 1.1

Find the variation of mean free path with density for air.

We turn to Vincenti and Kruger for numerical parameter values, which are seen to be $M = 28.9$ kg/kmole, $\mathcal{N} = 6.02252 \times 10^{23}$ molecule/mole, $d = 3.7 \times 10^{-10}$ m. Thus,

$$\lambda = \frac{\left(28.9 \ \frac{\text{kg}}{\text{kmole}}\right)\left(1 \ \frac{\text{kmole}}{1000 \ \text{mole}}\right)}{\sqrt{2}\pi\left(6.02252 \times 10^{23} \ \frac{\text{molecule}}{\text{mole}}\right)\rho\left(3.7 \times 10^{-10} \ \text{m}\right)^2}, \tag{1.7}$$

$$= \frac{7.8895 \times 10^{-8} \ \frac{\text{kg}}{\text{molecule m}^2}}{\rho}. \tag{1.8}$$

Note that the unit "molecule" is not really a dimension, but really is literally a "unit," which may well be thought of as dimensionless. Thus, we can safely say

$$\lambda = \frac{7.8895 \times 10^{-8} \ \frac{\text{kg}}{\text{m}^2}}{\rho}. \tag{1.9}$$

A plot of the variation of mean free path λ as a function of ρ is given in Fig. 1.2. Vincenti and Kruger go on to consider an atmosphere with density of $\rho = 1.288$ kg/m^3. For this density

$$\lambda = \frac{7.8895 \times 10^{-8} \ \frac{\text{kg}}{\text{m}^2}}{1.288 \ \frac{\text{kg}}{\text{m}^3}}, \tag{1.10}$$

$$= 6.125 \times 10^{-8} \ \text{m}, \tag{1.11}$$

$$= 6.125 \times 10^{-2} \mu\text{m}. \tag{1.12}$$

Vincenti and Kruger also show the mean molecular speed under these conditions is roughly $c = 500$ m/s, so the mean time between collisions, τ, is

$$\tau \sim \frac{\lambda}{c} = \frac{6.125 \times 10^{-8} \ \text{m}}{500 \ \frac{\text{m}}{\text{s}}} = 1.225 \times 10^{-10} \ \text{s}. \tag{1.13}$$

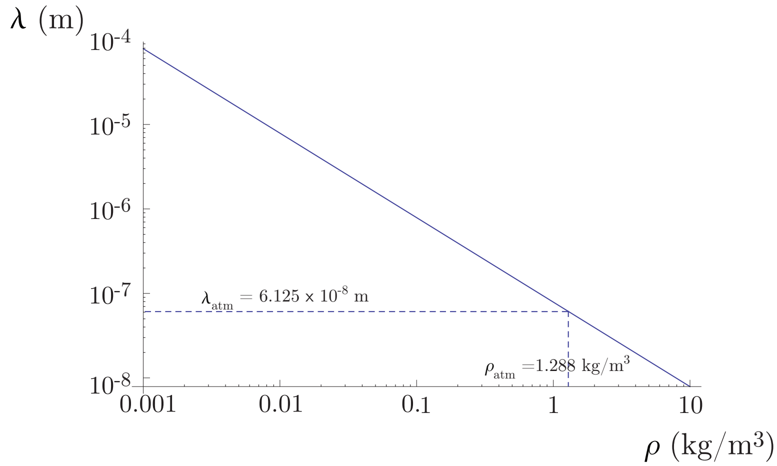

Figure 1.2: Mean free path length, λ, as a function of density, ρ, for air.

Density is an example of a *scalar* property. We shall have more to say later about scalars. For now we say that a scalar property associates a single number with each point in time and space. We can think of this by writing the usual notation $\rho(x, y, z, t)$, which indicates ρ has functional variation with position and time.

- Other properties are not scalar, but are *vector* properties. For example the velocity vector

$$\mathbf{v}(x, y, z, t) = u(x, y, z, t)\mathbf{i} + v(x, y, z, t)\mathbf{j} + w(x, y, z, t)\mathbf{k}, \tag{1.14}$$

associates three scalars u, v, w with each point in space and time. We will see that a vector can be characterized as a scalar associated with a particular direction in space. Here we use a boldfaced notation for a vector. This is known as Gibbs[8] notation. We will soon study an alternate notation, developed by Einstein, and known as Cartesian[9] index notation.

- Other properties are not scalar or vector, but are what is know as tensorial. The relevant properties are called *tensors*. The best known example is the stress tensor, whose physics and mathematics will be fully described in Sec. 1.4.2.2. One can think

[8] Josiah Willard Gibbs, 1839-1903, American physicist and chemist with a lifelong association with Yale University who made fundamental contributions to vector analysis, statistical mechanics, thermodynamics, and chemistry. Studied in Europe in the 1860s. Probably one of the few great American scientists of the nineteenth century.

[9] René Descartes, 1596-1650, French mathematician and philosopher of great influence. A great doubter of existence who nevertheless concluded, "I think, therefore I am." Developed analytic geometry.

of a tensor as a quantity which associates a vector with a plane inclined at a selected angle passing through a given point in space. An example is the viscous stress tensor $\boldsymbol{\tau}$, which is best expressed as a three by three matrix with nine components:

$$\boldsymbol{\tau}(x,y,z,t) = \begin{pmatrix} \tau_{xx}(x,y,z,t) & \tau_{xy}(x,y,z,t) & \tau_{xz}(x,y,z,t) \\ \tau_{yx}(x,y,z,t) & \tau_{yy}(x,y,z,t) & \tau_{yz}(x,y,z,t) \\ \tau_{zx}(x,y,z,t) & \tau_{zy}(x,y,z,t) & \tau_{zz}(x,y,z,t) \end{pmatrix} \tag{1.15}$$

1.2 Some necessary mathematics

Here we outline some fundamental mathematical principles which are necessary to understand continuum mechanics as it will be presented here.

1.2.1 Vectors and Cartesian tensors

1.2.1.1 Gibbs and Cartesian Index notation

Gibbs notation for vectors and tensors is the most familiar from undergraduate courses. It typically uses boldface, arrows, underscores, or overbars to denote a vector or a tensor. Unfortunately, it also hides some of the structures which are actually present in the equations. Einstein realized this in developing the theory of general relativity and developed a useful alternate, index notation. In these notes we will focus on what is known as Cartesian index notation, which is restricted to Cartesian coordinate systems. Einstein also developed a more general index system for non-Cartesian systems. We will briefly touch on this in our summaries of our equations later in this chapter but refer the reader to books such as that of Aris for a full exposition. While it can seem difficult at the outset, in the end many agree that the use of index notation actually simplifies many common notions in fluid mechanics. Moreover, its use in the archival literature is widespread, so to be conversant in fluid mechanics, one must know index notation. Table 1.1 summarizes the correspondences between Gibbs, Cartesian index, and matrix notation. Here we adopt a convention for the Gibbs notation, which we will find at times conflicts with other conventions, in which italics font (a) indicates a scalar, bold font ($\mathbf{a}$) indicates a vector, upper case sans serif (A) indicates a second order tensor, over-lined upper case sans serif ($\overline{\mathsf{A}}$) indicates a third order tensor, double over-lined upper case sans serif ($\overline{\overline{\mathsf{A}}}$) indicates a fourth order tensor. In Cartesian index notation, their is no need to use anything except italics, as all terms are thought of as scalar components of a more expansive structure, with the structure indicated by the presence of subscripts.

The essence of the Cartesian index notation is as follows. We can represent a three dimensional vector $\mathbf{a}$ as a linear combination of scalars and orthonormal basis vectors:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}. \tag{1.16}$$

We choose now to associate the subscript 1 with the x direction, the subscript 2 with the y direction, and the subscript 3 with the z direction. Further, we replace the orthonormal

Quantity	Common Parlance	Gibbs	Cartesian Index	Matrix
zeroth order tensor	scalar	a	a	$\left(a \right)$
first order tensor	vector	$\mathbf{a}$	a_i	$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$
second order tensor	tensor	A	a_{ij}	$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$
third order tensor	tensor	$\overline{\mathsf{A}}$	a_{ijk}	-
fourth order tensor	tensor	$\overline{\overline{\mathsf{A}}}$	a_{ijkl}	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	-

Table 1.1: Scalar, vector, and tensor notation conventions.

basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Then the vector $\mathbf{a}$ is represented by

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i = a_i\mathbf{e}_i = a_i = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \tag{1.17}$$

Following Einstein, we have adopted the convention that a summation is understood to exist when two indices, known as dummy indices, are repeated, and have further left the explicit representation of basis vectors out of our final version of the notation. We have also included a representation of $\mathbf{a}$ as a 3×1 column vector. We adopt the standard that all vectors can be thought of as column vectors. Often in matrix operations, we will need row vectors. They will be formed by taking the transpose, indicated by a superscript T, of a column vector. In the interest of clarity, full consistency with notions from matrix algebra, as well as transparent translation to the conventions of necessarily meticulous (as well as popular) software tools such as `MATLAB`, we will scrupulously use the transpose notation. This comes at the expense of a more cluttered set of equations at times. We also note that most authors do not *explicitly* use the transpose notation, but its use is implicit.

1.2.1.2 Rotation of axes

The Cartesian index notation is developed to be valid under transformations from one Cartesian coordinate system to another Cartesian coordinate system. It is not applicable to either general orthogonal systems (such as cylindrical or spherical) or non-orthogonal systems. It is straightforward, but tedious, to develop a more general system to handle generalized co-

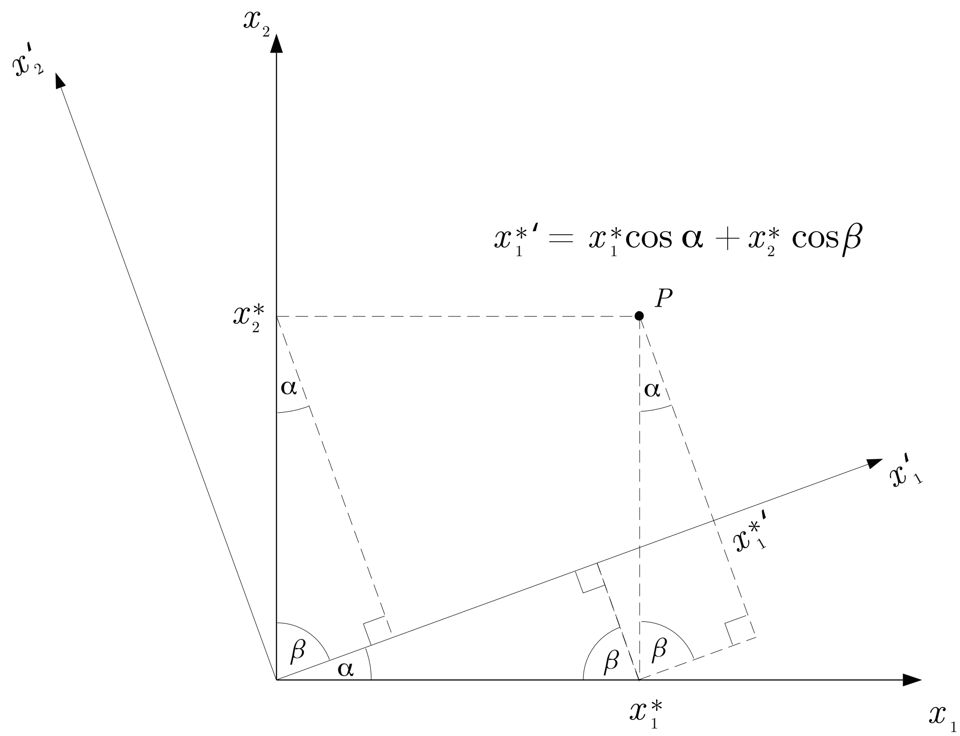

Figure 1.3: Sketch of coordinate transformation which is a rotation of axes.

ordinate transformations, and Einstein did just that as well. For our purposes however, the simpler Cartesian index notation will suffice.

We will consider a coordinate transformation which is a simple rotation of axes. This transformation preserves all angles; hence, right angles in the original Cartesian system will be right angles in the rotated, but still Cartesian system. It also preserves lengths of geometric features, with no stretching. We will require, ultimately, that whatever theory we develop must generate results in which physically relevant quantities such as temperature, pressure, density, and velocity magnitude, are independent of the particular set of coordinates with which we choose to describe the system. To motivate this, let us consider a two-dimensional rotation from an unprimed system to a primed system. So, we seek a transformation which maps $(x_1, x_2)^T \to (x_1', x_2')^T$. We will rotate the unprimed system counterclockwise through an angle α to achieve the primed system. The rotation is sketched in Figure 1.3. Note that it is easy to show that the angle $\beta = \pi/2 - \alpha$. Here a point P is identified by a particular set of coordinates (x_1^*, x_2^*). One of the keys to all of continuum mechanics is realizing that while the location (or velocity, or stress, ...) of P may be represented differently in various coordinate systems, ultimately it must represent the same physical reality. Straightforward geometry shows the following relation between the primed and unprimed coordinate systems for x_1'

$$x_1^{*'} = x_1^* \cos\alpha + x_2^* \cos\beta. \tag{1.18}$$

More generally, we can say for an arbitrary point that

$$x_1' = x_1 \cos\alpha + x_2 \cos\beta. \tag{1.19}$$

We adopt the following notation

- (x_1, x_1') denotes the angle between the x_1 and x_1' axes,
- (x_2, x_2') denotes the angle between the x_2 and x_2' axes,
- (x_3, x_3') denotes the angle between the x_3 and x_3' axes,
- (x_1, x_2') denotes the angle between the x_1 and x_2' axes,
- $\vdots$

Thus, in two-dimensions, we have

$$x_1' = x_1 \cos(x_1, x_1') + x_2 \cos(x_2, x_1'). \tag{1.20}$$

In three dimensions, this extends to

$$x_1' = x_1 \cos(x_1, x_1') + x_2 \cos(x_2, x_1') + x_3 \cos(x_3, x_1'). \tag{1.21}$$

Extending this analysis to calculate x_2' and x_3' gives

$$x_2' = x_1 \cos(x_1, x_2') + x_2 \cos(x_2, x_2') + x_3 \cos(x_3, x_2'), \tag{1.22}$$

$$x_3' = x_1 \cos(x_1, x_3') + x_2 \cos(x_2, x_3') + x_3 \cos(x_3, x_3'). \tag{1.23}$$

The above equations can be written in matrix form as

$$\left(x_1' \quad x_2' \quad x_3' \right) = \left(x_1 \quad x_2 \quad x_3 \right) \begin{pmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{pmatrix} \tag{1.24}$$

If we use the shorthand notation, for example, that $\ell_{11} = \cos(x_1, x_1')$, $\ell_{12} = \cos(x_1, x_2')$, etc., we have

$$\underbrace{\left(x_1' \quad x_2' \quad x_3' \right)}_{\mathbf{x}'^T} = \underbrace{\left(x_1 \quad x_2 \quad x_3 \right)}_{\mathbf{x}^T} \underbrace{\begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}}_{\mathsf{Q}} \tag{1.25}$$

In Gibbs notation, defining the matrix of ℓ's to be Q, and recalling that all vectors are taken to be column vectors, we can alternatively say $\mathbf{x}'^T = \mathbf{x}^T \cdot \mathsf{Q}$. Taking the transpose of both sides and recalling the useful identities that $(\mathbf{a} \cdot \mathbf{b})^T = \mathbf{b}^T \cdot \mathbf{a}^T$ and $(\mathbf{a}^T)^T = \mathbf{a}$, we can also say $\mathbf{x}' = \mathsf{Q}^T \cdot \mathbf{x}$.[10] We call $\mathsf{Q} = \ell_{ij}$ the matrix of direction cosines and $\mathsf{Q}^T = \ell_{ji}$ the rotation matrix. It can be shown that coordinate systems which satisfy the right hand rule require further that

$$\det \mathsf{Q} = 1. \tag{1.26}$$

Matrices Q that have $|\det \mathsf{Q}| = 1$ are associated with volume-preserving transformations. Matrices Q that have $\det \mathsf{Q} > 0$, are orientation-preserving transformations. Matrices Q that have $\det \mathsf{Q} = 1$ are thus volume- and orientation-preserving, and can be thought of a rotations. A matrix that had determinant -1 would be volume-preserving but not orientation-preserving. It could be considered as a reflection.

It can be shown that the transpose of an orthogonal matrix is its inverse:

$$\mathsf{Q}^T = \mathsf{Q}^{-1}. \tag{1.27}$$

Thus we have

$$\mathsf{Q} \cdot \mathsf{Q}^T = \mathsf{Q}^T \cdot \mathsf{Q} = \mathsf{I}. \tag{1.28}$$

The equation $\mathbf{x}'^T = \mathbf{x}^T \cdot \mathsf{Q}$ is really a set of three linear equations. For instance, the first is

$$x_1' = x_1 \ell_{11} + x_2 \ell_{21} + x_3 \ell_{31}. \tag{1.29}$$

More generally, we could say that

$$x_j' = x_1 \ell_{1j} + x_2 \ell_{2j} + x_3 \ell_{3j}. \tag{1.30}$$

Here j is a so-called "free index," which for three-dimensional space takes on values $j = 1, 2, 3$. Some rules of thumb for free indices are

- A free index can appear only once in each additive term.
- One free index (e.g. k) may replace another (e.g. j) as long as it is replaced in each additive term.

We can simplify Eq. (1.30) further by writing

$$x_j' = \sum_{i=1}^{3} x_i \ell_{ij}. \tag{1.31}$$

[10]The more commonly used alternate convention of not explicitly using the transpose notation for vectors would instead have our $\mathbf{x}'^T = \mathbf{x}^T \cdot \mathsf{Q}$ written as $\mathbf{x}' = \mathbf{x} \cdot \mathsf{Q}$. In fact, our use of the transpose notation is strictly viable only for Cartesian coordinate systems, while many will allow Gibbs notation to represent vectors in non-Cartesian coordinates, for which the transpose operation is ill-suited. However, realizing that these notes will primarily focus on Cartesian systems, and that such operations relying on the transpose are useful notions from linear algebra, it will be employed in an overly liberal fashion in these notes. The alternate convention still typically applies, where necessary, the transpose notation for tensors, so it would also hold that $\mathbf{x}' = \mathsf{Q}^T \cdot \mathbf{x}$.

This is commonly written in the following form:

$$x'_j = x_i \ell_{ij}. \tag{1.32}$$

We again note that it is to be understood that whenever an index is repeated, as has the index i above, that a summation from $i = 1$ to $i = 3$ is to be performed and that i is the "dummy index." Some rules of thumb for dummy indices are

- dummy indices can appear *only twice* in a given additive term,
- a pair of dummy indices, say i, i, can be exchanged for another, say j, j, in a given additive term with no need to change dummy indices in other additive terms.

We define the Kronecker[11] delta, δ_{ij} as

$$\delta_{ij} = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases} \tag{1.33}$$

This is effectively the identity matrix I:

$$\delta_{ij} = \mathsf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.34}$$

Direct substitution proves that what is effectively the law of cosines can be written as

$$\ell_{ij}\ell_{kj} = \delta_{ik}. \tag{1.35}$$

This is also equivalent to Eq. (1.28).

Example 1.2

Show for the two-dimensional system described in Figure 1.3 that $\ell_{ij}\ell_{kj} = \delta_{ik}$ holds.

Expanding for the two-dimensional system, we get

$$\ell_{i1}\ell_{k1} + \ell_{i2}\ell_{k2} = \delta_{ik}.$$

First, take $i = 1, k = 1$. We get then

$$\begin{aligned} \ell_{11}\ell_{11} + \ell_{12}\ell_{12} = \delta_{11} &= 1, \\ \cos\alpha\cos\alpha + \cos(\alpha + \pi/2)\cos(\alpha + \pi/2) &= 1, \\ \cos\alpha\cos\alpha + (-\sin(\alpha))(-\sin(\alpha)) &= 1, \\ \cos^2\alpha + \sin^2\alpha &= 1. \end{aligned}$$

[11] Leopold Kronecker, 1823-1891, German mathematician, critic of set theory, who stated "God made the integers; all else is the work of man."

This is obviously true. Next, take $i = 1, k = 2$. We get then

$$\begin{aligned} \ell_{11}\ell_{21} + \ell_{12}\ell_{22} = \delta_{12} &= 0, \\ \cos\alpha\cos(\pi/2 - \alpha) + \cos(\alpha + \pi/2)\cos(\alpha) &= 0, \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha &= 0. \end{aligned}$$

This is obviously true. Next, take $i = 2, k = 1$. We get then

$$\begin{aligned} \ell_{21}\ell_{11} + \ell_{22}\ell_{12} = \delta_{21} &= 0, \\ \cos(\pi/2 - \alpha)\cos\alpha + \cos\alpha\cos(\pi/2 + \alpha) &= 0, \\ \sin\alpha\cos\alpha + \cos\alpha(-\sin\alpha) &= 0. \end{aligned}$$

This is obviously true. Next, take $i = 2, k = 2$. We get then

$$\begin{aligned} \ell_{21}\ell_{21} + \ell_{22}\ell_{22} = \delta_{22} &= 1, \\ \cos(\pi/2 - \alpha)\cos(\pi/2 - \alpha) + \cos\alpha\cos\alpha &= 1, \\ \sin\alpha\sin\alpha + \cos\alpha\cos\alpha &= 1. \end{aligned}$$

Again, this is obviously true.

Using this, we can easily find the inverse transformation back to the unprimed coordinates via the following operations:

$$\ell_{kj}x'_j = \ell_{kj}x_i\ell_{ij}, \quad (1.36)$$

$$= \ell_{ij}\ell_{kj}x_i, \quad (1.37)$$

$$= \delta_{ik}x_i, \quad (1.38)$$

$$\ell_{kj}x'_j = x_k, \quad (1.39)$$

$$\ell_{ij}x'_j = x_i, \quad (1.40)$$

$$x_i = \ell_{ij}x'_j. \quad (1.41)$$

The Kronecker delta is also known as the substitution tensor as it has the property that application of it to a vector simply substitutes one index for another:

$$x_k = \delta_{ki}x_i. \quad (1.42)$$

For students familiar with linear algebra, it is easy to show that the matrix of direction cosines, ℓ_{ij}, is a rotation matrix. Each of its columns is a vector which is orthogonal to the other column vectors. Additionally, each column vector is itself normal. Such a matrix has a Euclidean norm of unity, and three eigenvalues which have magnitude of unity. Its determinant is $+1$, which renders it a rotation; in contrast a reflection matrix would have determinant of -1. Operation of a rotation matrix on a vector rotates it, but does not stretch it.

1.2.1.3 Vectors

Three scalar quantities v_i where $i = 1, 2, 3$ are scalar components of a *vector* if they transform according to the following rule

$$v'_j = v_i \ell_{ij} \tag{1.43}$$

under a rotation of axes characterized by direction cosines ℓ_{ij}. In Gibbs notation, we would say $\mathbf{v}'^T = \mathbf{v}^T \cdot \mathsf{Q}$, or alternatively $\mathbf{v}' = \mathsf{Q}^T \cdot \mathbf{v}$.

We can also say that a vector associates a scalar with a chosen direction in space by an expression that is linear in the direction cosines of the chosen direction.

Example 1.3

Consider the set of scalars which describe the velocity in a two dimensional Cartesian system:

$$v_i = \begin{pmatrix} v_x \\ v_y \end{pmatrix},$$

where we return to the typical x, y coordinate system. Determine if v_i is a vector.

In a rotated coordinate system, using the same notation of Figure 1.3, we find that

$$v'_x = v_x \cos\alpha + v_y \cos(\pi/2 - \alpha) = v_x \cos\alpha + v_y \sin\alpha,$$

$$v'_y = v_x \cos(\pi/2 + \alpha) + v_y \cos\alpha = -v_x \sin\alpha + v_y \cos\alpha.$$

This is linear in the direction cosines, and satisfies the definition for a vector.

Example 1.4

Do two arbitrary scalars, say the quotient of pressure and density and the product of specific heat and temperature, $(p/\rho, c_vT)^T$, form a vector?

If this quantity is a vector, then we can say

$$v_i = \begin{pmatrix} p/\rho \\ c_vT \end{pmatrix}.$$

This pair of numbers has an obvious physical meaning in our unrotated coordinate system. If the system were a calorically perfect ideal gas, the first component would represent the difference between the enthalpy and the internal energy, and the second component would represent the internal energy. And if we rotate through an angle α, we arrive at a transformed quantity of

$$v'_1 = \frac{p}{\rho}\cos\alpha + c_vT\cos(\pi/2 - \alpha).$$

$$v'_2 = \frac{p}{\rho}\cos(\pi/2 + \alpha) + c_vT\cos(\alpha).$$

This quantity does not have any known physical significance, and so it seems that these quantities do not form a vector.

We have the following vector algebra

- Addition
  - $w_i = u_i + v_i$ (Cartesian index notation)
  - $\mathbf{w} = \mathbf{u} + \mathbf{v}$ (Gibbs notation)
- Dot product (inner product)
  - $u_i v_i = b$ (Cartesian index notation)
  - $\mathbf{u}^T \cdot \mathbf{v} = b$ (Gibbs notation)
  - both notations require $u_1 v_1 + u_2 v_2 + u_3 v_3 = b$.

While u_i and v_i have scalar components which change under a rotation of axes, their inner product (or dot product) is a true scalar and is invariant under a rotation of axes. This is easily seen by subjecting vectors $\mathbf{u}$ and $\mathbf{v}$ to a rotation via Q so that $\mathbf{u}' = \mathsf{Q}^T \cdot \mathbf{u}$, $\mathbf{v}' = \mathsf{Q}^T \cdot \mathbf{v}$. Thus $\mathsf{Q} \cdot \mathbf{u}' = \mathsf{Q} \cdot \mathsf{Q}^T \cdot \mathbf{u} = \mathbf{u}$. and $\mathsf{Q} \cdot \mathbf{v}' = \mathsf{Q} \cdot \mathsf{Q}^T \cdot \mathbf{v} = \mathbf{v}$. Then consider the dot product

$$\mathbf{u}^T \cdot \mathbf{v} = b, \tag{1.44}$$

$$(\mathsf{Q} \cdot \mathbf{u}')^T \cdot (\mathsf{Q} \cdot \mathbf{v}') = b, \tag{1.45}$$

$$\mathbf{u}'^T \cdot \underbrace{\mathsf{Q}^T \cdot \mathsf{Q}}_{=\mathsf{I}} \cdot \mathbf{v}' = b, \tag{1.46}$$

$$\mathbf{u}'^T \cdot \mathsf{I} \cdot \mathbf{v}' = b, \tag{1.47}$$

$$\mathbf{u}'^T \cdot \mathbf{v}' = b. \tag{1.48}$$

The inner product is invariant under rotation.

Note that here we have in the Gibbs notation explicitly noted that the transpose is part of the inner product. Most authors in fact assume the inner product of two vectors implies the transpose and do not write it explicitly, writing the inner product simply as $\mathbf{u} \cdot \mathbf{v} \equiv \mathbf{u}^T \cdot \mathbf{v}$.

1.2.1.4 Tensors

1.2.1.4.1 Definition A second order tensor, or a rank two tensor, is nine scalar components that under a rotation of axes transformation according to the following rule:

$$T'_{ij} = \ell_{ki} \ell_{lj} T_{kl}. \tag{1.49}$$

Note we could also write this in an expanded form as

$$T'_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} \ell_{ki} \ell_{lj} T_{kl} = \sum_{k=1}^{3} \sum_{l=1}^{3} \ell_{ik}^T T_{kl} \ell_{lj}. \tag{1.50}$$

In the above expressions, i and j are both free indices; while k and l are dummy indices. The Gibbs notation for the above transformation is easily shown to be

$$\mathsf{T}' = \mathsf{Q}^T \cdot \mathsf{T} \cdot \mathsf{Q}. \tag{1.51}$$

Analogously to our conclusion for a vector, we say that a tensor associates a vector with each direction in space by an expression that is linear in the direction cosines of the chosen direction. For a given tensor T_{ij}, the first subscript is associated with the face of a unit cube (hence the memory device, "first-face"); the second subscript is associated with the vector components for the vector on that face.

Tensors can also be expressed as matrices. Note that all rank two tensors are two-dimensional matrices, but not all matrices are rank two tensors, as they do not necessarily satisfy the transformation rules. We can say

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \tag{1.52}$$

The first row vector, $(T_{11} \quad T_{12} \quad T_{13})$, is the vector associated with the 1 face. The second row vector, $(T_{21} \quad T_{22} \quad T_{23})$, is the vector associated with the 2 face. The third row vector, $(T_{31} \quad T_{32} \quad T_{33})$, is the vector associated with the 3 face.

We also have the following items associated with tensors.

1.2.1.4.2 Alternating unit tensor The alternating unit tensor, a tensor of rank 3, ϵ_{ijk} will soon be seen to be useful, especially when we introduce the vector cross product. It is defined as follows

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad ijk = 123, 231, \quad \text{or} \quad 312, \\ 0 & \text{if any two indices identical,} \\ -1 & \text{if} \quad ijk = 321, 213, \quad \text{or} \quad 132 \end{cases}. \tag{1.53}$$

Another way to remember this is to start with the sequence 123, which is positive. A sequential permutation, say from 123 to 231, retains the positive nature. A trade, say from 123 to 213, gives a negative value.

An identity which will be used extensively

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{1.54}$$

can be proved a number of ways, including the tedious way of direct substitution for all values of i, j, k, l, m.

1.2.1.4.3 Some secondary definitions

1.2.1.4.3.1 Transpose The transpose of a second rank tensor, denoted by a superscript T, is found by exchanging elements about the diagonal. In shorthand index notation, this is simply

$$(T_{ij})^T = T_{ji}. \tag{1.55}$$

Written out in full, if

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}, \tag{1.56}$$

then

$$T_{ij}^T = T_{ji} = \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}, \tag{1.57}$$

1.2.1.4.3.2 Symmetric A tensor D_{ij} is symmetric iff

$$D_{ij} = D_{ji}. \tag{1.58}$$

Note that a symmetric tensor has only six independent scalars. We will see that D is associated with the deformation of a fluid element.

1.2.1.4.3.3 Antisymmetric A tensor R_{ij} is anti-symmetric iff

$$R_{ij} = -R_{ji}. \tag{1.59}$$

Note that an anti-symmetric tensor must have zeroes on its diagonal, and only three independent scalars on off-diagonal elements. We will see that R is associated with the rotation of a fluid element.

1.2.1.4.3.4 Decomposition An arbitrary tensor T_{ij} can be separated into a symmetric and anti-symmetric pair of tensors:

$$T_{ij} = \frac{1}{2}T_{ij} + \frac{1}{2}T_{ij} + \frac{1}{2}T_{ji} - \frac{1}{2}T_{ji}. \tag{1.60}$$

Rearranging, we get

$$T_{ij} = \underbrace{\frac{1}{2}\left(T_{ij} + T_{ji}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(T_{ij} - T_{ji}\right)}_{\text{anti-symmetric}}. \tag{1.61}$$

The first term must be symmetric, and the second term must be anti-symmetric. This is easily seen by considering applying this to any matrix of actual numbers. If we define the symmetric part of the matrix T_{ij} by the following notation

$$T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right), \tag{1.62}$$

and the anti-symmetric part of the same matrix by the following notation

$$T_{[ij]} = \frac{1}{2}\left(T_{ij} - T_{ji}\right), \tag{1.63}$$

we then have

$$T_{ij} = T_{(ij)} + T_{[ij]}. \tag{1.64}$$

1.2.1.4.4 Tensor inner product The tensor inner product of two tensors T_{ij} and S_{ji} is defined as follows

$$T_{ij}S_{ji} = a, \tag{1.65}$$

where a is a scalar. In Gibbs notation, we would say

$$\mathsf{T} : \mathsf{S} = a. \tag{1.66}$$

It is easily shown, and will be important in upcoming derivations, that the tensor inner product of any symmetric tensor D with any anti-symmetric tensor R is the scalar *zero*:

$$D_{ij}R_{ji} = 0, \tag{1.67}$$

$$\mathsf{D} : \mathsf{R} = 0. \tag{1.68}$$

Further, if we decompose a tensor into its symmetric and anti-symmetric parts, $T_{ij} = T_{(ij)} + T_{[ij]}$ and take $T_{(ij)} = D_{ij} = \mathsf{D}$ and $T_{[ij]} = R_{ij} = \mathsf{R}$, so that $\mathsf{T} = \mathsf{D} + \mathsf{R}$, we note the following common term can be expressed as a tensor inner product with a dyadic product:

$$x_iT_{ij}x_j = \mathbf{x}^T \cdot \mathsf{T} \cdot \mathbf{x}, \tag{1.69}$$

$$x_i(T_{(ij)} + T_{[ij]})x_j = \mathbf{x}^T \cdot (\mathsf{D} + \mathsf{R}) \cdot \mathbf{x}, \tag{1.70}$$

$$x_iT_{(ij)}x_j = \mathbf{x}^T \cdot \mathsf{D} \cdot \mathbf{x}, \tag{1.71}$$

$$T_{(ij)}x_ix_j = \mathsf{D} : \mathbf{x}\mathbf{x}^T. \tag{1.72}$$

1.2.1.4.5 Dual vector of a tensor We define the dual vector, d_i, of a tensor T_{jk} as follows[12]

$$d_i = \frac{1}{2}\epsilon_{ijk}T_{jk} = \frac{1}{2}\underbrace{\epsilon_{ijk}T_{(jk)}}_{=0} + \frac{1}{2}\epsilon_{ijk}T_{[jk]}. \tag{1.73}$$

The term ϵ_{ijk} is anti-symmetric for any fixed i; thus when its tensor inner product is taken with the symmetric $T_{(jk)}$, the result must be the scalar zero. Hence, we also have

$$d_i = \frac{1}{2}\epsilon_{ijk}T_{[jk]}. \tag{1.74}$$

[12]There is a lack of uniformity in the literature in this area. First, note this definition differs from that given by Panton by a factor of 1/2. It is closer, but not identical, to the approach found in Aris, p. 25.

Let's find the inverse relation for d_i, Starting with Eq. (1.73), we take the inner product of d_i with ϵ_{ilm} to get

$$\epsilon_{ilm} d_i = \frac{1}{2} \epsilon_{ilm} \epsilon_{ijk} T_{jk}. \tag{1.75}$$

Employing Eq. (1.54) to eliminate the ϵ's in favor of δ's, we get

$$\begin{aligned} \epsilon_{ilm} d_i &= \frac{1}{2} \left(\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj} \right) T_{jk}, && (1.76) \\ &= \frac{1}{2} (T_{lm} - T_{ml}), && (1.77) \\ &= T_{[lm]}. && (1.78) \end{aligned}$$

Hence,

$$T_{[lm]} = \epsilon_{ilm} d_i. \tag{1.79}$$

Note that

$$T_{[lm]} = \epsilon_{1lm} d_1 + \epsilon_{2lm} d_2 + \epsilon_{3lm} d_3 = \begin{pmatrix} 0 & d_3 & -d_2 \\ -d_3 & 0 & d_1 \\ d_2 & -d_1 & 0 \end{pmatrix}. \tag{1.80}$$

And we can write the decomposition of an arbitrary tensor as the sum of its symmetric part and a factor related to the dual vector associated with its anti-symmetric part:

$$\underbrace{T_{ij}}_{\text{arbitrary tensor}} = \underbrace{T_{(ij)}}_{\text{symmetric part}} + \underbrace{\epsilon_{kij} d_k}_{\text{anti-symmetric part}}. \tag{1.81}$$

1.2.1.4.6 Tensor product: two tensors The tensor product between two arbitrary tensors yields a third tensor. For second order tensors, we have the tensor product in Cartesian index notation as

$$S_{ij} T_{jk} = P_{ik}. \tag{1.82}$$

Note that j is a dummy index, i and k are free indices, and that the free indices in each additive term are the same. In that sense they behave somewhat as dimensional units, which must be the same for each term. In Gibbs notation, the equivalent tensor product is written as

$$\mathsf{S} \cdot \mathsf{T} = \mathsf{P}. \tag{1.83}$$

Note that in contrast to the tensor inner product, which has two pairs of dummy indices and two dots, the tensor product has one pair of dummy indices and one dot. The tensor product is equivalent to matrix multiplication in matrix algebra.

An important property of tensors is that, in general, the tensor product *does not commute*, $\mathsf{S} \cdot \mathsf{T} \neq \mathsf{T} \cdot \mathsf{S}$. In the most formal manifestation of Cartesian index notation, one should also not commute the elements, and the dummy indices should appear next to another in adjacent terms as above. However, it is of no great consequence to change the order of terms so that we can write $S_{ij} T_{jk} = T_{jk} S_{ij}$. That is in Cartesian index notation, elements do commute.

But, in Cartesian index notation, the order of the indices is extremely important, and it is this order that does not commute: $S_{ij}T_{jk} \neq S_{ji}T_{jk}$ in general. The version presented above $S_{ij}T_{jk}$, in which the dummy index j is juxtaposed between each term, is slightly preferable as it maintains the order we find in the Gibbs notation.

1.2.1.4.7 Vector product: vector and tensor The product of a vector and tensor, again which does not in general commute, comes in two flavors, pre-multiplication and post-multiplication, both important, and given in Cartesian index and Gibbs notation below:

1.2.1.4.7.1 Pre-multiplication

$$u_j = v_i T_{ij} = T_{ij} v_i, \tag{1.84}$$

$$\mathbf{u}^T = \mathbf{v}^T \cdot \mathsf{T} \neq \mathsf{T} \cdot \mathbf{v}. \tag{1.85}$$

In the Cartesian index notation above the first form is preferred as it has a correspondence with the Gibbs notation, but both are correct representations given our summation convention.

1.2.1.4.7.2 Post-multiplication

$$w_i = T_{ij} v_j = v_j T_{ij}, \tag{1.86}$$

$$\mathbf{w} = \mathsf{T} \cdot \mathbf{v} \neq \mathbf{v}^T \cdot \mathsf{T}. \tag{1.87}$$

1.2.1.4.8 Dyadic product: two vectors As opposed to the inner product between two vectors, which yields a scalar, we also have the dyadic product, which yields a tensor. In Cartesian index and Gibbs notation, we have

$$T_{ij} = u_i v_j = v_j u_i, \tag{1.88}$$

$$\mathsf{T} = \mathbf{u}\mathbf{v}^T \neq \mathbf{v}\mathbf{u}^T. \tag{1.89}$$

Notice there is no dot in the dyadic product; the dot is reserved for the inner product.

1.2.1.4.9 Contraction We contract a general tensor, which has all of its subscripts different, by setting one subscript to be the same as the other. A single contraction will reduce the order of a tensor by two. For example the contraction of the second order tensor T_{ij} is T_{ii}, which indicates a sum is to be performed:

$$T_{ii} = T_{11} + T_{22} + T_{33}. \tag{1.90}$$

So, in this case the contraction yields a scalar. In matrix algebra, this particular contraction is the trace of the matrix.

1.2.1.4.10 Vector cross product The vector cross product is defined in Cartesian index and Gibbs notation as

$$w_i = \epsilon_{ijk} u_j v_k, \tag{1.91}$$
$$\mathbf{w} = \mathbf{u} \times \mathbf{v}. \tag{1.92}$$

Expanding for $i = 1, 2, 3$ gives

$$w_1 = \epsilon_{123} u_2 v_3 + \epsilon_{132} u_3 v_2 = u_2 v_3 - u_3 v_2, \tag{1.93}$$
$$w_2 = \epsilon_{231} u_3 v_1 + \epsilon_{213} u_1 v_3 = u_3 v_1 - u_1 v_3, \tag{1.94}$$
$$w_3 = \epsilon_{312} u_1 v_2 + \epsilon_{321} u_2 v_1 = u_1 v_2 - u_2 v_1. \tag{1.95}$$

1.2.1.4.11 Vector associated with a plane We often have to select a vector which is associated with a particular direction. Now for any direction we choose, there exists an associated unit vector and normal plane. Recall that our notation has been defined so that the first index is associated with a face or direction, and the second index corresponds to the components of the vector associated with that face. If we take n_i to be a unit normal vector associated with a given direction and normal plane, and we have been given a tensor T_{ij}, the vector t_j associated with that plane is given in Cartesian index and Gibbs notation by

$$t_j = n_i T_{ij}, \tag{1.96}$$
$$\mathbf{t}^T = \mathbf{n}^T \cdot \mathsf{T}, \tag{1.97}$$
$$\mathbf{t} = \mathsf{T}^T \cdot \mathbf{n}. \tag{1.98}$$

A sketch of a Cartesian element with the tensor components sketched on the proper face is shown in 1.4.

Example 1.5

Find the vector associated with the 1 face, $\mathbf{t}^{(1)}$, as shown in Figure 1.4,

We first choose the unit normal associated with the x_1 face, which is the vector $n_i = (1, 0, 0)^T$. The associated vector is found by doing the actual summation

$$t_j = n_i T_{ij} = n_1 T_{1j} + n_2 T_{2j} + n_3 T_{3j}. \tag{1.99}$$

Now $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$, so for this problem, we have

$$t_j^{(1)} = T_{1j}. \tag{1.100}$$

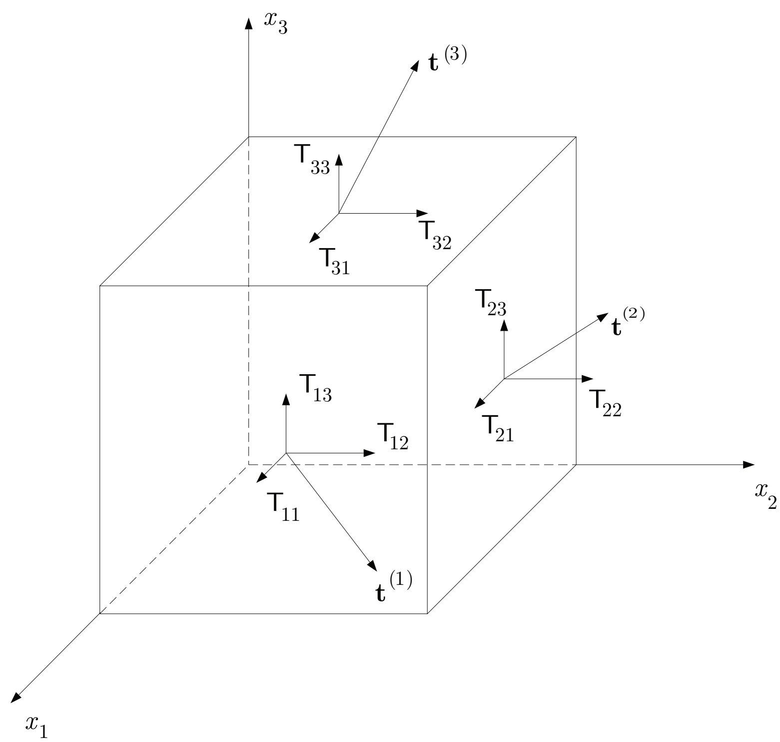

Figure 1.4: Sample Cartesian element which is aligned with coordinate axes, along with tensor components and vectors associated with each face.

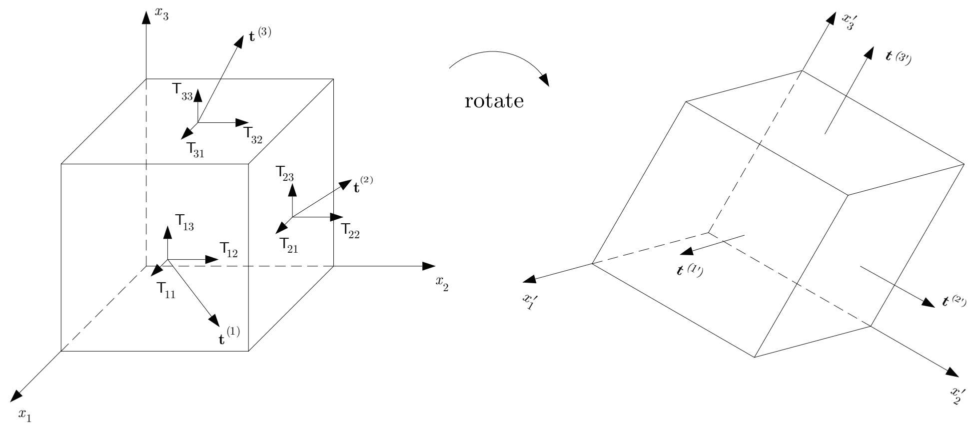

Figure 1.5: Sample Cartesian element which is rotated so that its faces have vectors which are aligned with the unit normals associated with the faces of the element.

1.2.2 Eigenvalues and eigenvectors

For a given tensor T_{ij}, it is possible to select a plane for which the vector from T_{ij} associated with that plane points in the same direction as the normal associated with the chosen plane. In fact for a three dimensional element, it is possible to choose three planes for which the vector associated with the given planes is aligned with the unit normal associated with those planes. We can think of this as finding a rotation as sketched in 1.5.

Mathematically, we can enforce this condition by requiring that

$$\underbrace{n_i T_{ij}}_{\text{vector associated with chosen direction}} = \underbrace{\lambda n_j}_{\text{scalar multiple of chosen direction}} . \tag{1.101}$$

Here λ is an as of yet unknown scalar. The vector n_i could be a unit vector, but does not have to be. We can rewrite this as

$$n_i T_{ij} = \lambda n_i \delta_{ij}. \tag{1.102}$$

In Gibbs notation, this becomes $\mathbf{n}^T \cdot \mathsf{T} = \lambda \mathbf{n}^T \cdot \mathsf{I}$. In mathematics, this is known as a left eigenvalue problem. Solutions n_i which are non-trivial are known as left eigenvectors. We can also formulate this as a right eigenvalue problem by taking the transpose of both sides to obtain $\mathsf{T}^T \cdot \mathbf{n} = \lambda \mathsf{I} \cdot \mathbf{n}$. Here we have used the fact that $\mathsf{I}^T = \mathsf{I}$. We note that the left eigenvectors of T are the right eigenvectors of T^T. Eigenvalue problems are quite general and arise whenever an operator operates on a vector to generate a vector which leaves the original unchanged except in magnitude.

We can rearrange to form

$$n_i \left(T_{ij} - \lambda \delta_{ij}\right) = 0. \tag{1.103}$$

In matrix notation, this can be written as

$$\begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}. \tag{1.104}$$

A trivial solution to this equation is $(n_1, n_2, n_3) = (0, 0, 0)$. But this is not interesting. We get a non-unique, non-trivial solution if we enforce the condition that the determinant of the coefficient matrix be zero. As we have an unknown parameter λ, we have sufficient degrees of freedom to accomplish this. So, we require

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0 \tag{1.105}$$

We know from linear algebra that such an equation for a third order matrix gives rise to a characteristic polynomial for λ of the form[13]

$$\lambda^3 - I_T^{(1)}\lambda^2 + I_T^{(2)}\lambda - I_T^{(3)} = 0, \tag{1.106}$$

where $I_T^{(1)}, I_T^{(2)}, I_T^{(3)}$ are scalars which are functions of all the scalars T_{ij}. The I_T's are known as the *invariants* of the tensor T_{ij}. They can be shown to be given by[14]

$$I_T^{(1)} = T_{ii} = \text{tr } \mathsf{T}, \tag{1.107}$$

$$I_T^{(2)} = \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ji}\right) = \frac{1}{2}\left((\text{tr } \mathsf{T})^2 - \text{tr } (\mathsf{T}\cdot\mathsf{T})\right) = (\det \mathsf{T})\left(\text{tr } \mathsf{T}^{-1}\right), \tag{1.108}$$

$$= \frac{1}{2}\left(T_{(ii)}T_{(jj)} + T_{[ij]}T_{[ij]} - T_{(ij)}T_{(ij)}\right), \tag{1.109}$$

$$I_T^{(3)} = \epsilon_{ijk}T_{1i}T_{2j}T_{3k} = \det \mathsf{T}. \tag{1.110}$$

Here "det" denotes the determinant. It can also be shown that if $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ are the three eigenvalues, then the invariants can also be expressed as

$$I_T^{(1)} = \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)}, \tag{1.111}$$

$$I_T^{(2)} = \lambda^{(1)}\lambda^{(2)} + \lambda^{(2)}\lambda^{(3)} + \lambda^{(3)}\lambda^{(1)}, \tag{1.112}$$

$$I_T^{(3)} = \lambda^{(1)}\lambda^{(2)}\lambda^{(3)}. \tag{1.113}$$

In general these eigenvalues, and consequently, the eigenvectors are complex. Additionally, in general the eigenvectors are non-orthogonal. If, however, the matrix we are considering is symmetric, which is often the case in fluid mechanics, it can be formally proven that

[13]We employ a slightly more common form here than the very similar Eq. (3.10.4) of Panton.

[14]Note the obvious error in the third of Panton's Eq. (3.10.5), where the indices j and q appear three times.

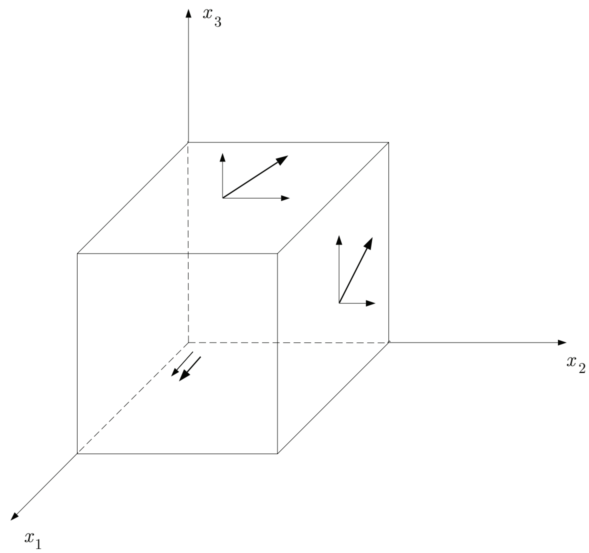

Figure 1.6: Sketch of stresses being applied to a cubical fluid element. The thinner lines with arrows are the components of the stress tensor; the thicker lines on each face represent the vector associated with the particular face.

all the eigenvalues are real and all the eigenvectors are real and orthogonal. If for instance, our tensor is the stress tensor, we will show that it is symmetric in the absence of external couples. The eigenvectors of the stress tensor can form the basis for an intrinsic coordinate system which has its axes aligned with the principal stress on a fluid element. The eigenvalues themselves give the value of the principal stress. This is actually a generalization of the familiar Mohr's circle from solid mechanics.

Example 1.6

Find the principal axes and principal values of stress if the stress tensor is

$$T_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}. \tag{1.114}$$

A sketch of these stresses is shown on the fluid element in Figure 1.6. We take the eigenvalue problem

$$n_i T_{ij} = \lambda n_j, \tag{1.115}$$

$$= \lambda n_i \delta_{ij}, \tag{1.116}$$

$$n_i (T_{ij} - \lambda \delta_{ij}) = 0. \tag{1.117}$$

This becomes for our problem

$$\begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}. \tag{1.118}$$

For a non-trivial solution for n_i, we must have

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0. \tag{1.119}$$

This gives rise to the polynomial equation

$$(1-\lambda)\left((1-\lambda)(1-\lambda)-4\right) = 0. \tag{1.120}$$

This has three solutions

$$\lambda = 1, \qquad \lambda = -1, \qquad \lambda = 3. \tag{1.121}$$

Notice all eigenvalues are real, which we expect since the tensor is symmetric.

Now let's find the eigenvectors (aligned with the principal axes of stress) for this problem First, it can easily be shown that when the vector product of a vector with a tensor commutes when the tensor is symmetric. Although this is not a crucial step, we will use it to write the eigenvalue problem in a slightly more familiar notation:

$$n_i\,(T_{ij}-\lambda\delta_{ij}) = 0 \Longrightarrow (T_{ij}-\lambda\delta_{ij})\,n_i = 0, \qquad \text{because scalar components commute.} \tag{1.122}$$

Because of symmetry, we can now commute the indices to get

$$(T_{ji}-\lambda\delta_{ji})\,n_i = 0, \text{because indices commute if symmetric.} \tag{1.123}$$

Expanding into matrix notation, we get

$$\begin{pmatrix} T_{11}-\lambda & T_{21} & T_{31} \\ T_{12} & T_{22}-\lambda & T_{32} \\ T_{13} & T_{23} & T_{33}-\lambda \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.124}$$

Note, we have taken the transpose of T in the above equation. Substituting for T_{ji} and considering the eigenvalue $\lambda = 1$, we get

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.125}$$

We get two equations $2n_2 = 0$, and $2n_3 = 0$, which require that $n_2 = n_3 = 0$. We can satisfy all equations with an arbitrary value of n_1. It is always the case that an eigenvector will have an arbitrary magnitude and a well-defined direction. Here we will choose to normalize our eigenvector and take $n_1 = 1$, so that the eigenvector is

$$n_j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{for} \qquad \lambda = 1. \tag{1.126}$$

Note, geometrically this means that the original 1 face already has an associated vector which is aligned with its normal vector.

Now consider the eigenvector associated with the eigenvalue $\lambda = -1$. Again substituting into the original equation, we get

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.127}$$

This is simply the system of equations

$$2n_1 = 0, \tag{1.128}$$
$$2n_2 + 2n_3 = 0, \tag{1.129}$$
$$2n_2 + 2n_3 = 0. \tag{1.130}$$

Clearly $n_1 = 0$. We could take $n_2 = 1$ and $n_3 = -1$ for a non-trivial solution. Alternatively, let's normalize and take

$$n_j = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}. \tag{1.131}$$

Finally consider the eigenvector associated with the eigenvalue $\lambda = 3$. Again substituting into the original equation, we get

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.132}$$

This is the system of equations

$$-2n_1 = 0, \tag{1.133}$$
$$-2n_2 + 2n_3 = 0, \tag{1.134}$$
$$2n_2 - 2n_3 = 0. \tag{1.135}$$

Clearly again $n_1 = 0$. We could take $n_2 = 1$ and $n_3 = 1$ for a non-trivial solution. Once again, let's normalize and take

$$n_j = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}. \tag{1.136}$$

In summary, the three eigenvectors and associated eigenvalues are

$$n_j^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \quad \lambda^{(1)} = 1, \tag{1.137}$$
$$n_j^{(2)} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} \end{pmatrix} \quad \text{for} \quad \lambda^{(2)} = -1, \tag{1.138}$$
$$n_j^{(3)} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \quad \text{for} \quad \lambda^{(3)} = 3. \tag{1.139}$$

Note that the eigenvectors are mutually orthogonal, as well as normal. We say they form an orthonormal set of vectors. Their orthogonality, as well as the fact that all the eigenvalues are real can be shown to be a direct consequence of the symmetry of the original tensor. A sketch of the principal stresses on the element rotated so that it is aligned with the principal axes of stress is shown on the fluid element in Figure 1.7.

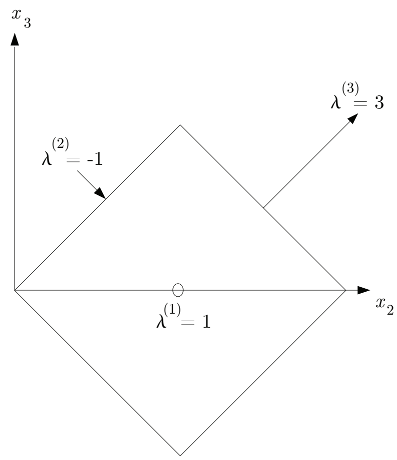

Figure 1.7: Sketch of fluid element rotated to be aligned with axes of principal stress, along with magnitude of principal stress. The 1 face projects out of the page.

Example 1.7

For a given stress tensor, which we will take to be symmetric though the theory applies to non-symmetric tensors as well,

$$T_{ij} = \mathsf{T} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix}, \tag{1.140}$$

find the three basic tensor invariants of stress $I_T^{(1)}$, $I_T^{(2)}$, and $I_T^{(3)}$, and show they are truly invariant when the tensor is subjected to a rotation with direction cosine matrix of

$$\ell_{ij} = \mathsf{Q} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.141}$$

Calculation reveals that

$$\det \mathsf{Q} = 1, \tag{1.142}$$

and that $\mathsf{Q} \cdot \mathsf{Q}^T = \mathsf{I}$, so that Q is a rotation matrix.

The eigenvalues of T, which are the principal values of stress are easily calculated to be

$$\lambda^{(1)} = 5.28675, \qquad \lambda^{(2)} = -3.67956, \qquad \lambda^{(3)} = 3.39281. \tag{1.143}$$

The three invariants of T_{ij} are

$$I_T^{(1)} = \text{tr } \mathsf{T} = \text{tr} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{pmatrix} = 1 + 3 + 1 = 5, \tag{1.144}$$

$$I_T^{(2)} = \frac{1}{2}\left((\text{tr } \mathsf{T})^2 - \text{tr } (\mathsf{T} \cdot \mathsf{T})\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\left(\text{tr}\begin{pmatrix}1 & 2 & 4\\ 2 & 3 & -1\\ 4 & -1 & 1\end{pmatrix}\right)^2 - \text{tr}\left(\begin{pmatrix}1 & 2 & 4\\ 2 & 3 & -1\\ 4 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 4\\ 2 & 3 & -1\\ 4 & -1 & 1\end{pmatrix}\right)\right),\\
&= \frac{1}{2}\left(5^2 - \text{tr}\begin{pmatrix}21 & 4 & 6\\ 4 & 14 & 4\\ 6 & 4 & 18\end{pmatrix}\right),\\
&= \frac{1}{2}(25-21-14-18),\\
&= -14, \qquad (1.145)\\
I_T^{(3)} &= \det \mathsf{T} = \det\begin{pmatrix}1 & 2 & 4\\ 2 & 3 & -1\\ 4 & -1 & 1\end{pmatrix} = -66. \qquad (1.146)
\end{aligned}
$$

Now when we rotate the tensor T, we get a transformed tensor given by

$$
\begin{aligned}
\mathsf{T}' = \mathsf{Q}^T\cdot\mathsf{T}\cdot\mathsf{Q} &= \begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1 & 2 & 4\\ 2 & 3 & -1\\ 4 & -1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{pmatrix},\\
&= \begin{pmatrix}4.10238 & 2.52239 & 1.60948\\ 2.52239 & -0.218951 & -2.91291\\ 1.60948 & -2.91291 & 1.11657\end{pmatrix}. \qquad (1.147)
\end{aligned}
$$

We then seek the tensor invariants of T'. Leaving out some of the details, which are the same as those for calculating the invariants of the T, we find the invariants indeed are invariant:

$$
\begin{aligned}
I_T^{(1)} &= 4.10238 - 0.218951 + 1.11657 = 5, \qquad (1.148)\\
I_T^{(2)} &= \frac{1}{2}(5^2 - 53) = -14, \qquad (1.149)\\
I_T^{(3)} &= -66. \qquad (1.150)
\end{aligned}
$$

Finally, we verify that the stress invariants are indeed related to the principal values (the eigenvalues of the stress tensor) as follows

$$
\begin{aligned}
I_T^{(1)} &= \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} = 5.28675 - 3.67956 + 3.39281 = 5, \qquad (1.151)\\
I_T^{(2)} &= \lambda^{(1)}\lambda^{(2)} + \lambda^{(2)}\lambda^{(3)} + \lambda^{(3)}\lambda^{(1)},\\
&= (5.28675)(-3.67956) + (-3.67956)(3.39281) + (3.39281)(5.28675) = -14, \qquad (1.152)\\
I_T^{(3)} &= \lambda^{(1)}\lambda^{(2)}\lambda^{(3)} = (5.28675)(-3.67956)(3.39281) = -66. \qquad (1.153)
\end{aligned}
$$

1.2.3 Grad, div, curl, etc.

Thus far, we have mainly dealt with the algebra of vectors and tensors. Now let us consider the calculus. For now, let us consider variables which are a function of the spatial vector x_i. We shall soon allow variation with time t also. We will typically encounter quantities such as

- $\phi(x_i) \rightarrow$ a scalar function of the position vector,
- $v_j(x_i) \rightarrow$ a vector function of the position vector, or
- $T_{jk}(x_i) \rightarrow$ a tensor function of the position vector.

1.2.3.1 Gradient operator

The gradient operator, sometimes denoted by "grad," is motivated as follows. Consider $\phi(x_i)$, which when written in full is

$$\phi(x_i) = \phi(x_1, x_2, x_3). \tag{1.154}$$

Taking a derivative using the chain rule gives

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3. \tag{1.155}$$

Following Panton, we define a non-traditional, but useful further notation ∂_i for the partial derivative

$$\partial_i \equiv \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \frac{\partial}{\partial x_3}\mathbf{e}_3 = \nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}, \tag{1.156}$$

so that the chain rule is actually

$$d\phi = \partial_1 \phi \; dx_1 + \partial_2 \phi \; dx_2 + \partial_3 \phi \; dx_3, \tag{1.157}$$

which is written using our summation convention as

$$d\phi = \partial_i \phi \; dx_i. \tag{1.158}$$

After commuting so as to juxtapose the i subscript, we have

$$d\phi = dx_i \; \partial_i \phi. \tag{1.159}$$

In Gibbs notation, we say

$$d\phi = d\mathbf{x}^T \cdot \nabla \phi = d\mathbf{x}^T \cdot \text{grad } \phi. \tag{1.160}$$

We can also take the transpose of both sides, recalling that the transpose of a scalar is the scalar itself, to obtain

$$(d\phi)^T = \left(d\mathbf{x}^T \cdot \nabla \phi\right)^T, \tag{1.161}$$

$$d\phi = (\nabla \phi)^T \cdot d\mathbf{x}, \tag{1.162}$$

$$d\phi = \nabla^T \phi \cdot d\mathbf{x}. \tag{1.163}$$

Here we expand ∇^T as $\nabla^T = (\partial_1, \partial_2, \partial_3)$. When ∂_i or ∇ operates on a scalar, it is known as the gradient operator. The gradient operator operating on a scalar function gives rise to a vector function.

We next describe the gradient operator operating on a vector. For vectors in Cartesian index and Gibbs notation, we have, following a similar analysis[15]

$$\begin{aligned} dv_i &= dx_j \partial_j v_i = \partial_j v_i \; dx_j, \\ d\mathbf{v}^T &= d\mathbf{x}^T \cdot \nabla \mathbf{v}^T, \\ d\mathbf{v} &= (\nabla \mathbf{v}^T)^T \cdot d\mathbf{x}, \\ d\mathbf{v} &= (\text{grad } \mathbf{v})^T \cdot d\mathbf{x}. \end{aligned} \tag{1.164}$$

Here the quantity $\partial_j v_i$ is the gradient of a vector, which is a tensor. So the gradient operator operating on a vector raises its order by one. Note that the Gibbs notation with transposes suggests properly that the gradient of a vector can be expanded as

$$\nabla \mathbf{v}^T = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} \partial_1 v_1 & \partial_1 v_2 & \partial_1 v_3 \\ \partial_2 v_1 & \partial_2 v_2 & \partial_2 v_3 \\ \partial_3 v_1 & \partial_3 v_2 & \partial_3 v_3 \end{pmatrix}. \tag{1.165}$$

Lastly we consider the gradient operator operating on a tensor. For tensors in Cartesian index notation, we have, following a similar analysis

$$dT_{ij} = dx_k \partial_k T_{ij} = \partial_k T_{ij} \; dx_k, \tag{1.166}$$

Here the quantity $\partial_k T_{ij}$ is a third order tensor. So the gradient operator operating on a tensor raises its order by one as well. The Gibbs notation is not straightforward as it can involve something akin to the transpose of a three-dimensional matrix.

1.2.3.2 Divergence operator

The contraction of the gradient operator on either a vector or a tensor is known as the divergence, sometimes denoted by "div." For the divergence of a vector, we have

$$\partial_i v_i = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 = \nabla^T \cdot \mathbf{v} = \text{div } \mathbf{v}. \tag{1.167}$$

The divergence of a vector is a scalar.

For the divergence of a second order tensor, we have

$$\partial_i T_{ij} = \partial_1 T_{1j} + \partial_2 T_{2j} + \partial_3 T_{3j} = \nabla^T \cdot \mathsf{T} = \text{div } \mathsf{T}. \tag{1.168}$$

[15] A more common approach, not using the transpose notation, would be to say here for the Gibbs notation that $d\mathbf{v} = d\mathbf{x} \cdot \nabla \mathbf{v}$. However, this is only works if we consider $d\mathbf{v}$ to be a row vector, as $d\mathbf{x} \cdot \nabla \mathbf{v}$ is a row vector. All in all, while at times clumsy, the transpose notation allows a for great deal of clarity and consistency with matrix algebra.

The divergence operator operating on a tensor gives rise to a row vector. We will sometimes have to transpose this row vector in order to arrive at a column vector, e.g. we will have need for the column vector $\left(\nabla^T \cdot \mathsf{T}\right)^T$. We note that, as with the vector inner product, most texts assume the transpose operation is understood and write the divergence of a vector or tensor simply as $\nabla \cdot \mathbf{v}$ or $\nabla \cdot \mathsf{T}$.

1.2.3.3 Curl operator

The curl operator is the derivative analog to the cross product. We write it in the following three ways:

$$\begin{aligned} \omega_i &= \epsilon_{ijk}\partial_j v_k, \\ \boldsymbol{\omega} &= \nabla \times \mathbf{v}, \\ \boldsymbol{\omega} &= \text{curl } \mathbf{v}. \end{aligned} \tag{1.169}$$

Expanding for $i = 1, 2, 3$ gives

$$\begin{aligned} \omega_1 &= \epsilon_{123}\partial_2 v_3 + \epsilon_{132}\partial_3 v_2 = \partial_2 v_3 - \partial_3 v_2, \\ \omega_2 &= \epsilon_{231}\partial_3 v_1 + \epsilon_{213}\partial_1 v_3 = \partial_3 v_1 - \partial_1 v_3, \\ \omega_3 &= \epsilon_{312}\partial_1 v_2 + \epsilon_{321}\partial_2 v_1 = \partial_1 v_2 - \partial_2 v_1. \end{aligned} \tag{1.170}$$

1.2.3.4 Laplacian operator

The Laplacian[16] operator can operate on a scalar, vector, or tensor function. It is a simple combination of first the gradient followed by the divergence. It yields a function of the same order as that which it operates on. For its most common operation on a scalar, it is denoted by as follows

$$\partial_i\partial_i\phi = \nabla^T \cdot \nabla\ \phi = \nabla^2\phi = \text{div grad } \phi. \tag{1.171}$$

In viscous fluid flow, we will have occasion to have the Laplacian operate on vector:

$$\partial_i\partial_i v_j = \left(\nabla^T \cdot \nabla\ \mathbf{v}^T\right)^T = \left(\nabla^2\mathbf{v}^T\right)^T = \nabla^2\mathbf{v} = \text{div grad } \mathbf{v}. \tag{1.172}$$

1.2.3.5 Relevant theorems

We will use several theorems which are developed in vector calculus. Here we give the simplest of motivations, and simply present them. The reader should consult a standard mathematics text for detailed derivations.

[16] Pierre-Simon Laplace, 1749-1827, Normandy-born French astronomer of humble origin. Educated at Caen, taught in Paris at École Militaire.

1.2.3.5.1 Fundamental theorem of calculus The fundamental theorem of calculus is as follows

$$\int_{x=a}^{x=b} f(x)\ dx = \int_{x=a}^{x=b} \left(\frac{d\phi}{dx}\right)\ dx = \phi(b) - \phi(a). \tag{1.173}$$

It effectively says that to find the integral of a function $f(x)$, which is the area under the curve, it suffices to find a function ϕ, whose derivative is f, and evaluate ϕ at each endpoint, and take the difference to find the area under the curve.

1.2.3.5.2 Gauss's theorem Gauss's[17] theorem is the analog of the fundamental theorem of calculus extended to volume integrals. It applies to tensor functions of arbitrary order and is as follows:

$$\int_R \partial_i \left(T_{jk\dots}\right)\ dV = \int_S n_i T_{jk\dots}\ dS \tag{1.174}$$

Here R is an arbitrary volume, dV is the element of volume, S is the surface that bounds V, n_i is the outward unit normal to S, and $T_{jk..}$ is an arbitrary tensor function. The surface integral is analogous to evaluating the function at the end points in the fundamental theorem of calculus.

Note if we take $T_{jk\dots}$ to be the scalar of unity (whose derivative must be zero), Gauss's theorem reduces to

$$\int_S n_i\ dS = 0. \tag{1.175}$$

That is the unit normal to the surface integrated over the surface, cancels to zero when the entire surface is included.

We will use Gauss's theorem extensively. It allows us to convert sometimes difficult volume integrals into easier interpreted surface integrals. It is often useful to use this theorem as a means of toggling back and forth from one form to another.

1.2.3.5.3 Stokes' theorem Stokes'[18] theorem is as follows.

$$\int_S n_i \epsilon_{ijk} \partial_j v_k\ dS = \oint_C \alpha_i v_i\ ds. \tag{1.176}$$

Once again S is a bounding surface and n_i is its outward unit normal. The integral with the circle through it denotes a closed contour integral with respect to arc length s, and α_i is the unit tangent vector to the bounding curve C.

In Gibbs notation, it is written as

$$\int_S \mathbf{n}^T \cdot \nabla \times \mathbf{v}\ dS = \oint_C \boldsymbol{\alpha}^T \cdot \mathbf{v}\ ds. \tag{1.177}$$

[17] Carl Friedrich Gauss, 1777-1855, Brunswick-born German mathematician, considered the founder of modern mathematics. Worked in astronomy, physics, crystallography, optics, biostatistics, and mechanics. Studied and taught at Göttingen.

[18] Sir George Gabriel Stokes, 1819-1903, Irish-born British physicist and mathematician, holder of the Lucasian chair of Mathematics at Cambridge University, developed, simultaneously with Navier, the governing equations of fluid motion, in a form which was more robust than that of Navier.

1.2.3.5.4 Kinetic energy divergence identity It is easy to show that a useful identity involving the divergence of specific kinetic energy holds:

$$v_j \partial_j v_i = \partial_i \left(\frac{1}{2} v_j v_j\right) - \epsilon_{ijk} v_j \omega_k, \tag{1.178}$$

$$\left(\mathbf{v}^T \cdot \nabla\right) \mathbf{v} = \nabla \left(\frac{1}{2} \mathbf{v}^T \cdot \mathbf{v}\right) - \mathbf{v} \times \boldsymbol{\omega}. \tag{1.179}$$

This is easily proved by considering the right hand side of Eq. (1.178), expanding, and using Eqs. (1.169) and then (1.54):

$$\partial_i \left(\frac{1}{2} v_j v_j\right) - \epsilon_{ijk} v_j \omega_k = v_j \partial_i v_j - \epsilon_{ijk} v_j \underbrace{\epsilon_{klm} \partial_l v_m}_{=\omega_k}, \tag{1.180}$$

$$= v_j \partial_i v_j - \epsilon_{kij} \epsilon_{klm} v_j \partial_l v_m, \tag{1.181}$$

$$= v_j \partial_i v_j - \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\right) v_j \partial_l v_m, \tag{1.182}$$

$$= \underbrace{v_j \partial_i v_j - v_j \partial_i v_j}_{=0} + v_j \partial_j v_i, \tag{1.183}$$

$$= v_j \partial_j v_i, \qquad \text{QED}. \tag{1.184}$$

1.2.3.5.5 Leibniz's rule Leibniz's[19] rule relates time derivatives of integral quantities to a form which distinguishes changes which are happening within the boundaries to changes due to fluxes through boundaries. It is a generalization of the more familiar control volume approach which uses the Reynolds[20] transport theorem. Leibniz's rule applied to an arbitrary tensorial function is as follows:

$$\frac{d}{dt} \int_{R(t)} T_{jk...}(x_i, t) \; dV = \int_{R(t)} \frac{\partial T_{jk...}}{\partial t} \; dV + \int_{S(t)} n_l w_l T_{jk...} \; dS. \tag{1.185}$$

- $R(t) \rightarrow$ arbitrary moving volume,
- $S(t) \rightarrow$ bounding surface of the arbitrary moving volume,
- $w_l \rightarrow$ velocity vector of points on the moving surface,
- $n_l \rightarrow$ unit normal to moving surface.

[19] Gottfried Wilhelm von Leibniz, 1646-1716, Leipzig-born German philosopher and mathematician. Invented calculus independent of Newton and employed a superior notation to that of Newton.

[20] Osborne Reynolds, 1842-1912, Belfast-born British engineer and physicist, educated in mathematics at Cambridge, first professor of engineering at Owens College, Manchester, did fundamental experimental work in fluid mechanics and heat transfer.

Say we have the very special case in which $T_{jk...} = 1$; then Leibniz's rule reduces to

$$\frac{d}{dt}\int_{R(t)} dV = \int_{R(t)} \frac{\partial}{\partial t}(1)\, dV + \int_{S(t)} n_k w_k(1)\, dS, \tag{1.186}$$

$$\frac{dV_R}{dt} = \int_{S(t)} n_k w_k\, dS. \tag{1.187}$$

This simply says the total volume of the region, which we call V_R, changes in response to net motion of the bounding surface.

1.2.3.5.6 Reynolds transport theorem Leibniz's rule reduces to the Reynolds transport theorem if we replace the tensor function $T_{jk...}$ with a scalar function, say f. Further, considering one-dimensional cases only, we can then say

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\, dx = \int_{x=a(t)}^{x=b(t)} \frac{\partial f}{\partial t}\, dx + \frac{db}{dt} f(b(t),t) - \frac{da}{dt} f(a(t),t). \tag{1.188}$$

As in the fundamental theorem of calculus, for the one-dimensional case, we do not have to evaluate a surface integral; instead, we simply must consider the function at its endpoints. Here db/dt and da/dt are the velocities of the bounding surface and analogous to w_k. The terms $f(b(t),t)$ and $f(a(t),t)$ are equivalent to evaluating $T_{jk..}$ on $S(t)$.

1.3 Kinematics

The previous section was in many ways a discussion of geometry or place. Here we will consider kinematics, the study of motion in space. Here we will pay no regard to what causes the motion. If we knew the position of every fluid particle as a function of time, then we could in principle also describe the velocity and acceleration of each particle. We could also make statements about how groups of particles translate, rotate, and deform. This is the essence of kinematics.

Fluid motion is generally a highly non-linear event. In this section, we will develop tools, using a local linear analysis, to break down the most complex fluid flows to a summation of fundamental motions.

1.3.1 Lagrangian description

A Lagrangian[21] description is similar to a classical description of motion in that each fluid particle is effectively labeled and tracked in terms of its initial position x_j^o and time $\hat{t}$. We take the position vector of a particle r_i to be

$$r_i = \tilde{r}_i(x_j^o, \hat{t}). \tag{1.189}$$

[21] Joseph-Louis Lagrange (originally Giuseppe Luigi Lagrangia), 1736-1813, Italian born, Italian-French mathematician. Worked on celestial mechanics and the three body problem. Worked in Berlin and Paris. Part of the committee which formulated the metric system.

The velocity v_i of a particular particle is the time derivative of its position, holding x_j^o fixed:

$$v_i = \left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o} \tag{1.190}$$

The acceleration a_i of a particular particle is the second time derivative of its position, holding x_j^o fixed:

$$a_i = \left.\frac{\partial^2 \tilde{r}_i}{\partial \hat{t}^2}\right|_{x_j^o} \tag{1.191}$$

We can also write other variables as functions of time and initial position, for example, we could have for pressure $p(x_j^o, \hat{t})$.

The Lagrangian description has important pedagogical value, but is only occasionally used in practice, except maybe where it can be useful to illustrate a particular point. In solid mechanics, it is often critically important to know the location of each solid element, and it is the method of choice.

1.3.2 Eulerian description

It is more common in fluid mechanics to use the Eulerian description of fluid motion. In this description, all variables are taken to be functions of time and *local* position, rather than initial position. Here, we will take the local position to be given by the position vector $x_i = r_i$. The transformation from Lagrangian coordinates to Eulerian coordinates is given by

$$\begin{aligned} x_i &= \tilde{r}_i(x_j^o, \hat{t}), \\ t &= \hat{t}. \end{aligned} \tag{1.192}$$

1.3.3 Material derivatives

The material derivative is the derivative following a fluid particle. It is also known as the substantial derivative or the total derivative. It is trivial in Lagrangian coordinates, since by definition, a Lagrangian description tracks a fluid particle. It is not as straightforward in the Eulerian viewpoint.

Consider a fluid property such as temperature or pressure, which we will call F here, which is function of position and time. We can characterize the position and time in either an Eulerian or Lagrangian fashion. Let the Lagrangian representation be $F = F_L(x_j^o, \hat{t})$ and the Eulerian representation be $F = F_E(x_i, t)$. Now both formulations must give the same result at the same time and position; applying our transformation between the two systems thus yields

$$F = F_L(x_j^o, \hat{t}) = F_E(x_i = \tilde{r}_i(x_j^o, \hat{t}), t = \hat{t}). \tag{1.193}$$

Now from basic calculus we have

$$dx_i = \left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o} d\hat{t} + \left.\frac{\partial \tilde{r}_i}{\partial x_j^o}\right|_{\hat{t}} dx_j^o. \tag{1.194}$$

From basic calculus, we also have

$$dF_L = \left.\frac{\partial F_L}{\partial \hat{t}}\right|_{x_j^o} d\hat{t} + \left.\frac{\partial F_L}{\partial x_j^o}\right|_{\hat{t}} dx_j^o, \tag{1.195}$$

$$dF_E = \left.\frac{\partial F_E}{\partial t}\right|_{x_i} dt + \left.\frac{\partial F_E}{\partial x_i}\right|_{t} dx_i. \tag{1.196}$$

Now, we must have $dF = dF_L = dF_E$ for the same fluid particle, so making substitutions from above, we get

$$\left.\frac{\partial F_L}{\partial \hat{t}}\right|_{x_j^o} d\hat{t} + \left.\frac{\partial F_L}{\partial x_j^o}\right|_{\hat{t}} dx_j^o = \left.\frac{\partial F_E}{\partial t}\right|_{x_i} dt + \left.\frac{\partial F_E}{\partial x_i}\right|_{t} \left(\left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o} d\hat{t} + \left.\frac{\partial \tilde{r}_i}{\partial x_j^o}\right|_{\hat{t}} dx_j^o \right). \tag{1.197}$$

For the variation of F of a particular particle, we hold x_j^o fixed, so that $dx_j^o = 0$. Using also the fact that $\hat{t} = t$, so $d\hat{t} = dt$, and dividing by $d\hat{t}$, we get

$$\left.\frac{\partial F_L}{\partial \hat{t}}\right|_{x_j^o} = \left.\frac{\partial F_E}{\partial t}\right|_{x_i} + \left.\frac{\partial F_E}{\partial x_i}\right|_{t} \left.\frac{\partial \tilde{r}_i}{\partial \hat{t}}\right|_{x_j^o}, \tag{1.198}$$

and using the definition of fluid particle velocity, Eq. (1.190), we get

$$\left.\frac{\partial F_L}{\partial \hat{t}}\right|_{x_j^o} = \left.\frac{\partial F_E}{\partial t}\right|_{x_i} + v_i \left.\frac{\partial F_E}{\partial x_i}\right|_{t}. \tag{1.199}$$

Ignoring the operand F, F_L, and F_E, we can write the derivative following a particle in the following manner as an operator

$$\left.\frac{\partial}{\partial \hat{t}}\right|_{x_j^o} = \left.\frac{\partial}{\partial t}\right|_{x_i} + v_i \left.\frac{\partial}{\partial x_i}\right|_{t} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \mathbf{v}^T \cdot \nabla = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \mathbf{v}^T \cdot \text{grad} \equiv \frac{D}{Dt} \equiv \frac{d}{dt} \tag{1.200}$$

We will generally use the following shorthand for the derivative following a particle:

$$\frac{d}{dt} = \partial_o + v_i \partial_i. \tag{1.201}$$

Here a second shorthand for the partial derivative with respect to time has been introduced: $\partial_o \equiv \partial/\partial t|_{x_i}$.

1.3.4 Streamlines

Streamlines are lines which are everywhere instantaneously parallel to velocity vectors. If a differential vector dx_k is parallel to a velocity vector v_j, then the cross product of the two vectors must be zero; hence for a streamline, we must have

$$\epsilon_{ijk} v_j \, dx_k = 0. \tag{1.202}$$

In Gibbs notation, we would say

$$\mathbf{v} \times d\mathbf{x} = \mathbf{0}. \tag{1.203}$$

Recalling that the cross product can be interpreted as a determinant, we get this condition to reduce to

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ dx_1 & dx_2 & dx_3 \end{vmatrix} = \mathbf{0}. \tag{1.204}$$

Expanding the determinant gives

$$\mathbf{e}_1(v_2 dx_3 - v_3 dx_2) + \mathbf{e}_2(v_3 dx_1 - v_1 dx_3) + \mathbf{e}_3(v_1 dx_2 - v_2 dx_1) = \mathbf{0}. \tag{1.205}$$

Since the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are linearly independent, the coefficient on each of them must be zero, giving rise to

$$v_2 dx_3 = v_3 dx_2, \quad \Rightarrow \quad \frac{dx_3}{v_3} = \frac{dx_2}{v_2}, \tag{1.206}$$

$$v_3 dx_1 = v_1 dx_3, \quad \Rightarrow \quad \frac{dx_1}{v_1} = \frac{dx_3}{v_3}, \tag{1.207}$$

$$v_1 dx_2 = v_2 dx_1, \quad \Rightarrow \quad \frac{dx_2}{v_2} = \frac{dx_1}{v_1}. \tag{1.208}$$

$$\tag{1.209}$$

Combining, we get

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}. \tag{1.210}$$

At a fixed instant in time, $t = t_o$, we set the above terms all equal to an arbitrary differential parameter $d\tau$ to obtain

$$\frac{dx_1}{v_1(x_1, x_2, x_3; t = t_o)} = \frac{dx_2}{v_2(x_1, x_2, x_3; t = t_o)} = \frac{dx_3}{v_3(x_1, x_2, x_3; t = t_o)} = d\tau. \tag{1.211}$$

Here τ should not be thought of as time, but just as a dummy parameter. Streamlines are only defined at a fixed time. While they will generally look different at different times, in the process of actually integrating to obtain them, time does not enter into the calculation.

We then divide each equation by $d\tau$ and find the above equations are equivalent to a system of differential equations of the autonomous form

$$\frac{dx_1}{d\tau} = v_1(x_1, x_2, x_3; t = t_o), \qquad x_1(\tau = 0) = x_{1o}, \tag{1.212}$$

$$\frac{dx_2}{d\tau} = v_2(x_1, x_2, x_3; t = t_o), \qquad x_2(\tau = 0) = x_{2o}, \tag{1.213}$$

$$\frac{dx_3}{d\tau} = v_3(x_1, x_2, x_3; t = t_o), \qquad x_3(\tau = 0) = x_{3o}. \tag{1.214}$$

After integration, which in general must be done numerically, we find

$$x_1(\tau; t_o, x_{1o}), \tag{1.215}$$

$$x_2(\tau; t_o, x_{2o}), \tag{1.216}$$

$$x_3(\tau; t_o, x_{3o}), \tag{1.217}$$

where we let the parameter τ vary over whatever domain we choose.

1.3.5 Pathlines

The pathlines are the locus of points traversed by a particular fluid particle. For an Eulerian description of motion where the velocity field is known as a function of space and time $v_j(x_i, t)$, we can get the pathlines by integrating the following set of three non-autonomous ordinary differential equations, with the associated initial conditions:

$$\frac{dx_1}{dt} = v_1(x_1, x_2, x_3, t), \qquad x_1(t = t_o) = x_{1o}, \tag{1.218}$$

$$\frac{dx_2}{dt} = v_2(x_1, x_2, x_3, t), \qquad x_2(t = t_o) = x_{2o}, \tag{1.219}$$

$$\frac{dx_3}{dt} = v_3(x_1, x_2, x_3, t), \qquad x_3(t = t_o) = x_{3o}. \tag{1.220}$$

In general these are non-linear equations, and often require full numerical solution, which gives us

$$x_1(t; x_{1o}), \tag{1.221}$$

$$x_2(t; x_{2o}), \tag{1.222}$$

$$x_3(t; x_{3o}). \tag{1.223}$$

1.3.6 Streaklines

A streakline is the locus of points that have passed through a particular point at some past time $t = \hat{t}$. Streaklines can be found by integrating a similar set of equations to those for pathlines.

$$\frac{dx_1}{dt} = v_1(x_1, x_2, x_3, t), \qquad x_1(t = \hat{t}) = x_{1o}, \tag{1.224}$$
$$\frac{dx_2}{dt} = v_2(x_1, x_2, x_3, t), \qquad x_2(t = \hat{t}) = x_{2o}, \tag{1.225}$$
$$\frac{dx_3}{dt} = v_3(x_1, x_2, x_3, t), \qquad x_3(t = \hat{t}) = x_{3o}. \tag{1.226}$$

After integration, which is generally done numerically, we get

$$x_1(t; x_{1o}, \hat{t}), \tag{1.227}$$
$$x_2(t; x_{2o}, \hat{t}), \tag{1.228}$$
$$x_3(t; x_{3o}, \hat{t}). \tag{1.229}$$

Then, if we fix time t and the particular point in which we are interested $(x_{1o}, x_{2o}, x_{3o})^T$, we get a parametric representation of a streakline

$$x_1(\hat{t}), \tag{1.230}$$
$$x_2(\hat{t}), \tag{1.231}$$
$$x_3(\hat{t}). \tag{1.232}$$

Example 1.8

If $v_1 = 2x_1 + t$, $v_2 = x_2 - 2t$, find a) the streamline through the point $(1,1)^T$ at $t = 1$, b) the pathline for the fluid particle which is at the point $(1,1)^T$ at $t = 1$, and c) the streakline through the point $(1,1)^T$ at $t = 1$.

a) streamline

For the streamline we have the following set of differential equations,

$$\frac{dx_1}{d\tau} = 2x_1 + t\big|_{t=1}, \qquad x_1(\tau = 0) = 1,$$
$$\frac{dx_2}{d\tau} = x_2 - 2t\big|_{t=1}, \qquad x_2(\tau = 0) = 1.$$

Here it is inconsequential where the parameter τ has its origin, as long as *some* value of τ corresponds to a streamline through $(1,1)^T$, so we have taken the origin for $\tau = 0$ to be the point $(1,1)^T$. These equations at $t = 1$ are

$$\frac{dx_1}{d\tau} = 2x_1 + 1, \qquad x_1(\tau = 0) = 1,$$
$$\frac{dx_2}{d\tau} = x_2 - 2, \qquad x_2(\tau = 0) = 1.$$

Solving, we get

$$x_1 = \frac{3}{2}e^{2\tau} - \frac{1}{2},$$
$$x_2 = -e^{\tau} + 2.$$

Solving for τ, we find

$$\tau = \frac{1}{2}\ln\left(\frac{2}{3}\left(x_1 + \frac{1}{2}\right)\right).$$

So, eliminating τ and writing $x_2(x_1)$, we get the streamline to be

$$x_2 = 2 - \sqrt{\frac{2}{3}\left(x_1 + \frac{1}{2}\right)}.$$

b) pathline

For the pathline we have the following equations

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + t, \qquad x_1(t=1) = 1, \\ \frac{dx_2}{dt} &= x_2 - 2t, \qquad x_2(t=1) = 1. \end{aligned}$$

These have solution

$$\begin{aligned} x_1 &= \frac{7}{4}e^{2(t-1)} - \frac{t}{2} - \frac{1}{4}, \\ x_2 &= -3e^{t-1} + 2t + 2. \end{aligned}$$

It is algebraically difficult to eliminate t so as to write $x_2(x_1)$ explicitly. However, the above certainly gives a parametric representation of the pathline, which can be plotted in x_1, x_2 space.

c) streakline

For the streakline we have the following equations

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + t, \qquad x_1(t=\hat{t}) = 1, \\ \frac{dx_2}{dt} &= x_2 - 2t, \qquad x_2(t=\hat{t}) = 1. \end{aligned}$$

These have solution

$$\begin{aligned} x_1 &= \frac{5+2\hat{t}}{4}e^{2(t-\hat{t})} - \frac{t}{2} - \frac{1}{4}, \\ x_2 &= -(1+2\hat{t})e^{t-\hat{t}} + 2t + 2. \end{aligned}$$

We evaluate the streakline at $t = 1$ and get

$$\begin{aligned} x_1 &= \frac{5+2\hat{t}}{4}e^{2(1-\hat{t})} - \frac{3}{4}, \\ x_2 &= -(1+2\hat{t})e^{1-\hat{t}} + 4. \end{aligned}$$

Once again, it is algebraically difficult to eliminate $\hat{t}$ so as to write $x_2(x_1)$ explicitly. However, the above gives a parametric representation of the streakline, which can be plotted in x_1, x_2 space.

A plot of the streamline, pathline, and streakline for this problem is shown in Figure 1.8. Note that at the point $(1,1)^T$, all three intersect with the same slope. This can also be deduced from the equations governing streamlines, pathlines, and streaklines.

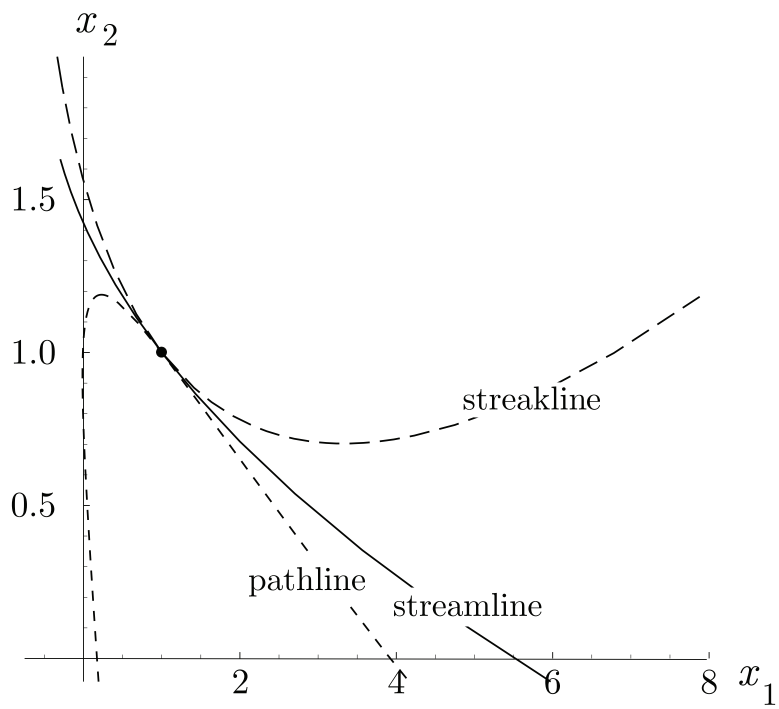

Figure 1.8: Streamline, pathlines, and streaklines for unsteady flow of example problem.

1.3.7 Kinematic decomposition of motion

In general the motion of a fluid is non-linear in nearly all respects. Certainly, it is common for particle pathlines to be far from straight lines; however, this is not actually a hallmark of non-linearity in that linear theories of fluid motion routinely predict pathlines with finite curvature. More to the point, we cannot in general use the method of superposition to add one flow to another to generate a third. One fundamental source of non-linearity is the non-linear operator $v_i \partial_i$, which we will see appears in most of our governing equations.

However, the local behavior of fluids is nearly always dominated by linear effects. By analyzing only the linear effects induced by small changes in velocity, which we will associate with the velocity gradient, we will learn a great deal about the richness of fluid motion. In the linear analysis, we will see that a fluid particle's motion can be described as a summation of a linear translation, rotation as a solid body, and straining of two types: extensional and shear. Both types of straining can be thought of as deformation rates. We use the word "straining" in contrast to "strain" to distinguish fluid and flexible solid behavior. Generally it is the rate of change of strain (that is the "straining") which has most relevance for a fluid, while it is the actual strain that has the most relevance for a flexible solid. This is because the stress in a flexible solid responds to strain, while the stress in a fluid responds to a strain rate. Nevertheless, while strain itself is associated with equilibrium configurations of a flexible solid, when its motion is decomposed, strain rate is relevant. In contrast, a rigid solid can be described by only a sum of linear translation and rotation. A point mass only

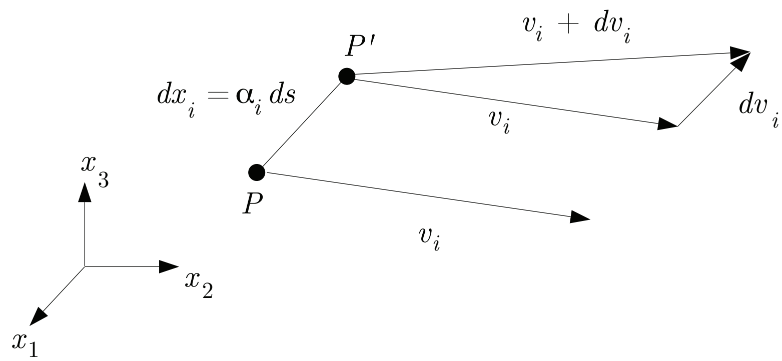

Figure 1.9: Sketch of fluid particle P in motion with velocity v_i and nearby neighbor particle P' with velocity $v_i + dv_i$.

translates; it cannot rotate or strain.

$$\begin{aligned}
\text{fluid motion} &= \text{translation} + \text{rotation} + \underbrace{\text{extensional straining} + \text{shear straining}}_{\text{straining}},\\
\text{flexible solid motion} &= \text{translation} + \text{rotation} + \underbrace{\text{extensional straining} + \text{shear straining}}_{\text{straining}},\\
\text{rigid solid motion} &= \text{translation} + \text{rotation},\\
\text{point mass motion} &= \text{translation}.
\end{aligned}$$

Let us consider in detail the configuration shown in Figure 1.9. Here we have a fluid particle at point P with coordinates x_i and velocity v_i. A small distance $dr_i = dx_i$ away is the fluid particle at point P', with coordinates $x_i + dx_i$. This particle moves with velocity $v_i + dv_i$. We can describe the difference in location by the product of a unit tangent vector α_i and a scalar differential distance magnitude $d\mathsf{s}$: $dr_i = dx_i = \alpha_i\ d\mathsf{s}$. Note that α_i is in general not aligned with the velocity vector, and the differential distance $d\mathsf{s}$ is not associated with the arc length along a particle path. Later in Sec. 1.3.12, we will select an alignment with the particle path, and thus choose $\alpha_i = \alpha_{ti}$ and $d\mathsf{s} = ds$, where α_{ti} is the unit tangent to the particle path and ds is the arc length.

1.3.7.1 Translation

We have the motion at P' to be $v_i + dv_i$. Obviously, the first term v_i represents translation.

1.3.7.2 Solid body rotation and straining

What remains is dv_i, and we shall see that it is appropriate to characterize this term by both a solid body rotation combined with straining.

We have from the chain rule that

$$dv_j = dx_i \partial_i v_j, \tag{1.233}$$
$$d\mathbf{v}^T = d\mathbf{x}^T \cdot \nabla \mathbf{v}^T, \tag{1.234}$$
$$d\mathbf{v} = \left(\nabla \mathbf{v}^T\right)^T \cdot d\mathbf{x}, \tag{1.235}$$
$$d\mathbf{v} = \mathsf{L}^T \cdot d\mathbf{x}. \tag{1.236}$$

Here $\partial_i v_j = \nabla \mathbf{v}^T \equiv \mathsf{L}$ is the velocity gradient tensor. We can break $\partial_i v_j$ into a symmetric and anti-symmetric part and say then

$$dv_j = \underbrace{dx_i \partial_{(i} v_{j)}}_{\text{Shear and extensional straining}} + \underbrace{dx_i \partial_{[i} v_{j]}}_{\text{Rotation}} \tag{1.237}$$

We also will find it useful to decompose the velocity gradient tensor L into a deformation tensor, D:

$$\mathsf{D} = D_{ij} \equiv \partial_{(i} v_{j)}, \tag{1.238}$$

a rotation tensor R:

$$\mathsf{R} = R_{ij} \equiv \partial_{[i} v_{j]}. \tag{1.239}$$

This yields

$$\mathsf{L} = \mathsf{D} + \mathsf{R}. \tag{1.240}$$

Thus,

$$dv_j = dx_i D_{ij} + dx_i R_{ij} = (\alpha_i D_{ij} + \alpha_i R_{ij})\, d\mathsf{s}, \tag{1.241}$$
$$d\mathbf{v}^T = d\mathbf{x}^T \cdot \mathsf{D} + d\mathbf{x}^T \cdot \mathsf{R} = \left(\boldsymbol{\alpha}^T \cdot \mathsf{D} + \boldsymbol{\alpha}^T \cdot \mathsf{R}\right) d\mathsf{s}, \tag{1.242}$$
$$d\mathbf{v} = \mathsf{D} \cdot d\mathbf{x} + \mathsf{R}^T \cdot d\mathbf{x} = \left(\mathsf{D} \cdot \boldsymbol{\alpha} + \mathsf{R}^T \cdot \boldsymbol{\alpha}\right) d\mathsf{s}. \tag{1.243}$$

Let

$$dv_j^{(s)} = dx_i \; \partial_{(i} v_{j)} = \alpha_i \; \partial_{(i} v_{j)} \; d\mathsf{s}, \tag{1.244}$$
$$d\mathbf{v}^{(s)T} = d\mathbf{x}^T \cdot \mathsf{D} = \boldsymbol{\alpha}^T \cdot \mathsf{D} \; d\mathsf{s}, \tag{1.245}$$
$$d\mathbf{v}^{(s)} = \mathsf{D} \cdot d\mathbf{x} = \mathsf{D} \cdot \boldsymbol{\alpha} \; d\mathsf{s}. \tag{1.246}$$

We will see this is associated with straining, both by shear and extension. We will call the symmetric tensor $\partial_{(i} v_{j)} = \mathsf{D}$ the strain rate or deformation tensor.

Further, let

$$dv_j^{(r)} = dx_i \; \partial_{[i} v_{j]} = \alpha_i \; \partial_{[i} v_{j]} \; d\mathsf{s}, \tag{1.247}$$
$$d\mathbf{v}^{(r)T} = d\mathbf{x}^T \cdot \mathsf{R} = \boldsymbol{\alpha}^T \cdot \mathsf{R} \; d\mathsf{s}, \tag{1.248}$$
$$d\mathbf{v}^{(r)} = \mathsf{R}^T \cdot d\mathbf{x} = \mathsf{R}^T \cdot \boldsymbol{\alpha} \; d\mathsf{s}. \tag{1.249}$$

We will see this is associated with rotation as a solid body, with $\partial_{[i} v_{j]} = \mathsf{R}$ as the rotation tensor.

1.3.7.2.1 Solid body rotation Let us examine $dv_j^{(r)}$. First, we define the vorticity vector ω_k as the curl of the velocity field

$$\omega_k = \epsilon_{kij}\partial_i v_j, \tag{1.250}$$
$$\boldsymbol{\omega} = \nabla \times \mathbf{v}. \tag{1.251}$$

Let us now split the velocity gradient $\partial_i v_j$ into its symmetric and anti-symmetric parts and recast the vorticity vector as

$$\omega_k = \underbrace{\epsilon_{kij}\partial_{(i}v_{j)}}_{=0} + \epsilon_{kij}\partial_{[i}v_{j]}. \tag{1.252}$$

The first term on the right side is zero because it is the tensor inner product of an anti-symmetric and symmetric tensor. In what remains, we see that half of the vorticity ω_k is actually the dual vector, Ω_k, associated with the anti-symmetric $\partial_{[i}v_{j]}$.

$$\omega_k = \epsilon_{kij}\partial_{[i}v_{j]} = \nabla \times \mathbf{v}, \tag{1.253}$$
$$\Omega_k = \frac{1}{2}\omega_k = \frac{1}{2}\epsilon_{kij}\partial_{[i}v_{j]} = \frac{1}{2}\nabla \times \mathbf{v}. \tag{1.254}$$

Using Eq. (1.79) to invert Eq. (1.254), we find

$$\partial_{[i}v_{j]} = \epsilon_{kij}\Omega_k = \frac{1}{2}\epsilon_{kij}\omega_k. \tag{1.255}$$

Thus we have

$$dv_j^{(r)} = dx_i\,\frac{1}{2}\epsilon_{kij}\omega_k, \tag{1.256}$$
$$= \epsilon_{kij}\left(\frac{\omega_k}{2}\right)dx_i, \tag{1.257}$$
$$= \epsilon_{jki}\left(\frac{\omega_k}{2}\right)dx_i, \tag{1.258}$$
$$= \frac{1}{2}\boldsymbol{\omega} \times d\mathbf{r} \qquad \text{and if} \qquad \boldsymbol{\Omega} = \frac{\boldsymbol{\omega}}{2}, \tag{1.259}$$
$$= \underbrace{\boldsymbol{\Omega} \times d\mathbf{r}}_{\text{Solid body rotation of one point about another}}. \tag{1.260}$$

By introducing the above definition for $\boldsymbol{\Omega}$, we see this term takes on the exact form for the differential velocity due to solid body rotation of P' about P from classical rigid body kinematics. Hence, we give it the same interpretation.

1.3.7.2.2 Straining Next we consider the remaining term, which we will associate with straining. First, let us further decompose this into what will be seen to be an extensional *(es)* straining and a shear straining *(ss)*:

$$dv_k^{(s)} = \underbrace{dv_k^{(es)}}_{\text{extension}} + \underbrace{dv_k^{(ss)}}_{\text{shear}}, \tag{1.261}$$
$$d\mathbf{v}^{(s)} = d\mathbf{v}^{(es)} + d\mathbf{v}^{(ss)}. \tag{1.262}$$

1.3.7.2.2.1 Extensional straining Let us define the extensional straining to be the component of straining in the direction of dx_j. To do this, we need to project $dv_j^{(s)}$ onto the unit vector α_j, then point the result in the direction of that same unit vector;

$$dv_k^{(es)} = \underbrace{\left(\alpha_j dv_j^{(s)}\right)}_{\text{projection of straining}} \alpha_k. \tag{1.263}$$

Now using the definition of $dv_j^{(s)}$, Eq. (1.244), we get

$$dv_k^{(es)} = \left(\alpha_j \underbrace{\left(\alpha_i \partial_{(i} v_{j)}\ d\mathsf{s}\right)}_{=dv_j^{(s)}}\right) \alpha_k, \tag{1.264}$$

$$= \left(\alpha_i \partial_{(i} v_{j)} \alpha_j\right) \alpha_k\ d\mathsf{s}, \tag{1.265}$$

$$d\mathbf{v}^{(es)} = \left(\boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}\right) \boldsymbol{\alpha}\ d\mathsf{s}. \tag{1.266}$$

Now, since $\alpha_i \alpha_j$ is symmetric, we can be led to a useful result. Consider the series of operations involving the velocity gradient, in general asymmetric, and a scalar quantity, ϕ:

$$\phi = \boldsymbol{\alpha}^T \cdot \mathsf{L} \cdot \boldsymbol{\alpha}, \tag{1.267}$$

$$= \boldsymbol{\alpha}^T \cdot (\mathsf{D} + \mathsf{R}) \cdot \boldsymbol{\alpha}, \tag{1.268}$$

$$= \boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha} + \underbrace{\boldsymbol{\alpha}^T \cdot \mathsf{R} \cdot \boldsymbol{\alpha}}_{=0}, \tag{1.269}$$

$$= \boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}. \tag{1.270}$$

Thus, we can recast Eq. (1.266) as

$$d\mathbf{v}^{(es)} = \left(\boldsymbol{\alpha}^T \cdot \mathsf{L} \cdot \boldsymbol{\alpha}\right) \boldsymbol{\alpha}\ d\mathsf{s}. \tag{1.271}$$

1.3.7.2.2.2 Shear straining What straining that is not aligned with the axis connecting P and P' must then be normal to that axis, and is easily visualized to represent a shearing between the two points. Hence the shear straining is

$$dv_j^{(ss)} = dv_j^{(s)} - dv_j^{(es)}, \tag{1.272}$$

$$= \left(\partial_{(j} v_{i)} \alpha_i - \alpha_i \partial_{(i} v_{k)} \alpha_k \alpha_j\right)\ d\mathsf{s}, \tag{1.273}$$

$$= \left(\partial_{(j} v_{i)} \alpha_i - \alpha_p \partial_{(p} v_{k)} \alpha_k \delta_{ji} \alpha_i\right)\ d\mathsf{s}, \tag{1.274}$$

$$= \left(\partial_{(j} v_{i)} - \left(\alpha_p \partial_{(p} v_{k)} \alpha_k\right) \delta_{ji}\right) \alpha_i\ d\mathsf{s}, \tag{1.275}$$

$$d\mathbf{v}^{(ss)} = \left(\mathsf{D} - \left(\boldsymbol{\alpha}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}\right) \mathsf{I}\right) \cdot \boldsymbol{\alpha}\ d\mathsf{s}. \tag{1.276}$$

1.3.7.2.2.3 Principal axes of strain rate We recall from our earlier discussion that the principal axes of stress are those axes for which the force associated with a given axis points in the same direction as that axis. We can extend this idea to straining, but develop it in a slightly different, but ultimately equivalent fashion based on notions from linear algebra. We first recall that most[22] arbitrary asymmetric square matrices L can be decomposed into a diagonal form as follows:

$$\mathsf{L} = \mathsf{P} \cdot \Lambda \cdot \mathsf{P}^{-1}. \tag{1.277}$$

Here P is a matrix of the same dimension as L which has in its columns the right eigenvectors of L. When L is symmetric, it can be shown that its eigenvalues are guaranteed to be real and its eigenvectors are guaranteed to be orthogonal. Further, since the eigenvectors can always be scaled by a constant and remain eigenvectors, we can choose to scale them in such a way that they are all normalized. In such a case in which the matrix P has orthonormal columns, the matrix is defined as *orthogonal* (though orthonormal would be a more accurate nomenclature). When P has been rendered orthogonal, we call it Q. So, when L is symmetric, such as when $\mathsf{L} = \mathsf{D}$, the symmetric part of the velocity gradient, we also have the following decomposition

$$\mathsf{D} = \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^{-1}. \tag{1.278}$$

Orthogonal matrices can be shown to have the remarkable property that their transpose is equal to their inverse, and so we also have the even more useful

$$\mathsf{D} = \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^{T}. \tag{1.279}$$

Geometrically Q is equivalent to a matrix of direction cosines; as we have seen before, its transpose Q^T is a rotation matrix which rotates but does not stretch a vector when it operates on the vector.

Now let us consider the straining component of the velocity difference; taking the symmetric $\partial_{(i}v_{j)} = \mathsf{D}$, which we further assume to be a constant for this analysis, we rewrite Eq. (1.246) using Gibbs notation as

$$\left(d\mathbf{v}^{(s)}\right)^T = d\mathbf{x}^T \cdot \mathsf{D}, \tag{1.280}$$

$$d\mathbf{v}^{(s)} = \mathsf{D}^T \cdot d\mathbf{x}, \tag{1.281}$$

$$d\mathbf{v}^{(s)} = \mathsf{D} \cdot d\mathbf{x}, \qquad \text{since } \mathsf{D} \text{ is symmetric.} \tag{1.282}$$

$$d\mathbf{v}^{(s)} = \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T \cdot d\mathbf{x}. \tag{1.283}$$

[22]Some matrices, which often do not have enough linearly independent eigenvectors, cannot be diagonalized; however, the argument can be extended through use of the singular value decomposition. The singular value decomposition can also be used to effectively diagonalize asymmetric matrices; however, in that case it can be shown there is no equivalent interpretation of the principal axes. Consequently, we will quickly focus the discussion on symmetric matrices.

Now let us select what amounts to a special axes rotation via matrix multiplication by the orthogonal matrix Q^T:

$$\mathsf{Q}^T \cdot d\mathbf{v}^{(s)} = \mathsf{Q}^T \cdot \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T \cdot d\mathbf{x}, \tag{1.284}$$

$$\mathsf{Q}^T \cdot d\mathbf{v}^{(s)} = \mathsf{Q}^{-1} \cdot \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T \cdot d\mathbf{x}, \tag{1.285}$$

$$\mathsf{Q}^T \cdot d\mathbf{v}^{(s)} = \Lambda \cdot \mathsf{Q}^T \cdot d\mathbf{x}, \tag{1.286}$$

$$d\underbrace{\left(\mathsf{Q}^T \cdot \mathbf{v}^{(s)}\right)}_{=\mathbf{v}'^{(s)}} = \Lambda \cdot d\underbrace{\left(\mathsf{Q}^T \cdot \mathbf{x}\right)}_{=\mathbf{x}'} \quad \text{since } \mathsf{D} \text{ and thus } \mathsf{Q}^T \text{ are assumed constant.} \tag{1.287}$$

Now we recall from the definition of vectors that $\mathsf{Q}^T \cdot \mathbf{v}^{(s)} = \mathbf{v}'^{(s)}$ and $\mathsf{Q}^T \cdot \mathbf{x} = \mathbf{x}'$. That is these are the representations of the vectors in a specially rotated coordinate system, so we have

$$d\mathbf{v}'^{(s)} = \Lambda \cdot d\mathbf{x}'. \tag{1.288}$$

Now since Λ is diagonal, we see that a perturbation in $\mathbf{x}'$ confined to any one of the rotated coordinate axes induces a change in velocity which lies in the same direction as that coordinate axis. For instance on the $1'$ axis, we have $dv'^{(s)}_1 = \Lambda_{11} dx'_1$. That is to say that *in this specially rotated frame, all straining is extensional; there is no shear straining.*

1.3.8 Expansion rate

Consider a small material region of fluid, also called a particle of fluid. *We define a material region as a region enclosed by a surface across which there is no flux of mass.* We shall later see by invoking the mass conservation axiom for a non-relativistic system, that the implication is that the mass of a material region is constant, but we need not yet consider this. In general the volume containing this particle can increase or decrease. It is useful to quantify the rate of this increase or decrease. Additionally, this will give a flavor of the analysis to come for the conservation axioms.

Taking the both MR and $R(t)$ to denote the same time-dependent finite material region in space, we must have

$$V_{MR} = \int_{R(t)} dV. \tag{1.289}$$

Using Leibniz's rule, Eq. (1.185), we take the time derivative of both sides and obtain

$$\frac{dV_{MR}}{dt} = \int_{R(t)} \frac{\partial}{\partial t}(1)\; dV + \int_{S(t)} n_i v_i \; dS, \tag{1.290}$$

$$= \int_{S(t)} n_i v_i \; dS, \tag{1.291}$$

$$= \int_{R(t)} \partial_i v_i \; dV \qquad \text{by Gauss's theorem,} \tag{1.292}$$

$$= \left(\partial_i v_i\right)_* V_{MR} \qquad \text{by the mean value theorem.} \tag{1.293}$$

In the analysis above, we note that the velocity of $S(t)$, in general w_i, has been set to the fluid velocity v_i since we have a material region. We also recall from calculus the mean value theorem which states that for any integral, a mean value can be defined, denoted by a $*$, as for example $\int_a^b f(x)\ dx = f_*(b-a)$. As we shrink the size of the material volume to zero, the mean value approaches the local value, so we get

$$\frac{1}{V_{MR}}\frac{dV_{MR}}{dt} = (\partial_i v_i)_*, \tag{1.294}$$

$$\lim_{V_{MR}\to 0}\frac{1}{V_{MR}}\frac{dV_{MR}}{dt} = \partial_i v_i = \nabla^T\cdot\mathbf{v} = \text{div }\mathbf{v} = \text{tr }\mathsf{D}. \tag{1.295}$$

Equation (1.295)describes the *relative expansion rate* also known as the *dilation rate* of a material fluid particle. A fluid particle for which $\partial_i v_i = 0$ must have a relative expansion rate of zero, and satisfies conditions to be an *incompressible fluid*.

1.3.9 Invariants of the strain rate tensor

The tensor associated with straining (also called the deformation rate tensor or strain rate tensor) $\partial_{(i}v_{j)}$ is symmetric. Consequently, it has three real eigenvalues, $\lambda_{\dot{\epsilon}}^{(i)}$, and an orientation for which the strain rate is aligned with the eigenvectors. As with stress, there are also three principal invariants of strain rate, namely

$$I_{\dot{\epsilon}}^{(1)} = \partial_{(i}v_{i)} = \partial_i v_i = \lambda_{\dot{\epsilon}}^{(1)} + \lambda_{\dot{\epsilon}}^{(2)} + \lambda_{\dot{\epsilon}}^{(3)}, \tag{1.296}$$

$$I_{\dot{\epsilon}}^{(2)} = \frac{1}{2}\left(\partial_{(i}v_{i)}\partial_{(j}v_{j)} - \partial_{(i}v_{j)}\partial_{(j}v_{i)}\right) = \lambda_{\dot{\epsilon}}^{(1)}\lambda_{\dot{\epsilon}}^{(2)} + \lambda_{\dot{\epsilon}}^{(2)}\lambda_{\dot{\epsilon}}^{(3)} + \lambda_{\dot{\epsilon}}^{(3)}\lambda_{\dot{\epsilon}}^{(1)}, \tag{1.297}$$

$$I_{\dot{\epsilon}}^{(3)} = \epsilon_{ijk}\partial_{(1}v_{i)}\partial_{(2}v_{j)}\partial_{(3}v_{k)} = \lambda_{\dot{\epsilon}}^{(1)}\lambda_{\dot{\epsilon}}^{(2)}\lambda_{\dot{\epsilon}}^{(3)}. \tag{1.298}$$

The physical interpretation for $I_{\dot{\epsilon}}^{(1)}$ is obvious in that it is equal to the relative rate of volume change for a material element, $\frac{1}{V}\frac{dV}{dt}$. Aris discusses how $I_{\dot{\epsilon}}^{(2)}$ is related to $\frac{1}{V}\frac{d^2V}{dt^2}$ and $I_{\dot{\epsilon}}^{(3)}$ is related to $\frac{1}{V}\frac{d^3V}{dt^3}$.

1.3.10 Invariants of the velocity gradient tensor

For completeness, the invariants of the more general velocity gradient tensor are included. They are

$$I_{\nabla\mathbf{v}}^{(1)} = \partial_i v_i = \lambda_{\nabla\mathbf{v}}^{(1)} + \lambda_{\nabla\mathbf{v}}^{(2)} + \lambda_{\nabla\mathbf{v}}^{(3)}, \tag{1.299}$$

$$I_{\nabla\mathbf{v}}^{(2)} = \frac{1}{2}\left((\partial_i v_i)(\partial_j v_j) - (\partial_i v_j)(\partial_j v_i)\right) = \lambda_{\nabla\mathbf{v}}^{(1)}\lambda_{\nabla\mathbf{v}}^{(2)} + \lambda_{\nabla\mathbf{v}}^{(2)}\lambda_{\nabla\mathbf{v}}^{(3)} + \lambda_{\nabla\mathbf{v}}^{(3)}\lambda_{\nabla\mathbf{v}}^{(1)}, \tag{1.300}$$

$$= \frac{1}{2}\left((\partial_i v_i)(\partial_j v_j) + \partial_{[i}v_{j]}\partial_{[i}v_{j]} - \partial_{(i}v_{j)}\partial_{(i}v_{j)}\right), \tag{1.301}$$

$$= \frac{1}{2}\left((\partial_i v_i)(\partial_j v_j) + \frac{1}{2}\omega_i\omega_i - \partial_{(i}v_{j)}\partial_{(i}v_{j)}\right), \tag{1.302}$$

$$I^{(3)}_{\nabla \mathbf{v}} = \epsilon_{ijk}\partial_1 v_i \partial_2 v_j \partial_3 v_k = \lambda^{(1)}_{\nabla \mathbf{v}}\lambda^{(2)}_{\nabla \mathbf{v}}\lambda^{(3)}_{\nabla \mathbf{v}}. \quad (1.303)$$

1.3.11 Two-dimensional kinematics

Next, consider some important two dimensional cases, first for general two-dimensional flows, and then for specific examples.

1.3.11.1 General two-dimensional flows

For two-dimensional motion, we have the velocity vector as $(v_1, v_2, v_3 = 0)$, and for the unit tangent of the vector separating two nearby particles $(\alpha_1, \alpha_2, \alpha_3 = 0)$.

1.3.11.1.1 Rotation Recalling that $dx_i = \alpha_i \; d\mathsf{s}$, for rotation, we have

$$dv_j^{(r)} = \partial_{[i} v_{j]} dx_i = \alpha_i \partial_{[i} v_{j]} \; d\mathsf{s}, \quad (1.304)$$

$$dv_j^{(r)} = \left(\alpha_1 \partial_{[1} v_{j]} + \alpha_2 \partial_{[2} v_{j]}\right) d\mathsf{s}, \quad (1.305)$$

$$dv_1^{(r)} = \left(\alpha_1 \underbrace{\partial_{[1} v_{1]}}_{=0} + \alpha_2 \partial_{[2} v_{1]}\right) d\mathsf{s}, \quad (1.306)$$

$$dv_1^{(r)} = \alpha_2 \partial_{[2} v_{1]} d\mathsf{s}, \quad (1.307)$$

$$dv_2^{(r)} = \left(\alpha_1 \partial_{[1} v_{2]} + \alpha_2 \underbrace{\partial_{[2} v_{2]}}_{=0}\right) d\mathsf{s}, \quad (1.308)$$

$$dv_2^{(r)} = \alpha_1 \partial_{[1} v_{2]} d\mathsf{s}, \quad (1.309)$$

rewriting in terms of the actual derivatives

$$dv_1^{(r)} = \frac{1}{2}\alpha_2 \left(\partial_2 v_1 - \partial_1 v_2\right) d\mathsf{s}, \quad (1.310)$$

$$dv_2^{(r)} = \frac{1}{2}\alpha_1 \left(\partial_1 v_2 - \partial_2 v_1\right) d\mathsf{s}. \quad (1.311)$$

Also for the vorticity vector, we get

$$\omega_k = \epsilon_{ijk} \partial_i v_j. \quad (1.312)$$

The only non-zero component is ω_3, which comes to

$$\omega_3 = \underbrace{\epsilon_{311}}_{=0} \partial_1 v_1 + \underbrace{\epsilon_{312}}_{=1} \partial_1 v_2 + \underbrace{\epsilon_{321}}_{=-1} \partial_2 v_1 + \underbrace{\epsilon_{322}}_{=0} \partial_2 v_2, \qquad \text{thus,} \quad (1.313)$$

$$= \partial_1 v_2 - \partial_2 v_1. \quad (1.314)$$

1.3.11.1.2 Extension

$$
\begin{aligned}
dv_k^{(es)} &= \alpha_k \alpha_i \alpha_j \partial_{(i} v_{j)} \; d\mathsf{s}, && (1.315)\\
&= \alpha_k \left(\alpha_1 \alpha_1 \partial_{(1} v_{1)} + \alpha_1 \alpha_2 \partial_{(1} v_{2)} + \alpha_2 \alpha_1 \partial_{(2} v_{1)} + \alpha_2 \alpha_2 \partial_{(2} v_{2)} \right) d\mathsf{s} && (1.316)\\
&= \alpha_k \left(\alpha_1^2 \partial_1 v_1 + \alpha_1 \alpha_2 \left(\partial_1 v_2 + \partial_2 v_1 \right) + \alpha_2^2 \partial_2 v_2 \right) d\mathsf{s}, \qquad \text{so} && (1.317)\\
dv_1^{(es)} &= \alpha_1 \left(\alpha_1^2 \partial_1 v_1 + \alpha_1 \alpha_2 \left(\partial_1 v_2 + \partial_2 v_1 \right) + \alpha_2^2 \partial_2 v_2 \right) d\mathsf{s}, && (1.318)\\
dv_2^{(es)} &= \alpha_2 \left(\alpha_1^2 \partial_1 v_1 + \alpha_1 \alpha_2 \left(\partial_1 v_2 + \partial_2 v_1 \right) + \alpha_2^2 \partial_2 v_2 \right) d\mathsf{s}. && (1.319)
\end{aligned}
$$

1.3.11.1.3 Shear

$$
\begin{aligned}
dv_j^{(ss)} &= dv_j^{(s)} - dv_j^{(es)}, && (1.320)\\
&= \left(\alpha_i \partial_{(i} v_{j)} - \alpha_j \alpha_i \alpha_k \partial_{(i} v_{k)} \right) d\mathsf{s}, && (1.321)\\
dv_1^{(ss)} &= \Bigg(\alpha_1 \partial_1 v_1 + \alpha_2 \left(\frac{\partial_2 v_1 + \partial_1 v_2}{2} \right) - \\
&\quad \alpha_1 \left(\alpha_1^2 \partial_1 v_1 + \alpha_1 \alpha_2 \left(\partial_1 v_2 + \partial_2 v_1 \right) + \alpha_2^2 \partial_2 v_2 \right) \Bigg) d\mathsf{s},\\
dv_2^{(ss)} &= \Bigg(\alpha_2 \partial_2 v_2 + \alpha_1 \left(\frac{\partial_1 v_2 + \partial_2 v_1}{2} \right) - \\
&\quad \alpha_2 \left(\alpha_1^2 \partial_1 v_1 + \alpha_1 \alpha_2 \left(\partial_1 v_2 + \partial_2 v_1 \right) + \alpha_2^2 \partial_2 v_2 \right) \Bigg) d\mathsf{s}.
\end{aligned}
$$

1.3.11.1.4 Expansion

$$
\frac{1}{V} \frac{dV}{dt} = \partial_1 v_1 + \partial_2 v_2. \qquad (1.322)
$$

1.3.11.2 Relative motion along 1 axis

Let us consider in detail the configuration shown in Figure 1.10 in which the particle separation is along the 1 axis. Hence $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0$.

- *Rotation*

$$
\begin{aligned}
dv_1^{(r)} &= 0, && (1.323)\\
dv_2^{(r)} &= \frac{1}{2} \left(\partial_1 v_2 - \partial_2 v_1 \right) \; d\mathsf{s} = \frac{\omega_3}{2} \; d\mathsf{s}. && (1.324)
\end{aligned}
$$

- *Extension*

$$
\begin{aligned}
dv_1^{(es)} &= \partial_1 v_1 \; d\mathsf{s}, && (1.325)\\
dv_2^{(es)} &= 0. && (1.326)
\end{aligned}
$$

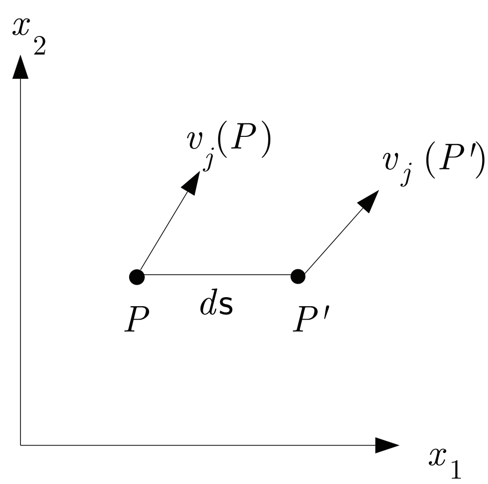

Figure 1.10: Sketch of fluid particle P in motion with velocity $v_j(P)$ and nearby neighbor particle P' with velocity $v_j(P')$.

- *Shear*

$$dv_1^{(ss)} = 0, \tag{1.327}$$

$$dv_2^{(ss)} = \frac{1}{2}\left(\partial_1 v_2 + \partial_2 v_1\right)\ d\mathsf{s} = \partial_{(1} v_{2)}\ d\mathsf{s}. \tag{1.328}$$

- *Expansion*: $\frac{1}{V}\frac{dV}{dt} = \partial_1 v_1 + \partial_2 v_2$.

1.3.11.3 Relative motion along 2 axis

Let us consider in detail the configuration shown in Figure 1.11 in which the particle separation is aligned with the 2 axis. Hence $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_3 = 0$.

- *Rotation*

$$dv_1^{(r)} = \frac{1}{2}\left(\partial_2 v_1 - \partial_1 v_2\right) d\mathsf{s} = -\frac{\omega_3}{2}\ d\mathsf{s}, \tag{1.329}$$

$$dv_2^{(r)} = 0. \tag{1.330}$$

- *Extension*

$$dv_1^{(es)} = 0, \tag{1.331}$$

$$dv_2^{(es)} = \partial_2 v_2\ d\mathsf{s}. \tag{1.332}$$

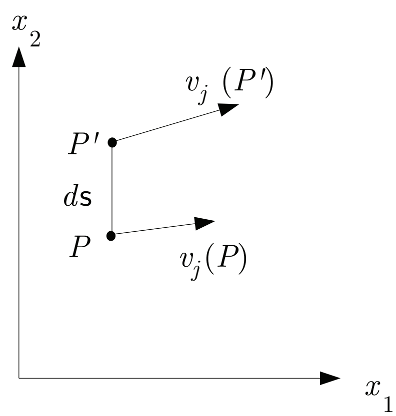

Figure 1.11: Sketch of fluid particle P in motion with velocity $v_i(P)$ and nearby neighbor particle P' with velocity $v_i(P')$.

- *Shear*

$$dv_1^{(ss)} = \frac{1}{2}\left(\partial_2 v_1 + \partial_1 v_2\right) d\mathsf{s} = \partial_{(1} v_{2)}\ d\mathsf{s}, \tag{1.333}$$

$$dv_2^{(ss)} = 0. \tag{1.334}$$

- *Expansion*

$$\frac{1}{V}\frac{dV}{dt} = \partial_1 v_1 + \partial_2 v_2. \tag{1.335}$$

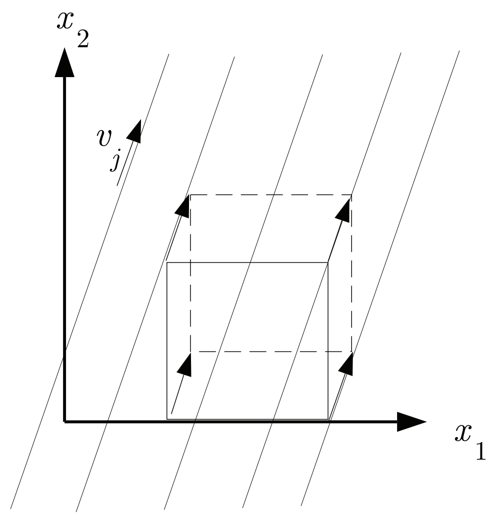

Figure 1.12: Sketch of uniform flow

1.3.11.4 Uniform flow

Consider the kinematics of a uniform two-dimensional flow in which

$$v_1 = k_1, \qquad v_2 = k_2, \qquad v_3 = 0, \tag{1.336}$$

as sketched in Figure 1.12.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}, \qquad \frac{dx_1}{k_1} = \frac{dx_2}{k_2}, \qquad x_1 = \left(\frac{k_1}{k_2}\right) x_2 + C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(k_1) - \partial_2(k_2) = 0.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = 0.$
  - *on 2-axis*: $\partial_2 v_2 = 0.$
- *Shear for unrotated element*: $\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) = 0.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0.$
- *Acceleration*:
  $\frac{dv_1}{dt} = \partial_o v_1 + v_1 \partial_1 v_1 + v_2 \partial_2 v_1 = 0 + k_1 \partial_1(k_1) + k_2 \partial_2(k_1) = 0.$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1 \partial_1 v_2 + v_2 \partial_2 v_2 = 0 + k_1 \partial_1(k_2) + k_2 \partial_2(k_2) = 0.$

For this very simple flow, the streamlines are straight lines, there is no rotation, no extension, no shear, no expansion, and no acceleration.

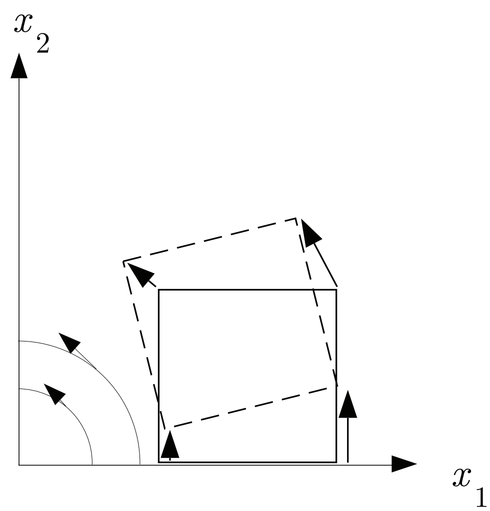

Figure 1.13: Sketch of pure rigid body rotation.

1.3.11.5 Pure rigid body rotation

Consider the kinematics of a two-dimensional flow in which

$$v_1 = -kx_2, \qquad v_2 = kx_1, \qquad v_3 = 0, \tag{1.337}$$

as sketched in Figure 1.13.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}, \qquad \frac{dx_1}{-kx_2} = \frac{dx_2}{kx_1}, \qquad x_1 dx_1 = -x_2 dx_2, \qquad x_1^2 + x_2^2 = C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(kx_1) - \partial_2(-kx_2) = 2k.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = 0.$
  - *on 2-axis*: $\partial_2 v_2 = 0.$
- *Shear for unrotated element*: $\frac{1}{2}\left(\partial_1(kx_1) + \partial_2(-kx_2)\right) = k - k = 0.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0 + 0 = 0.$
- *Acceleration*:
  $\frac{dv_1}{dt} = \partial_o v_1 + v_1\partial_1 v_1 + v_2 \partial_2 v_1 = 0 - kx_2\partial_1(-kx_2) + kx_1\partial_2(-kx_2) = -k^2 x_1.$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1\partial_1 v_2 + v_2 \partial_2 v_2 = 0 - kx_2\partial_1(kx_1) + kx_1\partial_2(kx_1) = -k^2 x_2.$

In this flow, the velocity magnitude grows linearly with distance from the origin. This is precisely how a rotating rigid body behaves. The streamlines are circles. The rotation is positive for positive k, hence counterclockwise, there is no deformation in extension or shear, and there is no expansion. The acceleration is pointed towards the origin.

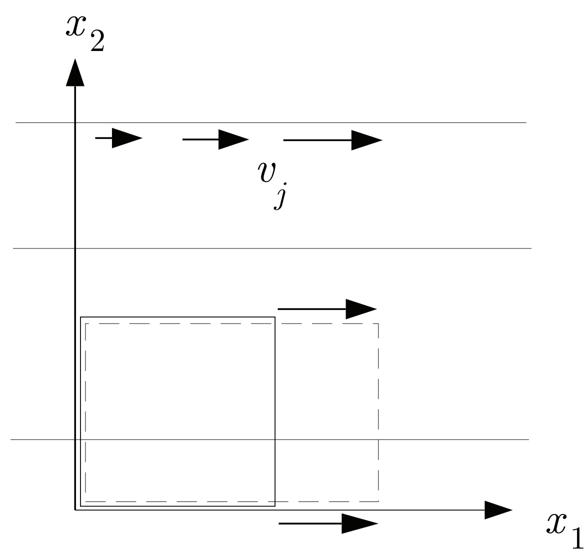

Figure 1.14: Sketch of extensional flow (1-D compressible)

1.3.11.6 Pure extensional motion (a compressible flow)

Consider the kinematics of a two-dimensional flow in which

$$v_1 = kx_1, \qquad v_2 = 0, \qquad v_3 = 0, \tag{1.338}$$

as sketched in Figure 1.14.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}$, $\qquad v_2\ dx_1 = v_1\ dx_2, \qquad 0 = kx_1\ dx_2, \qquad x_2 = C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(0) - \partial_2(kx_1) = 0.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = k.$
  - *on 2-axis*: $\partial_2 v_2 = 0.$
- *Shear for unrotated element*: $\frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) = \frac{1}{2}(\partial_1(0) + \partial_2(kx_1)) = 0.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = k.$
- *Acceleration*:
  $\frac{dv_1}{dt} = \partial_o v_1 + v_1 \partial_1 v_1 + v_2 \partial_2 v_1 = 0 + kx_1 \partial_1(kx_1) + 0\partial_2(kx_1) = k^2 x_1.$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1 \partial_1 v_2 + v_2 \partial_2 v_2 = 0 + kx_1 \partial_1(0) + 0\partial_2(0) = 0.$

In this flow, the streamlines are straight lines, there is no fluid rotation, there is extension (stretching) deformation along the 1-axis, but no shear deformation along this axis. The relative expansion rate is positive for positive k, indicating a compressible flow. The acceleration is confined to the x_1 direction.

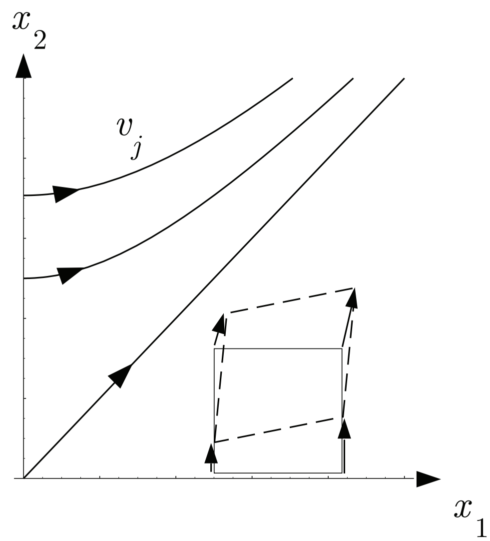

Figure 1.15: Sketch of pure shearing flow

1.3.11.7 Pure shear straining

Consider the kinematics of a two-dimensional flow in which

$$v_1 = kx_2, \qquad v_2 = kx_1, \qquad v_3 = 0, \tag{1.339}$$

as sketched in Figure 1.15.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}, \qquad \frac{dx_1}{kx_2} = \frac{dx_2}{kx_1}, \qquad x_1\ dx_1 = x_2\ dx_2, \qquad x_1^2 = x_2^2 + C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(kx_1) - \partial_2(kx_2) = k - k = 0.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = \partial_1(kx_2) = 0.$
  - *on 2-axis*: $\partial_2 v_2 = \partial_2(kx_1) = 0.$
- *Shear for unrotated element*: $\frac{1}{2}\left(\partial_1 v_2 + \partial_2 v_1\right) = \frac{1}{2}\left(\partial_1(kx_1) + \partial_2(kx_2)\right) = k.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0.$
- *Acceleration*:

  $\frac{dv_1}{dt} = \partial_o v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1 = 0 + kx_2\partial_1(kx_2) + kx_1\partial_2(kx_2) = k^2 x_1.$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2 = 0 + kx_2\partial_1(kx_1) + kx_1\partial_2(kx_1) = k^2 x_2.$

In this flow, the streamlines are hyperbolas, there is no rotation, no axial extension along the coordinate axes, positive shear deformation for an element aligned with the coordinate axes, and no expansion. So, the pure shear deformation preserves volume. The fluid is accelerating away from the origin.

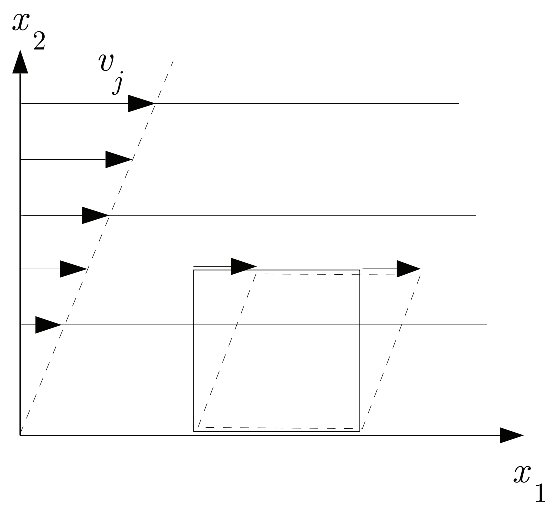

Figure 1.16: Sketch of Couette flow.

1.3.11.8 Couette flow: shear + rotation

Consider the kinematics of a two-dimensional flow in which

$$v_1 = kx_2, \qquad v_2 = 0, \qquad v_3 = 0, \tag{1.340}$$

as sketched in Figure 1.16. This is known as a Couette[23] flow.

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}, \qquad \frac{dx_1}{kx_2} = \frac{dx_2}{0}, \qquad 0 = kx_2\ dx_2, \qquad x_2 = C.$
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1(0) - \partial_2(kx_2) = -k.$
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = \partial_1(kx_2) = 0.$
  - *on 2-axis*: $\partial_2 v_2 = \partial_2(0) = 0.$
- *Shear for unrotated element*: $\frac{1}{2}\left(\partial_1 v_2 + \partial_2 v_1\right) = \frac{1}{2}\left(\partial_1(0) + \partial_2(kx_2)\right) = \frac{k}{2}.$
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0.$
- *Acceleration*:
  $\frac{dv_1}{dt} = \partial_o v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1 = 0 + kx_2\partial_1(kx_2) + 0\partial_2(kx_2) = 0.$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2 = 0 + kx_2\partial_1(0) + 0\partial_2(0) = 0.$

Here the streamlines are straight lines, and the flow is rotational (clockwise since $\omega < 0$ for $k > 0$)! The constant volume rotation is combined with a constant volume shear deformation for the element aligned with the coordinate axes. The fluid is not accelerating.

[23]Maurice Marie Alfred Couette, 1858-1943, French fluid mechanician, student of Joseph Valentin Boussinesq, and faculty member at Catholic University of Angers.

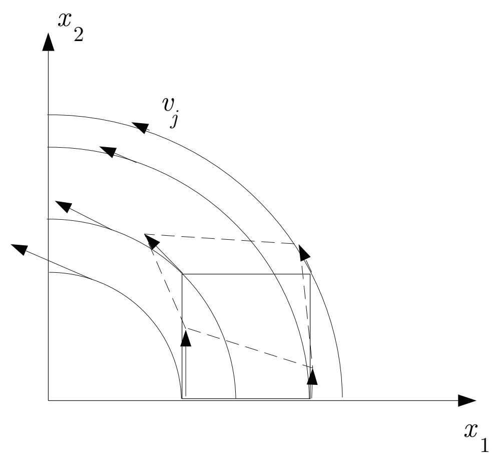

Figure 1.17: Sketch of ideal irrotational vortex.

1.3.11.9 Ideal point vortex: extension+shear

Consider the kinematics of a two-dimensional flow sketched in Figure 1.17.

$$v_1 = -k\frac{x_2}{x_1^2 + x_2^2}, \qquad v_2 = k\frac{x_1}{x_1^2 + x_2^2}, \qquad v_3 = 0, \tag{1.341}$$

- *Streamlines*: $\frac{dx_1}{v_1} = \frac{dx_2}{v_2}$, $\qquad \frac{dx_1}{-k\frac{x_2}{x_1^2+x_2^2}} = \frac{dx_2}{k\frac{x_1}{x_1^2+x_2^2}}$, $\qquad -\frac{dx_1}{x_2} = \frac{dx_2}{x_1}$, $\qquad x_1^2 + x_2^2 = C$.
- *Rotation*: $\omega_3 = \partial_1 v_2 - \partial_2 v_1 = \partial_1\left(k\frac{x_1}{x_1^2+x_2^2}\right) - \partial_2\left(-k\frac{x_2}{x_1^2+x_2^2}\right) = 0$.
- *Extension*
  - *on 1-axis*: $\partial_1 v_1 = \partial_1\left(-k\frac{x_2}{x_1^2+x_2^2}\right) = 2k\frac{x_1 x_2}{(x_1^2+x_2^2)^2}$.
  - *on 2-axis*: $\partial_2 v_2 = \partial_2\left(k\frac{x_1}{x_1^2+x_2^2}\right) = -2k\frac{x_1 x_2}{(x_1^2+x_2^2)^2}$.
- *Shear for unrotated element*: $\frac{1}{2}\left(\partial_1 v_2 + \partial_2 v_1\right) = k\frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}$.
- *Expansion*: $\partial_1 v_1 + \partial_2 v_2 = 0$.
- *Acceleration*:
  $\frac{dv_1}{dt} = \partial_o v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1 = -\frac{k^2 x_1}{(x_1^2+x_2^2)^2}$
  $\frac{dv_2}{dt} = \partial_o v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2 = -\frac{k^2 x_2}{(x_1^2+x_2^2)^2}$.

The streamlines are circles and the fluid element does not rotate about its own axis! It does rotate about the origin. It deforms by extension and shear in such a way that overall the volume is constant.

1.3.12 Kinematics as a dynamical system

Let us apply some standard notions from dynamical systems theory to fluid kinematics. Let us imagine that we are given a time-independent flow field, where the fluid velocity is known and is a function of position only. Then the motion of an individual fluid particle is governed by the following autonomous system of non-linear ordinary differential equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{X}. \tag{1.342}$$

Here, the initial position of the fluid particle is given by the constant vector $\mathbf{X}$. The solution of Eq. (1.342) can be expressed in general form

$$\mathbf{x} = \mathbf{x}(t; \mathbf{X}), \tag{1.343}$$

a function of time parameterized by the initial condition of the fluid particle. Such a solution is certainly a pathline, streamline, and streakline. It is also known as a trajectory in the dynamical systems literature.

Let us analyze Eq. (1.342) in some more detail. From the chain rule, see Eq. (1.236), we have

$$d\mathbf{v} = \underbrace{(\nabla \mathbf{v}^T)^T}_{\mathsf{L}^T} \cdot d\mathbf{x}, \tag{1.344}$$

$$d\mathbf{v} = \mathsf{L}^T \cdot d\mathbf{x}. \tag{1.345}$$

This gives the acceleration vector as

$$\frac{d\mathbf{v}}{dt} = \mathsf{L}^T \cdot \frac{d\mathbf{x}}{dt}, \tag{1.346}$$

$$= \mathsf{L}^T \cdot \mathbf{v}. \tag{1.347}$$

Example 1.9

Study the following non-linear autonomous system:

$$\frac{dx_1}{dt} = v_1(x_1, x_2, x_3) = 1 + x_1 x_2 x_3, \qquad x_1(0) = 0, \tag{1.348}$$

$$\frac{dx_2}{dt} = v_2(x_1, x_2, x_3) = x_1 + x_2^2 + x_1 x_2^3, \qquad x_2(0) = 0, \tag{1.349}$$

$$\frac{dx_3}{dt} = v_3(x_1, x_2, x_3) = 2 - x_1 + x_2 x_3, \qquad x_3(0) = 0. \tag{1.350}$$

Numerical solution of this nonlinear system of ordinary differential equations yields $x_1(t)$, $x_2(t)$, $x_3(t)$, which for this time-independent velocity field induces the particle pathlines, streamlines, and streaklines. All are plotted in Fig. 1.18. We could also apply the complete mathematical theory of dynamic systems to understand the system better.

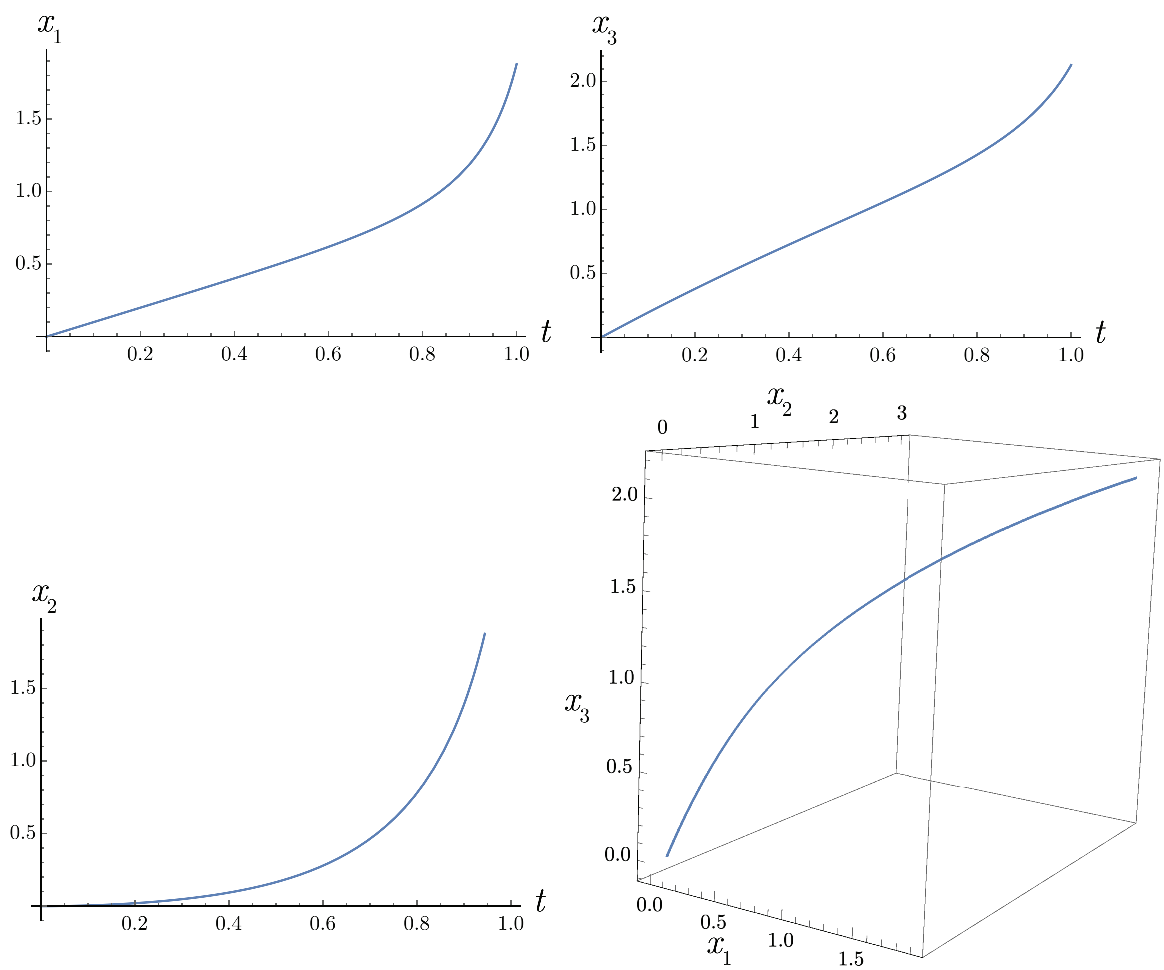

Figure 1.18: Plot of $x_1(t)$, $x_2(t)$, $x_3(t)$, along with the coincident pathline, streamline, and streakline for a steady three dimensional fluid particle that commences at the origin.

We can use Eq. (1.347) to calculate the acceleration vector field:

$$\begin{pmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{dv_3}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix}, \quad (1.351)$$

$$= \begin{pmatrix} x_2x_3 & x_1x_3 & x_1x_2 \\ 1+x_2^3 & 2x_2+3x_1x_2^2 & 0 \\ -1 & x_3 & x_2 \end{pmatrix} \begin{pmatrix} 1+x_1x_2x_3 \\ x_1+x_2^2+x_1x_2^3 \\ 2-x_1+x_2x_3 \end{pmatrix}, \quad (1.352)$$

$$= \begin{pmatrix} 2x_1x_2 - x_1^2x_2 + x_1^2x_3 + x_2x_3 + 2x_1x_2^2x_3 + x_1^2x_2^3x_3 + x_1x_2^2x_3^2 \\ 1 + 2x_1x_2 + 3x_1^2x_2^2 + 3x_2^3 + 5x_1x_2^4 + 3x_1^2x_2^5 + x_1x_2x_3 + x_1x_2^4x_3 \\ -1 + 2x_2 - x_1x_2 + x_1x_3 - x_1x_2x_3 + 2x_2^2x_3 + x_1x_2^3x_3 \end{pmatrix} \quad (1.353)$$

If we know the kinematics of a fluid particle, we know everything about its motion, including its acceleration. We shall soon discuss things like Newton's laws of motion that relate accelerations to force. If we know the acceleration, it is possible to induce what the force was that generated it by simply multiplying the acceleration by the mass. Rarely is this the case however. It is more common to know something about the forces and to use this to deduce what the motion is.

Now, we seek to analyze a particular pathline. Note that the velocity vector is tangent to the fluid particle trajectory. Let us study a unit vector which happens to be tangent to the velocity field:

$$\boldsymbol{\alpha}_t = \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (1.354)$$

Next, use the chain rule to examine how the unit tangent vector evolves with time:

$$\frac{d\boldsymbol{\alpha}_t}{dt} = \frac{1}{|\mathbf{v}|}\frac{d\mathbf{v}}{dt} - \frac{\mathbf{v}}{|\mathbf{v}|^2}\frac{d|\mathbf{v}|}{dt}. \quad (1.355)$$

We can scale Eq. (1.347) by $|\mathbf{v}|$ to get $(1/|\mathbf{v}|)d\mathbf{v}/dt = \mathsf{L}^T \cdot \mathbf{v}/|\mathbf{v}| = \mathsf{L}^T \cdot \boldsymbol{\alpha}_t$. Thus Eq. (1.355) can be rewritten as

$$\frac{d\boldsymbol{\alpha}_t}{dt} = \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \frac{\mathbf{v}}{|\mathbf{v}|^2}\frac{d|\mathbf{v}|}{dt}, \quad (1.356)$$

$$= \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \boldsymbol{\alpha}_t \frac{1}{|\mathbf{v}|}\frac{d|\mathbf{v}|}{dt}. \quad (1.357)$$

Next consider the following series of operations starting with Eq. (1.347):

$$\frac{d\mathbf{v}}{dt} = \mathsf{L}^T \cdot \mathbf{v}, \quad (1.358)$$

$$\mathbf{v}^T \cdot \frac{d\mathbf{v}}{dt} = \mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v}, \quad (1.359)$$

$$\frac{d}{dt}\left(\frac{\mathbf{v}^T \cdot \mathbf{v}}{2}\right) = \mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v}, \quad (1.360)$$

$$\frac{d}{dt}\left(\frac{|\mathbf{v}|^2}{2}\right) = \mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v}, \quad (1.361)$$

$$|\mathbf{v}|\frac{d}{dt}(|\mathbf{v}|) = \mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v}, \quad (1.362)$$

$$\frac{1}{|\mathbf{v}|}\frac{d}{dt}(|\mathbf{v}|) = \frac{\mathbf{v}^T}{|\mathbf{v}|} \cdot \mathsf{L}^T \cdot \frac{\mathbf{v}}{|\mathbf{v}|}, \quad (1.363)$$

$$\frac{1}{|\mathbf{v}|}\frac{d}{dt}(|\mathbf{v}|) = \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t. \quad (1.364)$$

Now substitute Eq. (1.364) into Eq. (1.357) to get

$$\frac{d\boldsymbol{\alpha}_t}{dt} = \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \left(\boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t\right)\boldsymbol{\alpha}_t. \quad (1.365)$$

As an aside, take the dot product of Eq. (1.365) with $\boldsymbol{\alpha}_t$ to get

$$\boldsymbol{\alpha}_t^T \cdot \frac{d\boldsymbol{\alpha}_t}{dt} = \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \left(\boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t\right)\underbrace{\boldsymbol{\alpha}_t^T \cdot \boldsymbol{\alpha}_t}_{=1}, \quad (1.366)$$

$$= \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \boldsymbol{\alpha}_t^T \cdot \mathsf{L}^T \cdot \boldsymbol{\alpha}_t, \quad (1.367)$$

$$= 0. \quad (1.368)$$

This must be an identity, because $\boldsymbol{\alpha}_t^T \cdot \boldsymbol{\alpha}_t = 1$, and its time derivative gives $\boldsymbol{\alpha}_t^T \cdot d\boldsymbol{\alpha}/dt = 0$.

Now recalling Eq. (1.240), and employing $\boldsymbol{\alpha}_t^T \cdot \mathsf{R}^T \cdot \boldsymbol{\alpha}_t = 0$, because of the antisymmetry of R, and $\mathsf{D}^T = \mathsf{D}$, because of the symmetry of D, Eq. (1.365) can be rewritten as

$$\frac{d\boldsymbol{\alpha}_t}{dt} = \mathsf{L}^T \cdot \boldsymbol{\alpha}_t - \left(\boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t\right)\boldsymbol{\alpha}_t. \quad (1.369)$$

Let us consider how a volume stretches in a direction aligned with the velocity vector. We first specialize the general differential arc length to that found along the particle path: $d\mathsf{s} = ds$. Now, recall from geometry that the square of the differential arc length must be

$$ds^2 = d\mathbf{x}^T \cdot d\mathbf{x}, \quad (1.370)$$

where $d\mathbf{x}$ is also confined to the particle path. Consider now how this quantity changes with time when we move with the particle:

$$\frac{d}{dt}(ds)^2 = \frac{d}{dt}\left(d\mathbf{x}^T \cdot d\mathbf{x}\right), \quad (1.371)$$

$$= d\mathbf{x}^T \cdot \frac{d}{dt}(d\mathbf{x}) + \left(\frac{d}{dt}(d\mathbf{x})\right)^T \cdot d\mathbf{x}, \quad (1.372)$$

$$= d\mathbf{x}^T \cdot d\left(\frac{d\mathbf{x}}{dt}\right) + \left(d\left(\frac{d\mathbf{x}}{dt}\right)\right)^T \cdot d\mathbf{x}, \quad (1.373)$$

$$= d\mathbf{x}^T \cdot d\mathbf{v} + d\mathbf{v}^T \cdot d\mathbf{x}, \quad (1.374)$$

$$
\begin{aligned}
&= 2d\mathbf{x}^T\cdot d\mathbf{v}, &(1.375)\\
&= 2d\mathbf{x}^T\cdot \mathsf{L}^T\cdot d\mathbf{x}, &(1.376)\\
2ds\frac{d}{dt}(ds) &= 2d\mathbf{x}^T\cdot \mathsf{L}^T\cdot d\mathbf{x}, &(1.377)\\
\frac{1}{ds}\frac{d}{dt}(ds) &= \frac{d\mathbf{x}}{ds}^T\cdot \mathsf{L}^T\cdot \frac{d\mathbf{x}}{ds}. &(1.378)
\end{aligned}
$$

Recall now that

$$
\begin{aligned}
\boldsymbol{\alpha}_t &= \frac{\mathbf{v}}{|\mathbf{v}|}, &(1.379)\\
&= \frac{\frac{d\mathbf{x}}{dt}}{\frac{ds}{dt}}, &(1.380)\\
&= \frac{d\mathbf{x}}{ds}. &(1.381)
\end{aligned}
$$

So, Eq. (1.378) can be rewritten as

$$
\begin{aligned}
\frac{1}{ds}\frac{d}{dt}(ds) &= \boldsymbol{\alpha}_t^T\cdot \mathsf{L}^T\cdot \boldsymbol{\alpha}_t, &(1.382)\\
\frac{d}{dt}(\ln ds) &= \boldsymbol{\alpha}_t^T\cdot (\mathsf{D}+\mathsf{R})^T\cdot \boldsymbol{\alpha}_t, &(1.383)\\
&= \boldsymbol{\alpha}_t^T\cdot \mathsf{D}\cdot \boldsymbol{\alpha}_t, &(1.384)\\
&= \mathsf{D} : \boldsymbol{\alpha}_t\boldsymbol{\alpha}_t^T. &(1.385)
\end{aligned}
$$

Note that this relative tangential stretching rate is closely related to the result of Eq. (1.266) for extensional strain rate. Specializing Eq. (1.266) for a particle pathline, and combining, we can say

$$
\begin{aligned}
d\mathbf{v}^{(es)} &= (\boldsymbol{\alpha}_t^T\cdot \mathsf{D}\cdot \boldsymbol{\alpha}_t)\boldsymbol{\alpha}_t\ ds, &(1.386)\\
\frac{d\mathbf{v}^{(es)}}{ds} &= (\boldsymbol{\alpha}_t^T\cdot \mathsf{D}\cdot \boldsymbol{\alpha}_t)\boldsymbol{\alpha}_t, &(1.387)\\
\boldsymbol{\alpha}_t^T\cdot\frac{d\mathbf{v}^{(es)}}{ds} &= (\boldsymbol{\alpha}_t^T\cdot \mathsf{D}\cdot \boldsymbol{\alpha}_t)\underbrace{\boldsymbol{\alpha}_t^T\cdot\boldsymbol{\alpha}_t}_{=1}, &(1.388)\\
&= \boldsymbol{\alpha}_t^T\cdot \mathsf{D}\cdot \boldsymbol{\alpha}_t = \frac{1}{ds}\frac{d}{dt}(ds), &(1.389)\\
&= \boldsymbol{\alpha}_t^T\cdot \mathsf{D}\cdot \boldsymbol{\alpha}_t = \frac{1}{ds}d\left(\frac{ds}{dt}\right), &(1.390)\\
&= \boldsymbol{\alpha}_t^T\cdot \mathsf{D}\cdot \boldsymbol{\alpha}_t = \frac{d|\mathbf{v}|}{ds}, &(1.391)\\
&= \mathsf{D} : \boldsymbol{\alpha}_t\boldsymbol{\alpha}_t^T = \frac{d|\mathbf{v}|}{ds}. &(1.392)
\end{aligned}
$$

Here, we invoked Eq. (1.384) to obtain Eq. (1.390). The quantity $\boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t = \mathsf{D} : \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T$ is a measure of how the magnitude of the velocity changes with respect to arc length along the particle path.

We can gain further insight into how velocity magnitude changes by a diagonal decomposition of $\mathsf{D} = \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T$, where Q is an orthogonal rotation matrix with the normalized eigenvectors of D in its columns, and Λ is the diagonal matrix with the eigenvalues of D in its diagonal. Thus

$$\frac{d|\mathbf{v}|}{ds} = \boldsymbol{\alpha}_t^T \cdot \underbrace{\mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T}_{\mathsf{D}} \cdot \boldsymbol{\alpha}_t, \tag{1.393}$$

$$= (\mathsf{Q}^T \cdot \boldsymbol{\alpha}_t)^T \cdot \Lambda \cdot (\mathsf{Q}^T \cdot \boldsymbol{\alpha}_t), \tag{1.394}$$

$$\tag{1.395}$$

The operation $\mathsf{Q}^T \cdot \boldsymbol{\alpha}_t \equiv \boldsymbol{\alpha}_s$ generates a new rotated unit vector $\boldsymbol{\alpha}_s = (\alpha_{s1}, \alpha_{s2}, \alpha_{s3})^T$. Thus we can state

$$\frac{d|\mathbf{v}|}{ds} = \alpha_{s1}^2 \lambda_1 + \alpha_{s2}^2 \lambda_2 + \alpha_{s3}^2 \lambda_3, \tag{1.396}$$

$$1 = \alpha_{s1}^2 + \alpha_{s2}^2 + \alpha_{s3}^2. \tag{1.397}$$

The rate of change of the velocity magnitude along a particle pathline can be understood to be a weighted average of the eigenvalues of the deformation tensor D. In the very special case in which $\boldsymbol{\alpha}_t$ is the i^{th} eigenvector of D, we simply get $d|\mathbf{v}|/ds = \lambda_i$, where λ_i is the corresponding eigenvalue.

Note now that if we extend Eq. (1.295) to differential material volumes, we could say the relative expansion rate is

$$\frac{1}{dV}\frac{d}{dt}(dV) = \text{tr } \mathsf{D}, \tag{1.398}$$

$$\frac{d}{dt}(\ln dV) = \text{tr } \mathsf{D}. \tag{1.399}$$

Now our differential volume can be formed by

$$dV = dA\ ds, \tag{1.400}$$

where dA is the cross-sectional area normal to the flow direction. Thus

$$\ln dV = \ln dA + \ln ds, \tag{1.401}$$

$$\ln dA = \ln dV - \ln ds, \tag{1.402}$$

$$\frac{d}{dt}(\ln dA) = \frac{d}{dt}(\ln dV) - \frac{d}{dt}(\ln ds), \tag{1.403}$$

$$\tag{1.404}$$

Substitute from Eqs. (1.384,1.399) to get the relative rate of change of the differential area normal to the flow direction:

$$\frac{d}{dt}(\ln dA) = \text{tr } \mathsf{D} - \boldsymbol{\alpha}_t^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_t. \tag{1.405}$$

Note this relation, while not identical, is similar to the expression for shear strain rate, Eq. (1.276). We can also use Eq. (1.72) to rewrite Eq. (1.405) as

$$\frac{d}{dt}(\ln dA) = \mathsf{D} : \mathsf{I} - \mathsf{D} : \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T, \tag{1.406}$$

$$= \mathsf{D} : \left(\mathsf{I} - \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T\right). \tag{1.407}$$

Now the matrix $\mathsf{I} - \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T$ has some surprising properties. It is singular and has rank two. Because it is symmetric, it has a set of three orthogonal eigenvectors which can be normalized to form an orthonormal set. Its three eigenvalues are 1, 1, and 0. Remarkably, the eigenvector associated with the zero eigenvalue must be parallel to and can be selected as $\boldsymbol{\alpha}_t$, the unit tangent to the curve. Thus the other two eigenvectors can be thought of as unit normals to the curve, which we label $\boldsymbol{\alpha}_{n1}$ and $\boldsymbol{\alpha}_{n2}$. Note that these eigenvectors are not unique; however, a set can always be found. We can summarize the decomposition in the following steps:

$$\mathsf{I} - \boldsymbol{\alpha}_t \boldsymbol{\alpha}_t^T = \mathsf{Q} \cdot \Lambda \cdot \mathsf{Q}^T, \tag{1.408}$$

$$= \begin{pmatrix} \vdots & \vdots & \vdots \\ \boldsymbol{\alpha}_{n1} & \boldsymbol{\alpha}_{n2} & \boldsymbol{\alpha}_t \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cdots & \boldsymbol{\alpha}_{n1}^T & \cdots \\ \cdots & \boldsymbol{\alpha}_{n2}^T & \cdots \\ \cdots & \boldsymbol{\alpha}_t^T & \cdots \end{pmatrix}, \tag{1.409}$$

$$= \boldsymbol{\alpha}_{n1} \boldsymbol{\alpha}_{n1}^T + \boldsymbol{\alpha}_{n2} \boldsymbol{\alpha}_{n2}^T. \tag{1.410}$$

Note that the two unit normals are orthogonal to each other, $\boldsymbol{\alpha}_{n1}^T \cdot \boldsymbol{\alpha}_{n2} = 0$. Thus, we have

$$\frac{d}{dt}(\ln dA) = \mathsf{D} : \left(\boldsymbol{\alpha}_{n1} \boldsymbol{\alpha}_{n1}^T + \boldsymbol{\alpha}_{n2} \boldsymbol{\alpha}_{n2}^T\right), \tag{1.411}$$

$$= \mathsf{D} : \boldsymbol{\alpha}_{n1} \boldsymbol{\alpha}_{n1}^T + \mathsf{D} : \boldsymbol{\alpha}_{n2} \boldsymbol{\alpha}_{n2}^T, \tag{1.412}$$

$$= \boldsymbol{\alpha}_{n1}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_{n1} + \boldsymbol{\alpha}_{n2}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_{n2}. \tag{1.413}$$

Comparing to Eq. (1.384) which has one mode associated with $\boldsymbol{\alpha}_t$ available for stretching of the one-dimensional arc length in the streamwise direction, there are two modes associated with $\boldsymbol{\alpha}_{n1}$, $\boldsymbol{\alpha}_{n2}$ available for stretching the two-dimensional area.

The form $\boldsymbol{\alpha}_{n1}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_{n1}$ suggests it determines the relative normal stretching rate in the direction of $\boldsymbol{\alpha}_{n1}$; a similar rate exists for the other normal direction. One might imagine that there exists a normal direction which yields extreme values for relative normal stretching rates. It is easily shown this achieved by the following. First, define a rectangular matrix, $\widehat{\mathsf{Q}}$, whose columns are populated by $\boldsymbol{\alpha}_{n1}$ and $\boldsymbol{\alpha}_{n2}$:

$$\widehat{\mathsf{Q}} = \begin{pmatrix} \vdots & \vdots \\ \boldsymbol{\alpha}_{n1} & \boldsymbol{\alpha}_{n2} \\ \vdots & \vdots \end{pmatrix}. \tag{1.414}$$

Then project the 3×3 matrix D onto this basis to form the 2×2 matrix $\widehat{\mathsf{D}}$ associated with stretching in the directions normal to the motion:

$$\widehat{\mathsf{D}} = \widehat{\mathsf{Q}}^T \cdot \mathsf{D} \cdot \widehat{\mathsf{Q}}. \tag{1.415}$$

The eigenvalues of $\widehat{\mathsf{D}}$ give the maximum and minimum values of the relative normal stretching rates, and the eigenvectors give the associated directions of extremal normal stretching.

Looked at another way and motivated by standard results from differential geometry, we can make special choices, $\boldsymbol{\alpha}_{n1} = \boldsymbol{\alpha}_{np}$, $\boldsymbol{\alpha}_{n2} = \boldsymbol{\alpha}_{nb}$, where $\boldsymbol{\alpha}_{np}$ is the so-called "principal normal unit vector" and $\boldsymbol{\alpha}_{nb}$ is the so-called "bi-normal unit vector." The following results are described in more detail in many sources, e.g. Sen and Powers and Sen, p. 89. We have the so-called "Frenet-Serret"[24] relations:

$$\frac{d\boldsymbol{\alpha}_t}{ds} = \kappa \boldsymbol{\alpha}_{np}, \tag{1.416}$$

$$\frac{d\boldsymbol{\alpha}_{np}}{ds} = -\kappa \boldsymbol{\alpha}_t - \tau \boldsymbol{\alpha}_{nb}, \tag{1.417}$$

$$\frac{d\boldsymbol{\alpha}_{nb}}{ds} = \tau \boldsymbol{\alpha}_{np}. \tag{1.418}$$

Here κ is the so-called "curvature," of the curve and τ is the so-called "torsion" of the curve. One can, with effort show that κ and τ are given by

$$\kappa = \frac{\sqrt{\left|\frac{d^2\mathbf{x}}{dt^2}\right|^2 \left|\frac{d\mathbf{x}}{dt}\right|^2 - \left(\frac{d\mathbf{x}}{dt}^T \cdot \frac{d^2\mathbf{x}}{dt^2}\right)^2}}{\left|\frac{d\mathbf{x}}{dt}\right|^3} = \frac{\left|\frac{d\mathbf{x}}{dt} \times \frac{d^2\mathbf{x}}{dt^2}\right|}{\left|\frac{d\mathbf{x}}{dt}\right|^3}, \tag{1.419}$$

$$\tau = \frac{-\left(\frac{d\mathbf{x}}{dt} \times \frac{d^2\mathbf{x}}{dt^2}\right)^T \cdot \frac{d^3\mathbf{x}}{dt^3}}{\left|\frac{d^2\mathbf{x}}{dt^2}\right|^2 \left|\frac{d\mathbf{x}}{dt}\right|^2 - \left(\frac{d\mathbf{x}}{dt}^T \cdot \frac{d^2\mathbf{x}}{dt^2}\right)^2} \tag{1.420}$$

Note κ and τ are expressed here as functions of time. This certainly the case for a particle moving along a path in time. But just as the intrinsic curvature of a mountain road is independent of the speed of the vehicle traveling on the road, despite the traveling vehicle experiencing a time-dependency of curvature, the curvature and torsion can be considered more fundamentally to be functions of position only, given that the velocity field is known as a function of position. Analysis reveals in fact that

$$\kappa = \frac{\sqrt{(\mathbf{v}^T \cdot \mathsf{L} \cdot \mathsf{L}^T \cdot \mathbf{v})(\mathbf{v}^T \cdot \mathbf{v}) - (\mathbf{v}^T \cdot \mathsf{L}^T \cdot \mathbf{v})^2}}{(\mathbf{v}^T \cdot \mathbf{v})^{3/2}} \tag{1.421}$$

One could also develop an expression for torsion which is explicitly dependent on position. The expression is complicated and requires the use of third order tensors to capture the higher order spatial variations.

[24] Jean Frédéric Frenet, 1816-1900, and Joseph Alfred Serret, 1819-1885, French mathematicians.

We can also use this intrinsic orthonormal basis to get

$$\frac{d}{dt}(\ln dA) = \mathsf{D} : \left(\boldsymbol{\alpha}_{np}\boldsymbol{\alpha}_{np}^T + \boldsymbol{\alpha}_{nb}\boldsymbol{\alpha}_{nb}^T\right), \quad (1.422)$$

$$= \mathsf{D} : \boldsymbol{\alpha}_{np}\boldsymbol{\alpha}_{np}^T + \mathsf{D} : \boldsymbol{\alpha}_{nb}\boldsymbol{\alpha}_{nb}^T, \quad (1.423)$$

$$= \boldsymbol{\alpha}_{np}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_{np} + \boldsymbol{\alpha}_{nb}^T \cdot \mathsf{D} \cdot \boldsymbol{\alpha}_{nb}. \quad (1.424)$$

1.4 Conservation axioms

A fundamental goal of this section is to take the verbal notions which embody the basic axioms of non-relativistic continuum mechanics into usable mathematical expressions. First, we must list those axioms. The axioms themselves are simply principles which have been observed to have wide validity as long as the particle velocity is small relative to the speed of light and length scales are sufficiently large to contain many molecules. Many of these axioms can be applied to molecules as well. The axioms cannot be proven. They are simply statements which have been useful in describing the universe.

A summary of the axioms in words is as follows

- *Mass conservation principle*: The time rate of change of mass of a material region is zero.

- *Linear momenta principle*: The time rate of change of the linear momenta of a material region is equal to the sum of forces acting on the region. This is Euler's generalization of Newton's second law of motion.

- *Angular momenta principle*: The time rate of change of the angular momenta of a material region is equal to the sum of the torques acting on the region. This was first formulated by Euler.

- *Energy conservation principle*: The time rate of change of energy within a material region is equal to the rate that energy is received by heat and work interactions. This is the first law of thermodynamics.

- *Entropy inequality*: The time rate of change of entropy within a material region is greater than or equal to the ratio of the rate of heat transferred to the region and the absolute temperature of the region. This is the second law of thermodynamics.

Some secondary concepts related to these axioms are as follows

- The local stress on one side of a surface is identically opposite that stress on the opposite side.

- Stress can be separated into *thermodynamic* and *viscous* stress.

- Forces can be separated into *surface* and *body* forces.

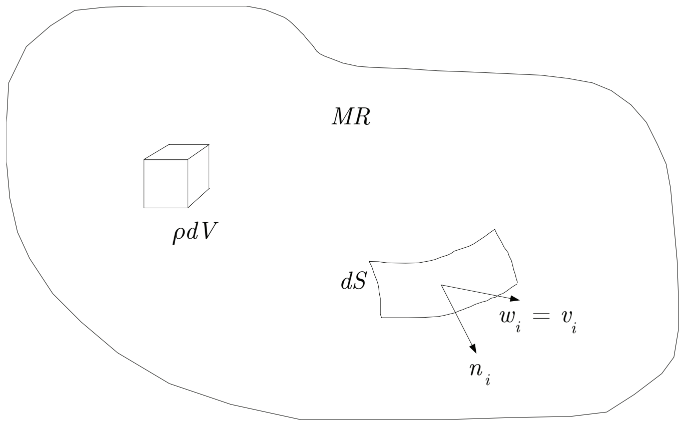

Figure 1.19: Sketch of finite material region MR, infinitesimal mass element ρdV, and infinitesimal surface element dS with unit normal n_i, and general velocity w_i equal to fluid velocity v_i.

- In the absence of body couples, the angular momenta principle reduces to a nearly trivial statement.
- The energy equation can be separated into mechanical and thermal components.

Next we shall systematically convert these words above into mathematical form.

1.4.1 Mass

The mass conservation axiom is simple to state mathematically. It is

$$\frac{d}{dt} m_{MR(t)} = 0. \tag{1.425}$$

Here $MR(t)$ stands for a material region which can evolve in time, and $m_{MR(t)}$ is the mass in the material region. A relevant material region is sketched in Figure 1.19. We can define the mass of the material region based upon the local value of density:

$$m_{MR(t)} = \int_{MR(t)} \rho \; dV. \tag{1.426}$$

So, the mass conservation axiom is

$$\frac{d}{dt} \int_{MR(t)} \rho \; dV = 0. \tag{1.427}$$

Recalling Leibniz's rule, Eq. (1.185), $\frac{d}{dt}\int_{R(t)} [\;]\; dV = \int_{R(t)} \frac{\partial}{\partial t}[\;]\; dV + \int_{S(t)} n_i w_i [\;]\; dS$, we take the arbitrary velocity $w_i = v_i$ as we are considering a material region so we get

$$\frac{d}{dt} \int_{MR(t)} \rho \; dV = \int_{MR(t)} \frac{\partial \rho}{\partial t} \; dV + \int_{MS(t)} n_i v_i \rho \; dS = 0. \tag{1.428}$$

Now we invoke Gauss's theorem, Eq. (1.174) $\int_{R(t)} \partial_i[\,]dV = \int_{S(t)} n_i[\,]dS$, to convert a surface integral to a volume integral to get the mass conservation axiom to read as

$$\int_{MR(t)} \frac{\partial \rho}{\partial t}\, dV + \int_{MR(t)} \partial_i(\rho v_i)\, dV = 0, \tag{1.429}$$

$$\int_{MR(t)} \left(\frac{\partial \rho}{\partial t} + \partial_i(\rho v_i) \right) dV = 0. \tag{1.430}$$

Now, in an important step, we realize that the only way for this integral, which has absolutely arbitrary limits of integration, to always be zero, is for the integrand itself to always be zero. Hence, we have

$$\frac{\partial \rho}{\partial t} + \partial_i(\rho v_i) = 0, \tag{1.431}$$

which we will write in Cartesian index, Gibbs, and full notation in what we call conservative or divergence form as

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{1.432}$$

$$\partial_o \rho + \nabla^T \cdot (\rho \mathbf{v}) = 0, \tag{1.433}$$

$$\partial_o \rho + \partial_1(\rho v_1) + \partial_2(\rho v_2) + \partial_3(\rho v_3) = 0. \tag{1.434}$$

There are several alternative forms for this axiom. Using the product rule, we can say also

$$\underbrace{\partial_o \rho + v_i \partial_i \rho}_{\text{material derivative of density}} + \rho \partial_i v_i = 0, \tag{1.435}$$

or, writing in what is called the non-conservative form,

$$\frac{d\rho}{dt} + \rho \partial_i v_i = 0, \tag{1.436}$$

$$\frac{d\rho}{dt} + \rho \nabla^T \cdot \mathbf{v} = 0, \tag{1.437}$$

$$(\partial_o \rho + v_1 \partial_1 \rho + v_2 \partial_2 \rho + v_3 \partial_3 \rho) + \rho\,(\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3) = 0. \tag{1.438}$$

So, we can also say

$$\underbrace{\frac{1}{\rho}\frac{d\rho}{dt}}_{\text{relative rate of density increase}} = - \underbrace{\partial_i v_i}_{\text{relative rate of particle volume expansion}}. \tag{1.439}$$

Thus the relative rate of density increase of a fluid particle is the negative of its relative rate of expansion, as expected. So, we also have

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{1}{V_{MR}}\frac{dV_{MR}}{dt}, \tag{1.440}$$

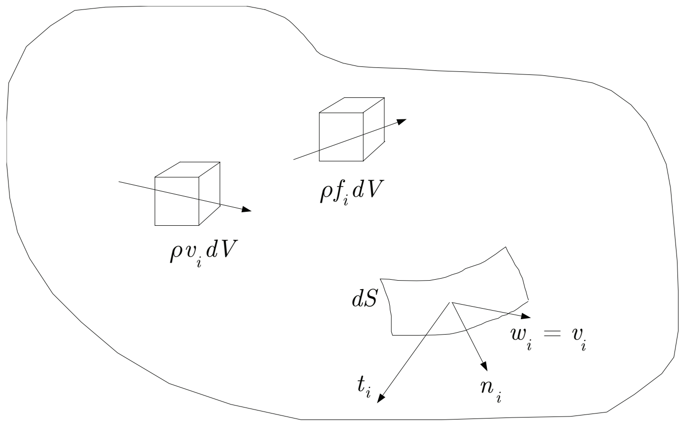

Figure 1.20: Sketch of finite material region MR, infinitesimal linear momenta element $\rho v_i dV$, infinitesimal body force element $\rho f_i dV$, and infinitesimal surface element dS with unit normal n_i, surface traction t_i and general velocity w_i equal to fluid velocity v_i.

$$\rho \frac{dV_{MR}}{dt} + V_{MR}\frac{d\rho}{dt} = 0, \quad (1.441)$$

$$\frac{d}{dt}(\rho V_{MR}) = 0, \quad (1.442)$$

$$\frac{d}{dt}(m_{MR}) = 0. \quad (1.443)$$

We note that in a relativistic system, in which mass-energy is conserved, but not mass, that we can have a material region, that is a region bounded by a surface across which there is no flux of mass, for which the mass can indeed change, thus violating our non-relativistic mass conservation axiom.

1.4.2 Linear momenta

1.4.2.1 Statement of the principle

The linear momenta conservation axiom is simple to state mathematically. It is

$$\underbrace{\frac{d}{dt}\int_{MR(t)} \rho v_i \; dV}_{\text{rate of change of linear momenta}} = \underbrace{\int_{MR(t)} \rho f_i \; dV}_{\text{body forces}} + \underbrace{\int_{MS(t)} t_i \; dS}_{\text{surface forces}}. \quad (1.444)$$

Again $MR(t)$ stands for a material region which can evolve in time. A relevant material region is sketched in Figure 1.20. The term f_i represents a body force per unit mass. An example of such a force would be the gravitational force acting on a body, which when scaled by mass, yields g_i. The term t_i is a traction, which is a vector representing force per unit area. A major challenge of this section will be to express the traction in terms of what is known as the stress tensor.

Consider first the left hand side, LHS, of the linear momenta principle

$$LHS = \int_{MR(t)} \partial_o(\rho v_i)\ dV + \int_{MS(t)} n_j \rho v_i v_j\ dS, \qquad \text{from Leibniz}, \tag{1.445}$$

$$= \int_{MR(t)} \left(\partial_o(\rho v_i) + \partial_j(\rho v_j v_i)\right)\ dV, \qquad \text{from Gauss}. \tag{1.446}$$

So, the linear momenta principle is

$$\int_{MR(t)} \left(\partial_o(\rho v_i) + \partial_j(\rho v_j v_i)\right)\ dV = \int_{MR(t)} \rho f_i\ dV + \int_{MS(t)} t_i\ dS. \tag{1.447}$$

These are all expressed in terms of volume integrals except for the term involving surface forces.

1.4.2.2 Surface forces

The surface force per unit area is a vector we call the traction t_j. It has the units of stress, but it is not formally a stress, which is a tensor. The traction is a function of both position x_i and surface orientation n_k: $t_j = t_j(x_i, n_k)$.

We intend to demonstrate the following: The traction can be stated in terms of a *stress tensor* T_{ij} as written below:

$$t_j = n_i T_{ij},$$
$$\mathbf{t}^T = \mathbf{n}^T \cdot \mathsf{T},$$
$$\mathbf{t} = \mathsf{T}^T \cdot \mathbf{n}. \tag{1.448}$$

The following excursions are necessary to show this.

- *Show force on one side of surface equal and opposite to that on the opposite side*

  Let us apply the principle of linear momenta to the material region is sketched in Figure 1.21. Here we indicate the dependency of the traction on orientation by notation such as $t_i\left(n_i^{II}\right)$. This does not indicate multiplication, nor that i is a dummy index here. In Figure 1.21, the thin pillbox has width Δl, circumference s, and a surface area for the circular region of ΔS. Surface I is a circular region; surface II is the opposite circular region, and surface III is the cylindrical side.

  We apply the mean value theorem to the linear momenta principle for this region and get

$$\left(\partial_o(\rho v_i) + \partial_j(\rho v_j v_i)\right)^* (\Delta S)(\Delta l) =$$
$$(\rho f_i)^*(\Delta S)(\Delta l) + t_i^*(n_i^I)\Delta S + t_i^*(n_i^{II})\Delta S + t_i^*(n_i^{III}) s(\Delta l). \tag{1.449}$$

  Now we let $\Delta l \to 0$, holding for now s and ΔS fixed to obtain

$$0 = \left(t_i^*(n_i^I) + t_i^*(n_i^{II})\right)\Delta S \tag{1.450}$$

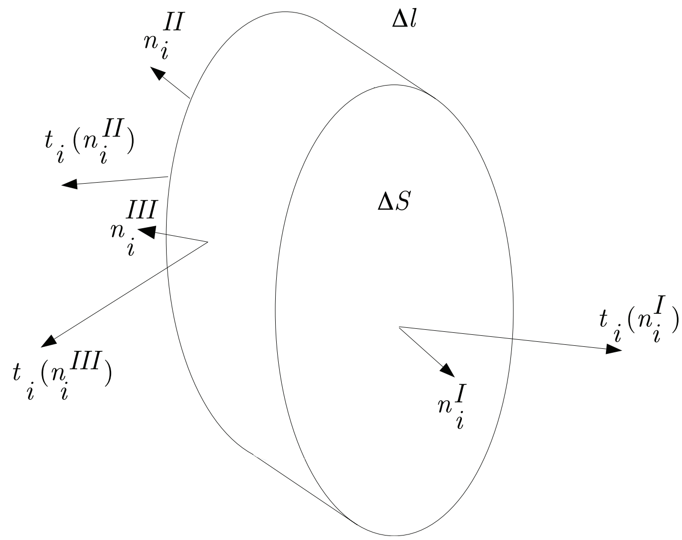

Figure 1.21: Sketch of pillbox element for stress analysis.

Now letting $\Delta S \to 0$, so that the mean value approaches the local value, and taking $n_i^I = -n_i^{II} \equiv n_i$, we get a useful result

$$t_i(n_i) = -t_i(-n_i). \tag{1.451}$$

At an infinitesimal length scale, the traction on one side of a surface is equal an opposite that on the other. That is, there is a local force balance. This applies even if there is velocity and acceleration of the material on a macroscale. On the microscale, surface forces dominate inertia and body forces. This is a useful general principle to remember. It the fundamental reason why microorganisms have very different propulsion systems that macro-organisms: they are fighting different forces.

- *Study stress on arbitrary plane and relate to stress on coordinate planes*

  Now let us consider a rectangular parallelepiped aligned with the Cartesian axes which has been sliced at an oblique angle to form a tetrahedron. We will apply the linear momenta principle to this geometry and make a statement about the existence of a stress tensor. The described material region is sketched in Figure 1.22. Let ΔL be a characteristic length scale of the tetrahedron. Also let four unit normals n_j exist, one for each surface. They will be $-n_1$, $-n_2$, $-n_3$ for the surfaces associated with each coordinate direction. They are negative because the outer normal points opposite to the direction of the axes. Let n_i be the normal associated with the oblique face. Let ΔS denote the surface area of each face.

  Now the volume of the tetrahedron must be of order L^3 and the surface area of order L^2. Thus applying the mean value theorem to the linear momenta principle, we obtain the form

$$(\text{inertia}) \times (\Delta L)^3 = (\text{body forces}) \times (\Delta L)^3 + (\text{surface forces}) \times (\Delta L)^2. \tag{1.452}$$

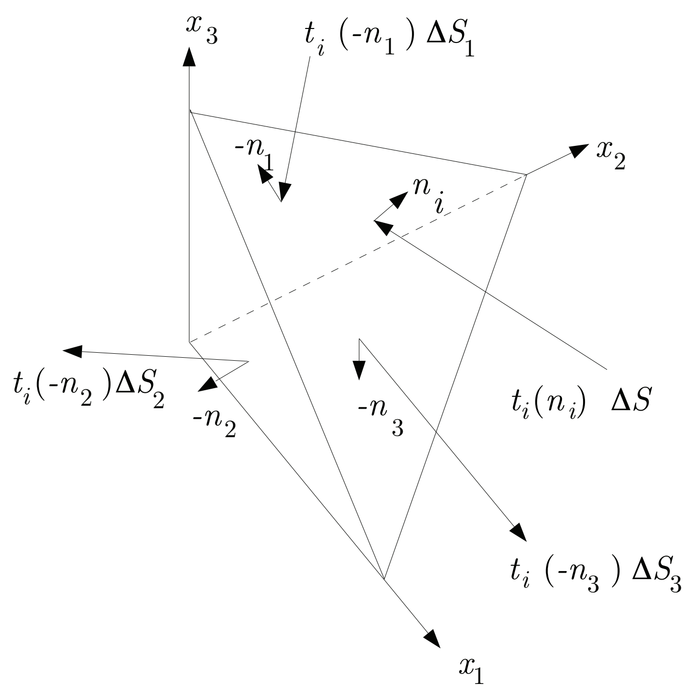

Figure 1.22: Sketch of tetrahedral element for stress analysis on an arbitrary plane.

As before, for small volumes, $\Delta L \to 0$, the linear momenta principle reduces to

$$\sum \text{surface forces} = 0. \tag{1.453}$$

Applying this to the configuration of Figure 1.22, we get

$$0 = t_i^*(n_i)\Delta S + t_i^*(-n_1)\Delta S_1 + t_i^*(-n_2)\Delta S_2 + t_i^*(-n_3)\Delta S_3. \tag{1.454}$$

But we know that $t_j(n_j) = -t_j(-n_j)$, so

$$t_i^*(n_i)\Delta S = t_i^*(n_1)\Delta S_1 + t_i^*(n_2)\Delta S_2 + t_i^*(n_3)\Delta S_3. \tag{1.455}$$

Now it is not a difficult geometry problem to show that $n_i\Delta S = \Delta S_i$, so we get

$$t_i^*(n_i)\Delta S = n_1 t_i^*(n_1)\Delta S + n_2 t_i^*(n_2)\Delta S + n_3 t_i^*(n_3)\Delta S, \tag{1.456}$$

$$t_i^*(n_i) = n_1 t_i^*(n_1) + n_2 t_i^*(n_2) + n_3 t_i^*(n_3). \tag{1.457}$$

Now we can consider terms like t_i is obviously a vector, and the indicator, for example (n_1), tells us with which surface the vector is associated. This is precisely what a tensor does, and in fact we can say

$$t_i(n_i) = n_1 T_{1i} + n_2 T_{2i} + n_3 T_{3i}. \tag{1.458}$$

In shorthand, we can say the same thing with

$$t_i = n_j T_{ji}, \qquad \text{or equivalently} \qquad t_j = n_i T_{ij}, \qquad \text{QED.} \tag{1.459}$$

Here T_{ij} is the component of stress in the j direction associated with the surface whose normal is in the i direction.

- *Consider pressure and the viscous stress tensor*

  Pressure is a familiar concept from thermodynamics and fluid statics. It is often tempting and sometimes correct to think of the pressure as the force per unit area normal to a surface and the force tangential to a surface being somehow related to frictional forces. We shall see that in general, this view is too simplistic.

  First recall from thermodynamics that what we will call p, the *thermodynamic pressure*, is for a simple compressible substance a function of at most two intensive thermodynamic variables, say $p = f(\rho, e)$, where e is the specific internal energy. Also recall that the thermodynamic pressure must be a normal stress, as thermodynamics considers formally only materials at rest, and viscous stresses are associated with moving fluids.

  To distinguish between thermodynamic stresses and other stresses, let us define the *viscous stress tensor* τ_{ij} as follows

$$\tau_{ij} = T_{ij} + p\delta_{ij}. \tag{1.460}$$

  Recall that T_{ij} is the *total stress tensor*. We obviously also have

$$T_{ij} = -p\delta_{ij} + \tau_{ij}. \tag{1.461}$$

  Note with this definition that pressure is positive in compression, while T_{ij} and τ_{ij} are positive in tension. Let us also define the *mechanical pressure*, $p^{(m)}$, as the negative of the average normal surface stress

$$p^{(m)} \equiv -\frac{1}{3}T_{ii} = -\frac{1}{3}(T_{11} + T_{22} + T_{33}). \tag{1.462}$$

  The often invoked *Stokes' assumption*, which remains a subject of widespread misunderstanding 150 years after it was first made, is often adopted for lack of a good alternative in answer to a question which will be addressed later in this chapter. It asserts that the thermodynamic pressure is equal to the mechanical pressure:

$$p = p^{(m)} = -\frac{1}{3}T_{ii}. \tag{1.463}$$

  Presumably a pressure measuring device in a moving flow field would actually measure the mechanical pressure, and not necessarily the thermodynamic pressure, so it is important to have this issue clarified for proper reconciliation of theory and measurement.

It will be seen that Stokes' assumption gives some minor aesthetic pleasure in certain limits, but it is not well-established, and is more a convenience than a requirement for most materials. It is the case that various incarnations of more fundamental kinetic theory under the assumption of a dilute gas composed of inert hard spheres give rise to the conclusion that Stokes' assumption is valid. At moderate densities, these hard sphere kinetic theory models predict that Stokes' assumption is invalid. However, none of the common kinetic theory models is able to predict results from experiments, which nevertheless also give indication, albeit indirect, that Stokes' assumption is invalid. Kinetic theories and experiments which consider polyatomic molecules, which can suffer vibrational and rotational effects as well, show further deviation from Stokes' assumption. It is often plausibly argued that these so-called non-equilibrium effects, that is molecular vibration and rotation, which are only important in high speed flow applications in which the flow velocity is on the order of the fluid sound speed, are the mechanisms which cause Stokes' assumption to be violated. Because they only are important in high speed applications, they are difficult to measure, though measurement of the decay of acoustic waves has provided some data. For liquids, there is little to no theory, and the limited data indicates that Stokes' assumption is invalid.

Now contracting Eq. (1.461), we get

$$T_{ii} = -p\delta_{ii} + \tau_{ii}. \tag{1.464}$$

Using the fact that $\delta_{ii} = 3$ and inserting Eq. (1.463) in Eq. (1.464), we find *for a fluid that obeys Stokes' assumption* that

$$T_{ii} = \frac{1}{3}T_{ii}(3) + \tau_{ii}, \tag{1.465}$$

$$0 = \tau_{ii}. \tag{1.466}$$

That is to say, the trace of the viscous stress tensor is zero. Moreover, for a fluid which obeys Stokes' assumption we can interpret the viscous stress as the deviation from the mean stress; that is, the viscous stress is a deviatoric stress:

$$\underbrace{T_{ij}}_{\text{total stress}} = -\underbrace{\frac{1}{3}T_{kk}\delta_{ij}}_{\text{mean stress}} + \underbrace{\tau_{ij}}_{\text{deviatoric stress}}, \qquad \text{(valid only if Stokes' assumption holds)} \tag{1.467}$$

If Stokes' assumption does not hold, then a portion of τ_{ij} will also contribute to the mean stress; that is, the viscous stress is not then entirely deviatoric.

Finally, let us note what the traction vector is when the fluid is static. For a static fluid, there is no viscous stress, so $\tau_{ij} = 0$, and we have

$$T_{ij} = -p\delta_{ij}, \qquad \text{static fluid.} \tag{1.468}$$

We get the traction vector on any surface with normal n_i by

$$t_j = n_i T_{ij} = -p n_i \delta_{ij} = -p n_j. \tag{1.469}$$

Changing indices, we see $t_i = -pn_i$, that is the traction vector must be oriented in the same direction as the surface normal; all stresses are normal to any arbitrarily oriented surface.

1.4.2.3 Final form of linear momenta equation

We are now prepared to write the linear momenta equation in final form. Substituting our expression for the traction vector, Eq. (1.459) into the linear momenta expression, Eq. (1.447), we get

$$\int_{MR(t)} (\partial_o(\rho v_i) + \partial_j(\rho v_j v_i))\ dV = \int_{MR(t)} \rho f_i\ dV + \int_{MS(t)} n_j T_{ji}\ dS. \tag{1.470}$$

Using Gauss's theorem, Eq. (1.174), to convert the surface integral into a volume integral, and combining all under one integral sign, we get

$$\int_{MR(t)} (\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) - \rho f_i - \partial_j T_{ji})\ dV = 0. \tag{1.471}$$

Making the same argument as before regarding arbitrary material volumes, this must then require that the integrand be zero (we actually must require all variables be continuous to make this work), so we obtain

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) - \rho f_i - \partial_j T_{ji} = 0. \tag{1.472}$$

Using then $T_{ij} = -p\delta_{ij} + \tau_{ij}$, we get in Cartesian index, Gibbs[25], and full notation

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{1.473}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \left(\nabla^T \cdot (\rho \mathbf{v}\mathbf{v}^T)\right)^T = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T, \tag{1.474}$$

$$\partial_o(\rho v_1) + \partial_1(\rho v_1 v_1) + \partial_2(\rho v_2 v_1) + \partial_3(\rho v_3 v_1) = \rho f_1 - \partial_1 p + \partial_1 \tau_{11} + \partial_2 \tau_{21} + \partial_3 \tau_{31}, \tag{1.475}$$

$$\partial_o(\rho v_2) + \partial_1(\rho v_1 v_2) + \partial_2(\rho v_2 v_2) + \partial_3(\rho v_3 v_2) = \rho f_2 - \partial_2 p + \partial_1 \tau_{12} + \partial_2 \tau_{22} + \partial_3 \tau_{32}, \tag{1.476}$$

$$\partial_o(\rho v_3) + \partial_1(\rho v_1 v_3) + \partial_2(\rho v_2 v_3) + \partial_3(\rho v_3 v_3) = \rho f_3 - \partial_3 p + \partial_1 \tau_{13} + \partial_2 \tau_{23} + \partial_3 \tau_{33}. \tag{1.477}$$

The form above is known as the linear momenta principle cast in conservative or divergence form. It is the first choice of forms for many numerical simulations, as discretizations of this form of the equation naturally preserve the correct values of global linear momenta, up to roundoff error.

However, there is a reduced, non-conservative form which makes some analysis and physical interpretation easier. Let us use the product rule to expand the linear momenta principle, then rearrange it, and use mass conservation and the definition of material derivative

[25] Here the transpose notation is particularly cumbersome and unfamiliar, though necessary for full consistency. One will more commonly see this equation written simply as $\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \mathbf{f} - \nabla p + \nabla \cdot \boldsymbol{\tau}$.

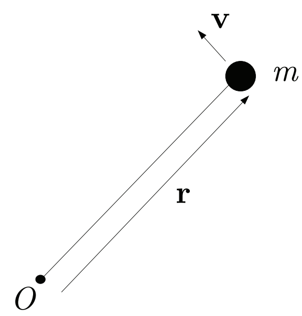

Figure 1.23: Sketch of particle of mass m velocity $\mathbf{v}$ rotating about an axis centered at point O, with radial distance vector $\mathbf{r}$.

to rewrite the expression:

$$\rho\partial_o v_i + v_i\partial_o\rho + v_i\partial_j(\rho v_j) + \rho v_j\partial_j v_i = \rho f_i - \partial_i p + \partial_j\tau_{ji}, \quad (1.478)$$

$$\rho(\partial_o v_i + v_j\partial_j v_i) + v_i(\underbrace{\partial_o\rho + \partial_j(\rho v_j)}_{=0 \text{ by mass conservation}}) = \rho f_i - \partial_i p + \partial_j\tau_{ji}, \quad (1.479)$$

$$\rho\underbrace{(\partial_o v_i + v_j\partial_j v_i)}_{=\frac{dv_i}{dt}} = \rho f_i - \partial_i p + \partial_j\tau_{ji}, \quad (1.480)$$

$$\rho\frac{dv_i}{dt} = \rho f_i - \partial_i p + \partial_j\tau_{ji}, \quad (1.481)$$

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{f} - \nabla p + \left(\nabla^T\cdot\boldsymbol{\tau}\right)^T, \quad (1.482)$$

$$\rho(\partial_o v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1 + v_3\partial_3 v_1) = \rho f_1 - \partial_1 p + \partial_1\tau_{11} + \partial_2\tau_{21} + \partial_3\tau_{31}(1.483)$$

$$\rho(\partial_o v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2 + v_3\partial_3 v_2) = \rho f_2 - \partial_2 p + \partial_1\tau_{12} + \partial_2\tau_{22} + \partial_3\tau_{32}(1.484)$$

$$\rho(\partial_o v_3 + v_1\partial_1 v_3 + v_2\partial_2 v_3 + v_3\partial_3 v_3) = \rho f_3 - \partial_3 p + \partial_1\tau_{13} + \partial_2\tau_{23} + \partial_3\tau_{33}(1.485)$$

So, we see that particles accelerate due to body forces and unbalanced surface forces. If the surface forces are non-zero but uniform, they will have no gradient or divergence, and hence not contribute to accelerating a particle.

1.4.3 Angular momenta

It is often easy to overlook the angular momenta principle, and its consequence is so simple that, it is often just asserted without proof. In fact in classical rigid body mechanics, it is redundant with the linear momenta principle. It is, however, an independent axiom for continuous deformable media.

Let us first recall some notions from classical rigid body mechanics, while referring to the sketch of Figure 1.23. We have the angular momenta vector $\mathbf{L}$ for the particle of Figure 1.23

$$\mathbf{L} = \mathbf{r}\times(m\mathbf{v}). \quad (1.486)$$

Any force $\mathbf{F}$ which acts on m with lever arm $\mathbf{r}$ induces a torque $\hat{\mathbf{T}}$ which is

$$\hat{\mathbf{T}} = \mathbf{r} \times \mathbf{F}. \tag{1.487}$$

Now let us apply these notions for an infinitesimal fluid particle with differential mass ρdV.

$$\begin{aligned}
\text{Angular momenta} &= \mathbf{r} \times (\rho\ dV)\mathbf{v} = \rho \epsilon_{ijk} r_j v_k\ dV, && (1.488)\\
\text{Torque of body force} &= \mathbf{r} \times \mathbf{f}(\rho\ dV) = \rho \epsilon_{ijk} r_j f_k\ dV, && (1.489)\\
\text{Torque of surface force} &= \mathbf{r} \times \mathbf{t}\ dS = \epsilon_{ijk} r_j t_k\ dS, &&\\
&= \mathbf{r} \times (\mathbf{n}^T \cdot \mathsf{T})\ dS = \epsilon_{ijk} r_j n_p T_{pk}\ dS, && (1.490)\\
\text{Angular momenta from surface couples} &= \mathbf{n}^T \cdot \mathsf{H}\ dS = n_k H_{ki}\ dS. && (1.491)
\end{aligned}$$

Now the principle, which in words says the time rate of change of angular momenta of a material region is equal to the sum of external couples (or torques) on the system becomes mathematically,

$$\underbrace{\frac{d}{dt} \int_{MR(t)} \rho \epsilon_{ijk} r_j v_k\ dV}_{\text{Apply Leibniz then Gauss}} = \int_{MR(t)} \rho \epsilon_{ijk} r_j f_k\ dV + \underbrace{\int_{MS(t)} \left(\epsilon_{ijk} r_j n_p T_{pk} + n_k H_{ki} \right)\ dS}_{\text{apply Gauss}}. \tag{1.492}$$

We apply Leibniz's and Gauss's theorem to the indicated terms and let the volume of the material region shrink to zero now. First with Leibniz, we get

$$\int_{MR(t)} \partial_o \rho \epsilon_{ijk} r_j v_k\ dV + \int_{MS(t)} \epsilon_{ijk} \rho r_j v_k n_p v_p\ dS =$$
$$\int_{MR(t)} \rho \epsilon_{ijk} r_j f_k\ dV + \int_{MS(t)} \left(\epsilon_{ijk} r_j n_p T_{pk} + n_k H_{ki} \right)\ dS. \tag{1.493}$$

Next with Gauss we get

$$\int_{MR(t)} \partial_o \rho \epsilon_{ijk} r_j v_k\ dV + \int_{MR(t)} \epsilon_{ijk} \partial_p (\rho r_j v_k v_p)\ dV =$$
$$\int_{MR(t)} \rho \epsilon_{ijk} r_j f_k\ dV + \int_{MR(t)} \epsilon_{ijk} \partial_p (r_j T_{pk})\ dV + \int_{MR(t)} \partial_k H_{ki}\ dV. \tag{1.494}$$

As the region is arbitrary, the integrand formed by placing all terms under the same integral must be zero, which yields

$$\epsilon_{ijk} \left(\partial_o (\rho r_j v_k) + \partial_p (\rho r_j v_p v_k) - \rho r_j f_k - \partial_p (r_j T_{pk}) \right) = \partial_k H_{ki}. \tag{1.495}$$

Using the product rule to expand some of the derivatives, we get

$$\epsilon_{ijk} \left(r_j \partial_o (\rho v_k) + \rho v_k \underbrace{\partial_o r_j}_{=0} + r_j \partial_p (\rho v_p v_k) + \rho v_p v_k \underbrace{\partial_p r_j}_{\delta_{pr}} - r_j \rho f_k - r_j \partial_p T_{pk} - T_{pk} \underbrace{\partial_p r_j}_{\delta_{pr}} \right) = \partial_k H_{ki}. \tag{1.496}$$

Applying the simplifications indicated above and rearranging, we get

$$\epsilon_{ijk} r_j \underbrace{\left(\partial_o(\rho v_k) + \partial_p(\rho v_p v_k) - \rho f_k - \partial_p T_{pk}\right)}_{=0 \text{ by linear momenta}} = \partial_k H_{ki} - \rho \epsilon_{ijk} v_j v_k + \epsilon_{ijk} T_{jk}. \quad (1.497)$$

So, we can say,

$$\partial_k H_{ki} = \epsilon_{ijk}(\rho v_j v_k - T_{jk}) = \underbrace{\epsilon_{ijk}}_{\text{anti-sym.}} \left(\underbrace{\rho v_j v_k}_{\text{sym.}} - \underbrace{T_{(jk)}}_{\text{sym.}} - \underbrace{T_{[jk]}}_{\text{anti-sym.}} \right), \quad (1.498)$$

$$= -\epsilon_{ijk} T_{[jk]}. \quad (1.499)$$

We have utilized the fact that the tensor inner product of any anti-symmetric tensor with any symmetric tensor must be zero. Now, if we have the case where there are no externally imposed angular momenta fields, such as could be the case when electromagnetic forces are important, we have the common condition of $H_{ki} = 0$, and the angular momenta principle reduces to the simple statement that

$$T_{[ij]} = 0. \quad (1.500)$$

That is, the anti-symmetric part of the stress tensor must be zero. Hence, the stress tensor, absent any body or surface couples, must be symmetric, and we get in Cartesian index and Gibbs notation:

$$T_{ij} = T_{ji}, \quad (1.501)$$

$$\mathsf{T} = \mathsf{T}^T. \quad (1.502)$$

1.4.4 Energy

We recall the first law of thermodynamics, which states the time rate of change of a material region's internal and kinetic energy is equal to the rate of heat transferred to the material region less the rate of work done by the material region. Mathematically, this is stated as

$$\frac{dE}{dt} = \frac{dQ}{dt} - \frac{dW}{dt}. \quad (1.503)$$

In this case (though this is not uniformly enforced in these notes), the upper case letters denote extensive thermodynamic properties. For example, E is total energy, inclusive of internal and kinetic, with SI units of J. We could have included potential energy in E, but will instead absorb it into the work term W. Let us consider each term in the first law of thermodynamics in detail and then write the equation in final form.

1.4.4.1 Total energy term

For a fluid particle, the differential amount of total energy is

$$dE = \rho\left(e + \frac{1}{2}v_j v_j\right)\,dV, \tag{1.504}$$

$$= \underbrace{\rho\,dV}_{\text{mass}}\quad\underbrace{\left(e + \frac{1}{2}v_j v_j\right)}_{\text{specific internal +kinetic energy}}. \tag{1.505}$$

1.4.4.2 Work term

Recall that work is done when a force acts through a distance, and a work rate arises when a force acts through a distance at a particular rate in time (hence, a velocity is involved). Recall also that it is the dot product (inner product) of the force vector with the position or velocity that gives the true work or work rate. In shorthand, we could say

$$dW = d\mathbf{x}^T \cdot \mathbf{F}, \tag{1.506}$$

$$\frac{dW}{dt} = \frac{d\mathbf{x}^T}{dt}\cdot\mathbf{F} = \mathbf{v}^T\cdot\mathbf{F}. \tag{1.507}$$

Here W has the SI units of J, and $\mathbf{F}$ has the SI units of N. We contrast this with our expression for body force per unit mass $\mathbf{f}$, which has SI units of N/kg = m/s^2. Now for the materials we consider, we must describe work done by two types of forces: 1) body, and 2) surface.

- *Work rate done by a body force*

$$\text{Work rate done } by \text{ force on fluid} = (\rho\,dV)(f_i)v_i, \tag{1.508}$$

$$\text{Work rate done } by \text{ fluid} = -\rho v_i f_i\,dV. \tag{1.509}$$

- *Work rate done by a surface force*

$$\text{Work rate done } by \text{ force on fluid} = (t_i\,dS)v_i = ((n_j T_{ji})\,dS)v_i, \tag{1.510}$$

$$\text{Work rate done } by \text{ fluid} = -n_j T_{ji} v_i\,dS. \tag{1.511}$$

1.4.4.3 Heat transfer term

The only thing confusing about the heat transfer rate is the sign convention. We recall that heat transfer *to* a body is associated with an increase in that body's energy. Now following the scenario sketched in the material region of Figure 1.24, we define the heat flux vector q_i as a vector which points in the direction of thermal energy flow which has units of energy per area per time; in SI this would be W/m^2. So, we have

$$\text{heat transfer rate } from \text{ body through } dS = n_i q_i\,dS, \tag{1.512}$$

$$\text{heat transfer rate } to \text{ body through } dS = -n_i q_i\,dS. \tag{1.513}$$

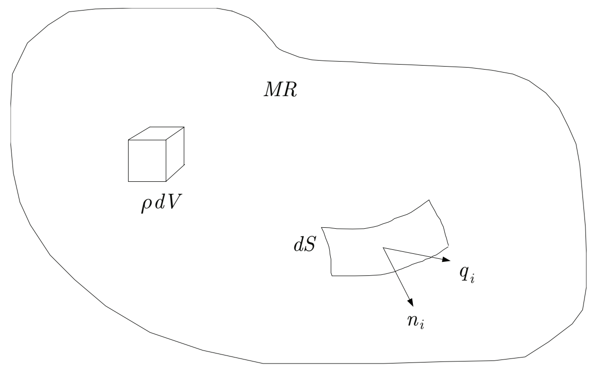

Figure 1.24: Sketch of finite material region MR, infinitesimal mass element ρdV, and infinitesimal surface element dS with unit normal n_i, and heat flux vector q_i.

1.4.4.4 Conservative form of the energy equation

Putting the words of the first law into equation form, we get

$$\frac{d}{dt}\int_{MR(t)} \rho\left(e+\frac{1}{2}v_jv_j\right) dV = -\int_{MS(t)} n_iq_i\, dS + \int_{MS(t)} n_iT_{ij}v_j\, dS + \int_{MR(t)} \rho f_iv_i\, dS. \quad (1.514)$$

Skipping the details of an identical application of Leibniz's and Gauss's theorems, and shrinking the volume to approach zero, we obtain the differential equation of energy in *conservative* or divergence form (in first Cartesian index then Gibbs notation):

$$\underbrace{\partial_o\left(\rho\left(e+\frac{1}{2}v_jv_j\right)\right)}_{\text{rate of change of total energy}} + \underbrace{\partial_i\left(\rho v_i\left(e+\frac{1}{2}v_jv_j\right)\right)}_{\text{advection of total energy}} = - \underbrace{\partial_i q_i}_{\text{diffusive heat flux}} + \underbrace{\partial_i(T_{ij}v_j)}_{\text{surface force work rate}} + \underbrace{\rho v_i f_i}_{\text{body force work rate}}, \quad (1.515)$$

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) + \nabla^T\cdot\left(\rho\mathbf{v}\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) = -\nabla^T\cdot\mathbf{q} + \nabla^T\cdot(\mathsf{T}\cdot\mathbf{v}) + \rho\mathbf{v}^T\cdot\mathbf{f}. \quad (1.516)$$

Note that this is a scalar equation as there are no free indices.

We can segregate the work done by the surface forces into that done by pressure forces and that done by viscous forces by rewriting this in terms of p and τ_{ij} as follows

$$\partial_o\left(\rho\left(e+\frac{1}{2}v_jv_j\right)\right) + \partial_i\left(\rho v_i\left(e+\frac{1}{2}v_jv_j\right)\right) = -\partial_i q_i - \partial_i(pv_i) + \partial_i(\tau_{ij}v_j) + \rho v_i f_i, \quad (1.517)$$

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) + \nabla^T\cdot\left(\rho\mathbf{v}\left(e+\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)\right) = -\nabla^T\cdot\mathbf{q} - \nabla^T\cdot(p\mathbf{v}) + \nabla^T\cdot(\boldsymbol{\tau}\cdot\mathbf{v}) + \rho\mathbf{v}^T\cdot\mathbf{f}. \quad (1.518)$$

1.4.4.5 Secondary forms of the energy equation

While the energy equation just derived is perfectly valid for all continuous materials, it is common to see other forms. They will be described here. The first, the mechanical energy equation, actually has no foundation in the first law of thermodynamics; instead, it is entirely a consequence of the linear momenta principle. It is the type of energy that is often considered in classical Newtonian particle mechanics, a world in which energy is either potential or kinetic but not thermal. We include it here because it is closely related to other forms of energy.

1.4.4.5.1 Mechanical energy equation The mechanical energy equation, a pure consequence of the linear momenta principle, is obtained by taking the dot product (inner product) of the velocity vector with the linear momenta principle:

$$\mathbf{v}^T \cdot \text{linear momenta.}$$

In detail, we get

$$v_j\left(\rho\partial_o v_j + \rho v_i \partial_i v_j\right) = \rho v_j f_j - v_j \partial_j p + (\partial_i \tau_{ij}) v_j, \tag{1.519}$$

$$\rho\partial_o\left(\frac{v_j v_j}{2}\right) + \rho v_i \partial_i\left(\frac{v_j v_j}{2}\right) = \rho v_j f_j - v_j \partial_j p + (\partial_i \tau_{ij}) v_j, \tag{1.520}$$

$$\tfrac{v_j v_j}{2}\ \text{mass}: \frac{v_j v_j}{2}\partial_o \rho + \frac{v_j v_j}{2}\partial_i(\rho v_i) = 0. \tag{1.521}$$

We add Eqs. (1.520) and (1.521) and use the product rule to get

$$\partial_o\left(\rho\frac{v_j v_j}{2}\right) + \partial_i\left(\rho v_i \frac{v_j v_j}{2}\right) = \rho v_j f_j - v_j \partial_j p + (\partial_i \tau_{ij}) v_j. \tag{1.522}$$

$$\frac{\partial}{\partial t}\left(\rho\frac{\mathbf{v}^T\cdot\mathbf{v}}{2}\right) + \nabla^T\cdot\left(\rho\mathbf{v}\frac{\mathbf{v}^T\cdot\mathbf{v}}{2}\right) = \rho\mathbf{v}^T\cdot\mathbf{f} - \mathbf{v}^T\cdot\nabla p + \left(\nabla^T\cdot\boldsymbol{\tau}\right)\cdot\mathbf{v}. \tag{1.523}$$

The term $\rho v_j v_j/2$ represents the volume averaged kinetic energy, with SI units J/m^3. Note that the mechanical energy equation, Eq. (1.522), predicts the kinetic energy increases due to three effects:

- fluid motion in the direction of a body force,
- fluid motion in the direction of *decreasing* pressure, or
- fluid motion in the direction of *increasing* viscous stress.

Note that body forces themselves affect mechanical energy, while it is imbalances in surface forces which affect mechanical energy.

1.4.4.5.2 Thermal energy equation If we take the conservative form of the energy equation (1.517) and subtract from it the mechanical energy equation (1.522), we get an equation for the evolution of thermal energy:

$$\partial_o(\rho e) + \partial_i(\rho v_i e) = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \quad (1.524)$$

$$\frac{\partial}{\partial t}(\rho e) + \nabla^T \cdot (\rho \mathbf{v} e) = -\nabla^T \cdot \mathbf{q} - p\nabla^T \cdot \mathbf{v} + \boldsymbol{\tau} : \nabla \mathbf{v}^T. \quad (1.525)$$

Here ρe is the volume averaged internal energy with SI units J/m^3. Note that the thermal energy equation (1.524) predicts thermal energy (or internal energy) increases due to

- negative gradients in heat flux (more heat enters than leaves),
- pressure force accompanied by a mean negative volumetric deformation (that is, a uniform compression; note that $\partial_i v_i$ is the relative expansion rate), or
- viscous force associated with a deformation[26] (we'll worry about the sign later).

Note that in contrast to mechanical energy, thermal energy changes do not require surface force imbalances; instead they require kinematic deformation. Moreover, body forces have no influence on thermal energy. The work done by a body force is partitioned entirely to the mechanical energy of a body.

1.4.4.5.3 Non-conservative energy equation We can obtain the commonly used non-conservative form of the energy equation, also known as the energy equation following a fluid particle, by the following operations. First expand the thermal energy equation (1.524):

$$\rho\partial_o e + e\partial_o\rho + \rho v_i\partial_i e + e\partial_i(\rho v_i) = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j. \quad (1.526)$$

Then regroup and notice terms common from the mass conservation equation:

$$\rho \underbrace{(\partial_o e + v_i\partial_i e)}_{\frac{de}{dt}} + e \underbrace{(\partial_o\rho + \partial_i(\rho v_i))}_{=0 \text{ by mass}} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \quad (1.527)$$

so we get

$$\rho\frac{de}{dt} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \quad (1.528)$$

$$\rho\frac{de}{dt} = -\nabla^T \cdot \mathbf{q} - p\nabla^T \cdot \mathbf{v} + \boldsymbol{\tau} : \nabla \mathbf{v}^T. \quad (1.529)$$

[26]For a general fluid, this includes a mean volumetric deformation as well as a deviatoric deformation. If the fluid satisfies Stokes' assumption, it is only the deviatoric deformation that induces a change in internal energy in the presence of viscous stress.

We can get an equation which is reminiscent of elementary thermodynamics, valid for small volumes V by multiplying Eq. (1.528) by V and using Eq. (1.295 to replace $\partial_i v_i$ by its known value in terms of the relative expansion rate to obtain

$$\rho V \frac{de}{dt} = -V\partial_i q_i - p\frac{dV}{dt} + V\tau_{ij}\partial_i v_j. \tag{1.530}$$

The only term not usually found in elementary thermodynamics texts is the third on the right hand side, which is a viscous work term.

1.4.4.5.4 Energy equation in terms of enthalpy Often the energy equation is cast in terms of enthalpy. This is generally valid, but especially useful in constant pressure environments. Recall from elementary thermodynamics the specific enthalpy h is defined as

$$h = e + \frac{p}{\rho}. \tag{1.531}$$

Now starting with the energy equation following a particle (1.528), we can use one form of the mass equation, Eq. (1.439), to eliminate the relative expansion rate $\partial_i v_i$ in favor of density derivatives to get

$$\rho\frac{de}{dt} = -\partial_i q_i + \frac{p}{\rho}\frac{d\rho}{dt} + \tau_{ij}\partial_i v_j. \tag{1.532}$$

Rearranging, we get

$$\rho\left(\frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}\right) = -\partial_i q_i + \tau_{ij}\partial_i v_j. \tag{1.533}$$

Now differentiating Eq. (1.531), we find

$$\begin{aligned}
dh &= de - \frac{p}{\rho^2}d\rho + \frac{1}{\rho}dp, && (1.534)\\
\frac{dh}{dt} &= \frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\rho}\frac{dp}{dt}, && (1.535)\\
\frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} &= \frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt}, && (1.536)\\
\rho\frac{de}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} &= \rho\frac{dh}{dt} - \frac{dp}{dt}. && (1.537)
\end{aligned}$$

So, using Eq. (1.537) to eliminate de/dt in Eq. (1.533) in favor of dh/dt, the energy equation in terms of enthalpy becomes

$$\begin{aligned}
\rho\frac{dh}{dt} &= \frac{dp}{dt} - \partial_i q_i + \tau_{ij}\partial_i v_j, && (1.538)\\
\rho\frac{dh}{dt} &= \frac{dp}{dt} - \nabla^T\cdot\mathbf{q} + \boldsymbol{\tau} : \nabla\mathbf{v}^T. && (1.539)
\end{aligned}$$

1.4.4.5.5 Energy equation in terms of entropy By using standard relations from thermodynamics, we can write the energy equation in terms of entropy. It is important to note that this is just an algebraic substitution. The physical principle which this equation will represent is still *energy* conservation.

Recall the Gibbs equation from thermodynamics, which serves to define entropy s:

$$T\ ds = de + p\ d\hat{v}. \tag{1.540}$$

Here T is the absolute temperature, and $\hat{v}$ is the specific volume, $\hat{v} = V/m = 1/\rho$. In terms of ρ, the Gibbs equation is

$$T\ ds = de - \frac{p}{\rho^2}\ d\rho. \tag{1.541}$$

Taking the material derivative of Eq. (1.541) , which is operationally equivalent to dividing by dt, and solving for de/dt,we get

$$\frac{de}{dt} = T\frac{ds}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt}. \tag{1.542}$$

This is still essentially a thermodynamic definition of s. Now use Eq. (1.542) in the non-conservative energy equation (1.528) to get an alternate expression for the first law:

$$\rho T\frac{ds}{dt} + \frac{p}{\rho}\frac{d\rho}{dt} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j. \tag{1.543}$$

Recalling Eq. (1.439), $-\partial_i v_i = (1/\rho)(d\rho/dt)$, we have

$$\rho T\frac{ds}{dt} = -\partial_i q_i + \tau_{ij}\partial_i v_j, \tag{1.544}$$

$$\rho\frac{ds}{dt} = -\frac{1}{T}\partial_i q_i + \frac{1}{T}\tau_{ij}\partial_i v_j. \tag{1.545}$$

Using the fact that from the quotient rule we have $\partial_i(q_i/T) = (1/T)\partial_i q_i - (q_i/T^2)\partial_i T$, we can then say

$$\rho\frac{ds}{dt} = -\partial_i\left(\frac{q_i}{T}\right) - \frac{1}{T^2}q_i\partial_i T + \frac{1}{T}\tau_{ij}\partial_i v_j, \tag{1.546}$$

$$\rho\frac{ds}{dt} = -\nabla^T\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{1}{T^2}\mathbf{q}^T\cdot\nabla T + \frac{1}{T}\boldsymbol{\tau}:\nabla\mathbf{v}^T. \tag{1.547}$$

From this statement, we can conclude from the *first law* of thermodynamics that the entropy of a fluid particle changes due to heat transfer and to deformation in the presence of viscous stress. We will make a more precise statement about entropy changes after we introduce the second law of thermodynamics.

The energy equation in terms of entropy can be written in conservative or divergence form by adding the product of s and the mass equation, $s\partial_o\rho + s\partial_i(\rho v_i) = 0$, to Eq. (1.546) to obtain

$$\partial_o(\rho s) + \partial_i(\rho v_i s) = -\partial_i\left(\frac{q_i}{T}\right) - \frac{1}{T^2}q_i\partial_i T + \frac{1}{T}\tau_{ij}\partial_i v_j, \tag{1.548}$$

$$\frac{\partial}{\partial t}(\rho s) + \nabla^T\cdot(\rho\mathbf{v}s) = -\nabla^T\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{1}{T^2}\mathbf{q}^T\cdot\nabla T + \frac{1}{T}\boldsymbol{\tau}:\nabla\mathbf{v}^T. \tag{1.549}$$

1.4.5 Entropy inequality

Let us use a non-rigorous method to suggest a form of the entropy inequality which is consistent with classical thermodynamics. Recall the mathematical statement of the entropy inequality from classical thermodynamics:

$$dS \geq \frac{dQ}{T}. \tag{1.550}$$

Here S is the extensive entropy, with SI units J/K, and Q is the heat energy into a system with SI units of J. Notice that entropy can go up or down in a process, depending on the heat transferred. If the process is adiabatic, $dQ = 0$, and the entropy can either remain fixed or rise. Now for our continuous material we have

$$dS = \rho s \, dV, \tag{1.551}$$
$$dQ = -q_i n_i \, dA \, dt. \tag{1.552}$$

Here we have used s for the specific entropy, which has SI units J/kg/K. We have also changed, for obvious reasons, the notation for our element of surface area, now dA, rather than the previous dS. Notice we must be careful with our sign convention. When the heat flux vector is aligned with the outward normal, heat leaves the system. Since we want positive dQ to represent heat into a system, we need the negative sign.

The second law becomes then

$$\rho s \, dV \geq -\frac{q_i}{T} n_i \, dA \, dt. \tag{1.553}$$

Now integrate over the finite geometry: on the left side this is a volume integral and the right side this is an area integral.

$$\int_{MR(t)} \rho s \, dV \geq \left(\int_{MS(t)} -\frac{q_i}{T} n_i \, dA \right) dt. \tag{1.554}$$

Differentiating with respect to time and then applying our typical machinery to the second law gives rise to

$$\frac{d}{dt} \int_{MR(t)} \rho s \, dV \geq \int_{MS(t)} -\frac{q_i}{T} n_i \, dA, \tag{1.555}$$

$$\int_{MR(t)} \partial_o(\rho s) \, dV + \int_{MS(t)} \rho s v_i n_i \, dA \geq \int_{MS(t)} -\frac{q_i}{T} n_i \, dA, \tag{1.556}$$

$$\int_{MR(t)} \left(\partial_o(\rho s) + \partial_i(\rho s v_i) \right) \, dV \geq \int_{MR(t)} -\partial_i \left(\frac{q_i}{T} \right) \, dV, \tag{1.557}$$

$$\int_{MR(t)} \left(\partial_o(\rho s) + \partial_i(\rho s v_i) \right) \, dV = \int_{MR(t)} -\partial_i \left(\frac{q_i}{T} \right) \, dV + \int_{MR(t)} I \, dV, \tag{1.558}$$

where irreversibility $I \geq 0$,

$$\partial_o(\rho s) + \partial_i(\rho s v_i) = -\partial_i \left(\frac{q_i}{T} \right) + I, \tag{1.559}$$

$$\rho \frac{ds}{dt} = -\partial_i \left(\frac{q_i}{T} \right) + I. \tag{1.560}$$

This is the second law. Now if we subtract from this the first law written in terms of entropy, Eq. (1.546), we get the result

$$I = -\frac{1}{T^2} q_i \partial_i T + \frac{1}{T} \underbrace{\tau_{ij} \partial_i v_j}_{\Phi}. \tag{1.561}$$

As an aside, we have defined the commonly used *viscous dissipation function* Φ as

$$\Phi = \tau_{ij} \partial_i v_j. \tag{1.562}$$

For symmetric stress tensors, we also have $\Phi = \tau_{ij}\ \partial_{(i} v_{j)}$. Now since $I \geq 0$, we can view the entirety of the second law as the following constraint, sometimes called the weak form of the *Clausius-Duhem*[27][28] inequality:

$$-\frac{1}{T^2} q_i \partial_i T + \frac{1}{T} \tau_{ij} \partial_i v_j \geq 0, \tag{1.563}$$

$$-\frac{1}{T^2} \mathbf{q}^T \cdot \nabla T + \frac{1}{T} \boldsymbol{\tau} : \nabla \mathbf{v}^T \geq 0, \tag{1.564}$$

$$-\frac{1}{T^2} \mathbf{q}^T \cdot \nabla T + \frac{1}{T} \boldsymbol{\tau} : \mathsf{L} \geq 0. \tag{1.565}$$

Recalling that τ_{ij} is symmetric by the angular momenta principle for no external body couples, and, consequently, that its tensor inner product with the velocity gradient only has a contribution from the symmetric part of the velocity gradient (that is, the deformation rate or strain rate tensor), the entropy inequality reduces slightly to

$$-\frac{1}{T^2} q_i \partial_i T + \frac{1}{T} \tau_{ij} \partial_{(i} v_{j)} \geq 0, \tag{1.566}$$

$$-\frac{1}{T^2} \mathbf{q}^T \cdot \nabla T + \frac{1}{T} \boldsymbol{\tau} : \left(\frac{\nabla \mathbf{v}^T + \left(\nabla \mathbf{v}^T \right)^T}{2} \right) \geq 0, \tag{1.567}$$

$$-\frac{1}{T^2} \mathbf{q}^T \cdot \nabla T + \frac{1}{T} \boldsymbol{\tau} : \mathsf{D} \geq 0. \tag{1.568}$$

We shall see in upcoming sections that we will be able to specify q_i and τ_{ij} in such a fashion that is both consistent with experiment and satisfies the entropy inequality.

The more restrictive (and in some cases, *overly* restrictive) strong form of the Clausius-Duhem inequality requires each term to be greater than or equal to zero. For our system the strong form, realizing that the absolute temperature $T > 0$, is

$$-q_i \partial_i T \geq 0, \qquad \underbrace{\tau_{ij} \partial_{(i} v_{j)}}_{\Phi} \geq 0, \tag{1.569}$$

[27]Rudolf Clausius, 1822-1888, Prussian-born German mathematical physicist, key figure in making thermodynamics a science, author of well-known statement of the second law of thermodynamics, taught at Zürich Polytechnikum, University of Würzburg, and University of Bonn.

[28]Pierre Maurice Marie Duhem, 1861-1916, French physicist, mathematician, and philosopher, taught at Lille, Rennes, and the University of Bordeaux.

$$-\mathbf{q}^T \cdot \nabla T \geq 0, \qquad \boldsymbol{\tau} : \left(\frac{\nabla \mathbf{v}^T + \left(\nabla \mathbf{v}^T\right)^T}{2} \right) \;\geq\; 0. \tag{1.570}$$

It is straightforward to show that terms which generate entropy due to viscous work also dissipate mechanical energy. This can be cleanly demonstrated by considering the mechanisms which cause mechanical energy the change within a finite fixed control volume V. First consider a restatement of the mechanical energy equation, Eq. (1.520) in terms of the material derivative of specific kinetic energy:

$$\rho \frac{d}{dt} \left(\frac{v_j v_j}{2} \right) = \rho v_j f_j - v_j \partial_j (p) + v_j \partial_i \tau_{ij}. \tag{1.571}$$

Now use the product rule to restate the pressure and viscous work terms so as to achieve

$$\rho \frac{d}{dt} \left(\frac{v_j v_j}{2} \right) = \rho v_j f_j - \partial_j (v_j p) + p \partial_j v_j + \partial_i (\tau_{ij} v_j) - \underbrace{\tau_{ij} \partial_i v_j}_{= \Phi \geq 0}. \tag{1.572}$$

So, here we see what induces *local* changes in mechanical energy. We see that body forces, pressure forces and viscous forces in general can induce the mechanical energy to rise or fall. However that part of the viscous stresses which is associated with the viscous dissipation, Φ, is guaranteed to induce a *local decrease* in mechanical energy.

To study global changes in mechanical energy, we consider the conservative form of the mechanical energy equation, Eq. (1.522), here written in the same way which takes advantage of application of the product rule to the pressure and viscous terms:

$$\partial_o \left(\rho \frac{v_j v_j}{2} \right) + \partial_i \left(\rho v_i \frac{v_j v_j}{2} \right) = \rho v_j f_j - \partial_j (v_j p) + p \partial_j v_j + \partial_i (\tau_{ij} v_j) - \tau_{ij} \partial_i v_j. \tag{1.573}$$

Now integrate over the fixed control volume, so that

$$\begin{aligned} \int_V \partial_o \left(\rho \frac{v_j v_j}{2} \right) \, dV + \int_V \partial_i \left(\rho v_i \frac{v_j v_j}{2} \right) \, dV \;&=\; \int_V \rho v_j f_j \, dV - \int_V \partial_j (v_j p) \, dV + \int_V p \partial_j v_j \, dV \\ &\quad + \int_V \partial_i (\tau_{ij} v_j) \, dV - \int_V \tau_{ij} \partial_i v_j \, dV. \end{aligned} \tag{1.574}$$

Applying Leibniz's rule and Gauss's law, we get

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho \frac{v_j v_j}{2} \, dV + \int_S n_i \rho v_i \frac{v_j v_j}{2} \, dS \;&=\; \int_V \rho v_j f_j \, dV - \int_S n_j v_j p \, dS + \int_V p \partial_j v_j \, dV \\ &\quad + \int_S n_i (\tau_{ij} v_j) \, dS - \int_V \tau_{ij} \partial_i v_j \, dV. \end{aligned} \tag{1.575}$$

Now on the surface of the fixed volume, the velocity is zero, so we get

$$\frac{\partial}{\partial t} \int_V \rho \frac{v_j v_j}{2} \, dV = \int_V \rho v_j f_j \, dV + \int_V p \partial_j v_j \, dV - \int_V \underbrace{\tau_{ij} \partial_i v_j}_{\text{positive}} \, dV. \tag{1.576}$$

Now the strong form of the second law requires that $\tau_{ij}\partial_i v_j = \tau_{ij}\partial_{(i}v_{j)} \geq 0$. So, we see for a finite fixed volume of fluid that a body force and pressure force in conjunction with local volume changes can cause the global mechanical energy to either grow or decay, the viscous stress always induces a decay of global mechanical energy; in other words it is a dissipative effect.

1.4.6 Summary of axioms in differential form

Here we pause to summarize the mathematical form of our axioms. We give the Cartesian index, Gibbs, and the full non-orthogonal index notation. All details of development of the non-orthogonal index notation are omitted, and the reader is referred to Aris for a full development. We will first present the conservative form and then the non-conservative form.

1.4.6.1 Conservative form

1.4.6.1.1 Cartesian index form

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \quad (1.577)$$

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \quad (1.578)$$

$$\tau_{ij} = \tau_{ji}, \quad (1.579)$$

$$\partial_o\left(\rho\left(e + \frac{1}{2}v_j v_j\right)\right) + \partial_i\left(\rho v_i\left(e + \frac{1}{2}v_j v_j\right)\right) = -\partial_i q_i - \partial_i(p v_i) + \partial_i(\tau_{ij} v_j) + \rho v_i f_i, \quad (1.580)$$

$$\partial_o(\rho s) + \partial_i(\rho s v_i) \geq -\partial_i\left(\frac{q_i}{T}\right). \quad (1.581)$$

1.4.6.1.2 Gibbs form

$$\frac{\partial \rho}{\partial t} + \nabla^T \cdot (\rho \mathbf{v}) = 0, \quad (1.582)$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \left(\nabla^T \cdot (\rho \mathbf{v}\mathbf{v}^T)\right)^T = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T, \quad (1.583)$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}^T, \quad (1.584)$$

$$\frac{\partial}{\partial t}\left(\rho\left(e + \frac{1}{2}\mathbf{v}^T \cdot \mathbf{v}\right)\right) + \nabla^T \cdot \left(\rho \mathbf{v}\left(e + \frac{1}{2}\mathbf{v}^T \cdot \mathbf{v}\right)\right) = -\nabla^T \cdot \mathbf{q} - \nabla^T \cdot (p\mathbf{v}) + \nabla^T \cdot (\boldsymbol{\tau} \cdot \mathbf{v}) + \rho \mathbf{v}^T \cdot \mathbf{f}, \quad (1.585)$$

$$\frac{\partial \rho s}{\partial t} + \nabla^T \cdot (\rho s \mathbf{v}) \geq -\nabla^T \cdot \left(\frac{\mathbf{q}}{T}\right). \quad (1.586)$$

1.4.6.1.3 Non-orthogonal index form Here we introduce, following Aris and many others, some standard notation from tensor analysis. In this notation, both sub- and superscripts are needed to distinguish between what are known as covariant and contravariant vectors, which are really different mathematical representations of the same quantity, just cast onto different basis vectors. In brief, we have the *metric tensor* $g_{ij} =$

$\frac{\partial \xi^k}{\partial x^i}\frac{\partial \xi^k}{\partial x^j}$, where ξ^k is a Cartesian coordinate and x^i is a non-Cartesian coordinate. We also have $g^{ij} = \frac{1}{2}\epsilon^{imn}\epsilon^{jpq}g_{mp}g_{nq}$, $\sqrt{g} = \det \frac{\partial \xi^k}{\partial x^i}$. The *Christoffel*[29] *symbols* are given by $\Gamma^m_{ij} = \frac{1}{2}g^{mk}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$. We note also that few texts give a proper exposition of the conservative form of the equations in non-orthogonal coordinates. Here we have extended the development of Vinokur[30] to include the effects of momentum and energy diffusion. This extension has been guided by general notions found in standard works such as Aris as well as Liseikin.

$$\frac{\partial}{\partial t}\left(\sqrt{g}\ \rho\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \rho v^k\right) = 0, \tag{1.587}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\sqrt{g}\ \rho v^j \frac{\partial \xi^i}{\partial x^j}\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \rho v^j v^k \frac{\partial \xi^i}{\partial x^j}\right) &= \sqrt{g}\ \rho f^j \frac{\partial \xi^i}{\partial x^j} \\ &\quad - \frac{\partial}{\partial x^k}\left(\sqrt{g}\ p g^{jk}\frac{\partial \xi^i}{\partial x^j}\right) \\ &\quad + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \tau^{jk}\frac{\partial \xi^i}{\partial x^j}\right),\end{aligned} \tag{1.588}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\sqrt{g}\ \rho\left(e + \frac{1}{2}g_{ij}v^i v^j\right)\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \rho v^k\left(e + \frac{1}{2}g_{ij}v^i v^j\right)\right) &= -\frac{\partial}{\partial x^k}\left(\sqrt{g}\ q^k\right) \\ &\quad - \frac{\partial}{\partial x^k}\left(\sqrt{g}\ p v^k\right) \\ &\quad + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ g_{ij}v^j \tau^{ik}\right) \\ &\quad + \sqrt{g}\ \rho g_{ij} v^j f^i,\end{aligned} \tag{1.589}$$

$$\frac{\partial}{\partial t}\left(\sqrt{g}\ \rho s\right) + \frac{\partial}{\partial x^k}\left(\sqrt{g}\ \rho s v^k\right) \geq -\frac{\partial}{\partial x^k}\left(\sqrt{g}\ \frac{q^k}{T}\right). \tag{1.590}$$

1.4.6.2 Non-conservative form

1.4.6.2.1 Cartesian index form

$$\frac{d\rho}{dt} = -\rho \partial_i v_i, \tag{1.591}$$

$$\rho \frac{dv_i}{dt} = \rho f_i - \partial_i p + \partial_j \tau_{ji}, \tag{1.592}$$

$$\tau_{ij} = \tau_{ji}, \tag{1.593}$$

$$\rho \frac{de}{dt} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \tag{1.594}$$

[29]Elwin Bruno Christoffel, 1829-1900, German mathematician and physicist.

[30]Vinokur, M., 1974, "Conservation Equations of Gasdynamics," *Journal of Computational Physics*, 14(2): 105-125.

$$\rho \frac{ds}{dt} \geq -\partial_i \left(\frac{q_i}{T} \right). \tag{1.595}$$

1.4.6.2.2 Gibbs form

$$\frac{d\rho}{dt} = -\rho \nabla^T \cdot \mathbf{v}, \tag{1.596}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \boldsymbol{\tau} \right)^T, \tag{1.597}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}^T, \tag{1.598}$$

$$\rho \frac{de}{dt} = -\nabla^T \cdot \mathbf{q} - p \nabla^T \cdot \mathbf{v} + \boldsymbol{\tau} : \nabla \mathbf{v}^T, \tag{1.599}$$

$$\rho \frac{ds}{dt} \geq -\nabla^T \cdot \left(\frac{\mathbf{q}}{T} \right). \tag{1.600}$$

1.4.6.2.3 Non-orthogonal index form *These have not been checked carefully!*

$$\frac{\partial \rho}{\partial t} + v^i \frac{\partial \rho}{\partial x^i} = -\frac{\rho}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\; v^i \right), \tag{1.601}$$

$$\rho \left(\frac{\partial v^i}{\partial t} + v^j \left(\frac{\partial v^i}{\partial x^j} + \Gamma^i_{jl} v^l \right) \right) = \rho f^i - g^{ij} \frac{\partial p}{\partial x^j} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g}\; \tau^{ij} \right) + \Gamma^i_{jk} \tau^{jk}, \tag{1.602}$$

$$\rho \left(\frac{\partial e}{\partial t} + v^i \frac{\partial e}{\partial x^i} \right) = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\; q^i \right) - \frac{p}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\; v^i \right) + g_{ik} \tau^{kj} \left(\frac{\partial v^i}{\partial x^j} + \Gamma^i_{jl} v^l \right), \tag{1.603}$$

$$\rho \left(\frac{\partial s}{\partial t} + v^i \frac{\partial s}{\partial x^i} \right) \geq -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\; \frac{q^i}{T} \right). \tag{1.604}$$

1.4.6.3 Physical interpretations

Each term in the governing axioms represents a physical mechanism. This approach is emphasized in the classical text by Bird, Stewart, and Lightfoot on transport processes. In general, the equations which are partial differential equations can be represented in the following form:

$$\text{local change} = \text{advection} + \text{diffusion} + \text{source}. \tag{1.605}$$

Here we consider advection and diffusion to be types of transport phenomena. If we have a fixed volume of material, a property of that material, such as its thermal energy, can change because an outside flow sweeps energy in from outside. That is advection. It can change because random molecular motions allow slow leakage to the outside or leakage in from the outside. That is diffusion. Or the material can undergo intrinsic changes inside, such as viscous work, which converts kinetic energy into thermal energy.

Let us write the Gibbs form of the non-conservative equations of mass, linear momentum, and energy in a slightly different way to illustrate these mechanisms:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} = & \quad \text{local change in mass} \\
& -\mathbf{v}^T \cdot \nabla \rho \quad \text{advection of mass} \\
& +0 \quad \text{diffusion of mass} \\
& -\rho \nabla^T \cdot \mathbf{v}, \quad \text{volume expansion source}
\end{aligned} \tag{1.606}$$

$$\begin{aligned}
\rho \frac{\partial \mathbf{v}}{\partial t} = & \quad \text{local change in linear momenta} \\
& -\rho \left(\mathbf{v}^T \cdot \nabla\right) \mathbf{v} \quad \text{advection of linear momenta} \\
& + \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T, \quad \text{diffusion of linear momenta} \\
& +\rho \mathbf{f} \quad \text{body force source of linear momenta} \\
& -\nabla p \quad \text{pressure force source of linear momenta}
\end{aligned} \tag{1.607}$$

$$\begin{aligned}
\rho \frac{\partial e}{\partial t} = & \quad \text{local change in thermal energy} \\
& -\rho \mathbf{v}^T \cdot \nabla e \quad \text{advection of thermal energy} \\
& -\nabla^T \cdot \mathbf{q} \quad \text{diffusion of thermal energy} \\
& -p \nabla^T \cdot \mathbf{v} \quad \text{pressure work thermal energy source} \\
& +\boldsymbol{\tau} : \nabla \mathbf{v}^T \quad \text{viscous work thermal energy source.}
\end{aligned} \tag{1.608}$$

Briefly considering the second law, we note that the irreversibility I is solely associated with diffusion of linear momenta and diffusion of energy. This makes sense in that diffusion is associated with random molecular motions and thus disorder. Convection is associated with an ordered motion of matter in that we retain knowledge of the position of the matter. Pressure volume work is a reversible work and does not contribute to entropy changes. A portion of the heat transfer can be considered to be reversible. All of the work done by the viscous forces is irreversible work.

1.4.7 Complete system of equations?

The beauty of these axioms is that they are valid for any material which can be modeled as a continuum under the influence of the forces we have mentioned. Specifically, they are valid for both solid and fluid mechanics, which is remarkable.

While the axioms are complete, the equations are not! Note that we have twenty-three unknowns here $\rho(1), v_i(3), f_i(3), p(1), \tau_{ij}(9), e(1), q_i(3), T(1), s(1)$, and only eight equations (one mass, three linear momenta, three independent angular momenta, one energy). We cannot really count the second law as an equation, as it is an inequality. Whatever result we get must be consistent with it. Whatever the case we are short a number of equations. We will see in a later section how we use constitutive equations, equations founded in empiricism, which in some sense model sub-continuum effects that we have ignored, to complete our system.

Before we go onto this, however, we will in the next section discuss integral "control volume" forms of the governing equations.

1.4.8 Integral forms

- Our governing equations are formulated based upon laws which apply to a *material element*.
- We are not often interested in an actual material element but in some other fixed of moving region in space.
- Rules for such systems can by formulated with Leibniz's rule in conjunction with the differential forms of our axioms.

Let us first apply Leibniz's rule (1.185) to an arbitrary function f over a time dependent arbitrary region $AR(t)$:

$$\frac{d}{dt}\int_{AR(t)} f\ dV = \int_{AR(t)} \frac{\partial f}{\partial t}\ dV + \int_{AS(t)} n_i w_i f\ dS. \tag{1.609}$$

Recall that w_i is the velocity of the arbitrary surface, not necessarily the particle velocity.

1.4.8.1 Mass

Rewriting the mass equation as

$$\partial_o \rho = -\partial_i(\rho v_i), \tag{1.610}$$

Now let's use this, and let $f = \rho$ in Leibniz's rule to get

$$\frac{d}{dt}\int_{AR(t)} \rho\ dV = \int_{AR(t)} \partial_o \rho\ dV + \int_{AS(t)} n_i w_i \rho\ dS, \tag{1.611}$$

$$\frac{d}{dt}\int_{AR(t)} \rho\ dV = \int_{AR(t)} (-\partial_i(\rho v_i))\ dV + \int_{AS(t)} n_i w_i \rho\ dS, \tag{1.612}$$

$$\text{with Gauss} \tag{1.613}$$

$$\frac{d}{dt}\int_{AR(t)} \rho\ dV = \int_{AS(t)} n_i \rho(w_i - v_i)\ dS. \tag{1.614}$$

Now consider three special cases.

1.4.8.1.1 Fixed region We take $w_i = 0$.

$$\frac{d}{dt}\int_{AR(t)} \rho\ dV = -\int_{AS(t)} n_i \rho v_i\ dS, \tag{1.615}$$

$$\frac{d}{dt}\int_{AR(t)} \rho\ dV + \int_{AS(t)} n_i \rho v_i\ dS = 0. \tag{1.616}$$

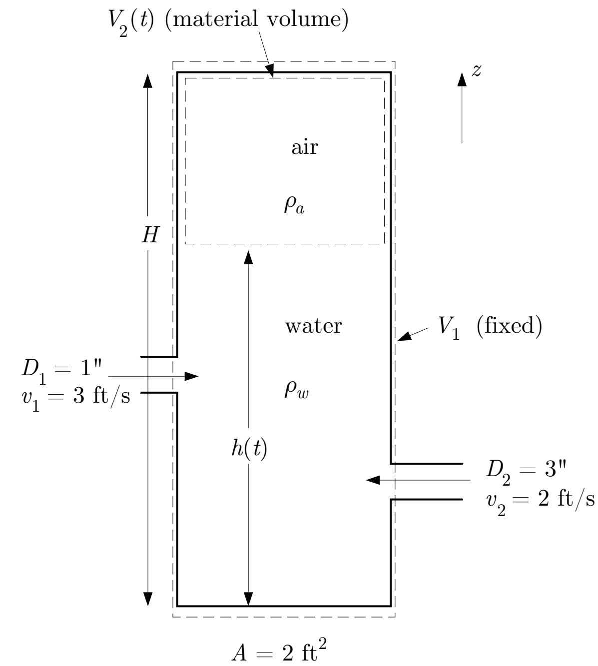

Figure 1.25: Sketch of volume with water and air being filled with water.

1.4.8.1.2 Material region Here we take $w_i = v_i$.

$$\frac{d}{dt}\int_{MR(t)} \rho \, dV = 0. \tag{1.617}$$

1.4.8.1.3 Moving solid enclosure with holes Say the region considered is a solid enclosure with holes through with fluid can enter and exit. The our arbitrary surface $AS(t)$ can be specified as

$$AS(t) = A_e(t) \quad \text{area of entrances and exits} \tag{1.618}$$
$$+A_s(t) \quad \text{solid moving surface with } w_i = v_i \tag{1.619}$$
$$+A_s \quad \text{fixed solid surface with } w_i = v_i = 0. \tag{1.620}$$

Then we get

$$\frac{d}{dt}\int_{AR(t)} \rho \, dV + \int_{A_e(t)} \rho n_i (v_i - w_i) \, dS = 0. \tag{1.621}$$

Example 1.10

Consider the volume sketched in Figure 1.25. Water enters a circular hole of diameter $D_1 = 1$" with velocity $v_1 = 3$ ft/s. Water enters another circular hole of diameter $D_2 = 3$" with velocity $v_2 = 2$ ft/s. The cross sectional area of the cylindrical tank is $A = 2$ ft^2. The tank has height H. Water at density ρ_w exists in the tank at height $h(t)$. Air at density ρ_a fills the remainder of the tank. Find the rate of rise of the water $\frac{dh}{dt}$.

Consider two control volumes

- V_1: the fixed region enclosing the entire tank, and
- $V_2(t)$: the material region attached to the air.

First, let us write mass conservation for the material region 2:

$$\frac{d}{dt}\int_{V_2} \rho_a \, dV = 0, \tag{1.622}$$

$$\frac{d}{dt}\int_{h(t)}^{H} \rho_a A \, dz = 0. \tag{1.623}$$

Mass conservation for V_1 is

$$\frac{d}{dt}\int_{V_1} \rho \, dV + \int_{A_e} \rho v_i n_i \, dS = 0. \tag{1.624}$$

Now break up V_1 and write A_e explicitly

$$\frac{d}{dt}\int_{0}^{h(t)} \rho_w A \, dz + \underbrace{\frac{d}{dt}\int_{h(t)}^{H} \rho_a A \, dz}_{=0} = -\int_{A_1} \rho_w v_i n_i \, dS - \int_{A_2} \rho_w v_i n_i \, dS, \tag{1.625}$$

$$\frac{d}{dt}\int_{0}^{h(t)} \rho_w A \, dz = \rho_w v_1 A_1 + \rho_2 v_2 A_2, \tag{1.626}$$

$$= \frac{\rho_w \pi}{4}(v_1 D_1^2 + v_2 D_2^2), \tag{1.627}$$

$$\rho_w A \frac{d}{dt}\int_{0}^{h(t)} dz = \frac{\rho_w \pi}{4}(v_1 D_1^2 + v_2 D_2^2), \tag{1.628}$$

$$\frac{dh}{dt} = \frac{\pi}{4A}(v_1 D_1^2 + v_2 D_2^2) \tag{1.629}$$

$$= \frac{\pi}{4(2 \text{ ft}^2)}\left(\left(3 \frac{\text{ft}}{\text{s}}\right)\left(\frac{1}{12} \text{ ft}\right)^2 + \left(2 \frac{\text{ft}}{\text{s}}\right)\left(\frac{3}{12} \text{ ft}\right)^2\right), \tag{1.630}$$

$$= \frac{7\pi}{384} \frac{\text{ft}}{\text{s}} = 0.057 \frac{\text{ft}}{\text{s}}. \tag{1.631}$$

1.4.8.2 Linear momenta

Let us perform the same exercise for the linear momenta equation. First, in a strictly mathematical step, apply Leibniz's rule to linear momenta, ρv_i:

$$\frac{d}{dt}\int_{AR(t)} \rho v_i \, dV = \int_{AR(t)} \partial_o(\rho v_i) \, dV + \int_{AS(t)} n_j w_j \rho v_i \, dS. \tag{1.632}$$

Now invoke the physical linear momenta axiom. Here the axiom gives us an expression for $\partial_o(\rho v_i)$. We will also convert volume integrals to surface integrals via Gauss's theorem to get

$$\frac{d}{dt}\int_{AR(t)} \rho v_i \; dV = -\int_{AS(t)} (\rho n_j (v_j - w_j) v_i + n_i p - n_j \tau_{ij}) \; dS + \int_{AR(t)} \rho f_i \; dV. \tag{1.633}$$

Now momentum flux terms only have values at entrances and exits (at solid surfaces we get $v_i = w_i$, so we can say

$$\frac{d}{dt}\int_{AR(t)} \rho v_i \; dV + \int_{A_e(t)} \rho n_j (v_j - w_j) v_i \; dS = -\int_{AS(t)} n_i p \; dS + \int_{AS(t)} n_j \tau_{ij} \; dS + \int_{AR(t)} \rho f_i \; dV. \tag{1.634}$$

Note that the surface forces are evaluated along all surfaces, not just entrances and exits.

1.4.8.3 Energy

Applying the same analysis to the energy equation, we obtain

$$\begin{aligned} \frac{d}{dt}\int_{AR(t)} \rho \left(e + \frac{1}{2} v_j v_j\right) \; dV \;\; = \;\; & -\int_{AS(t)} \rho n_i (v_i - w_i) \left(e + \frac{1}{2} v_j v_j\right) \; dS \\ & -\int_{AR(t)} n_i q_i \; dS \\ & -\int_{AS(t)} (n_i v_i p - n_i \tau_{ij} v_j) \; dS \\ & +\int_{AR(t)} \rho v_i f_i \; dV. \end{aligned} \tag{1.635}$$

1.4.8.4 General expression

If we have a governing equation from a physical principle which is of form

$$\partial_o f_j + \partial_i (v_i f_j) = \partial_i g_j + h_j, \tag{1.636}$$

then we can say for an arbitrary volume that

$$\underbrace{\frac{d}{dt}\int_{AR(t)} f_j \; dV}_{\text{change of } f_j} = \underbrace{-\int_{AS(t)} n_i f_j (v_i - w_i) \; dS}_{\text{flux of } f_j} + \underbrace{\int_{AS(t)} n_i g_j \; dS}_{\text{effect of } g_j} + \underbrace{\int_{AR(t)} h_j \; dV}_{\text{effect of } h_j}. \tag{1.637}$$

1.5 Constitutive equations

We now return to the problem of completing our set of equations. We recall we have too many unknowns and not enough equations. Constitutive equations are additional equations which are not as fundamental as the previously developed axioms. They can be rather *ad hoc* relations which in some sense model the sub-continuum nano-structure. In some cases, for example, the sub-continuum kinetic theory of gases, we can formally show that when the sub-continuum is formally averaged, that we obtain commonly used constitutive equations. In most cases however, constitutive equations simply represent curve fits to basic experimental results, which can vary widely from material to material. As is briefly discussed below, constitutive equations are not completely arbitrary. Whatever is proposed must allow our final equations to be invariant under Galilean[31] transformations and rotations as well as satisfy the entropy inequality.

For example, we might hope to develop a constitutive equation for the heat flux vector q_i. Being naive, we might in general expect it to be a function of a large number of variables:

$$q_i = q_i(\rho, p, T, v_i, \tau_{ij}, f_i, e, s, \ldots). \tag{1.638}$$

The principles of continuum mechanics will rule out some possibilities, but still allow a broad range of forms.

1.5.1 Frame and material indifference

Our choice of a constitutive law must be invariant under a Galilean transformation (frame invariance) a rotation (material indifference). Say for example, we propose that the heat flux vector is proportional to the velocity vector

$$q_i = a v_i, \qquad \text{trial constitutive relation.} \tag{1.639}$$

If we changed frames such that velocities in the moving frame were $u_i = v_i - V$, we would have $q_i = a(u_i + V)$. With this constitutive law, we find a physical quantity is dependent on the frame velocity, which we observe to be non-physical; hence we rule out this trial constitutive relation.

A commonly used constitutive law for stress in a one-dimensional experiment is

$$\tau_{12} = b(\partial_1 v_2)^a (\partial_1 u_2)^b, \tag{1.640}$$

where u_2 is the displacement of particle. While this may fit one-dimensional data well, it is in no way clear how one could simply extend this to write an expression for τ_{ij}, and many propositions will fail to satisfy material indifference.

1.5.2 Second law restrictions and Onsager relations

The entropy inequality from the second law of thermodynamics provides additional restrictions on the form of constitutive equations. Recall the second law (equivalently, the weak

[31] Galileo Galilei, 1564-1642, Pisa-born Italian astronomer, physicist, and developer of experimental methods, first employed a pendulum to keep time, builder and user of telescopes used to validate the Copernican view of the universe, developer of the principle of inertia and relative motion.

form of the Clausius-Duhem inequality, Eq. (1.566)) tells us that

$$-\frac{1}{T^2}q_i\partial_i T + \frac{1}{T}\tau_{ij}\partial_{(i}v_{j)} \geq 0. \tag{1.641}$$

We would like to find forms of q_i and τ_{ij} which are consistent with the above weak form of the entropy inequality.

1.5.2.1 Weak form of the Clausius-Duhem inequality

The weak form suggests that we may want to consider both q_i and τ_{ij} to be functions involving the temperature gradient $\partial_i T$ and the deformation tensor $\partial_{(i}v_{j)}$.

1.5.2.1.1 Non-physical motivating example To see that this is actually too general of an assumption, it suffices to consider a one-dimensional limit. In the one-dimensional limit, the weak form of the entropy inequality, Eq. (1.566), reduces to

$$-\frac{1}{T^2}q\frac{\partial T}{\partial x} + \frac{1}{T}\tau\frac{\partial u}{\partial x} \geq 0. \tag{1.642}$$

We can write this in a vector form as

$$\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \begin{pmatrix} \frac{q}{T} \\ \frac{\tau u}{T} \end{pmatrix} \geq 0. \tag{1.643}$$

Note that a factor of u/u was introduced to the viscous stress term. This allows for a necessary dimensional consistency in that q/T has the same units as $\tau u/T$. Let us then hypothesize a linear relationship exists between the generalized fluxes q/T and $\tau u/T$ and the generalized driving gradients $-\frac{1}{T}\frac{\partial T}{\partial x}$ and $\frac{1}{u}\frac{\partial u}{\partial x}$:

$$\frac{q}{T} = C_{11}\left(-\frac{1}{T}\frac{\partial T}{\partial x}\right) + C_{12}\frac{1}{u}\frac{\partial u}{\partial x}, \tag{1.644}$$

$$\frac{\tau u}{T} = C_{21}\left(-\frac{1}{T}\frac{\partial T}{\partial x}\right) + C_{22}\frac{1}{u}\frac{\partial u}{\partial x}, \tag{1.645}$$

$$\tag{1.646}$$

In matrix form this becomes

$$\begin{pmatrix} \frac{q}{T} \\ \frac{\tau u}{T} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{pmatrix}. \tag{1.647}$$

We then substitute this hypothesized relationship into the entropy inequality to obtain

$$\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{pmatrix} \geq 0. \tag{1.648}$$

We next segregate the matrix C_{ij} into a symmetric and anti-symmetric part to get

$$\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \left(\left(\begin{array}{cc} C_{11} & \frac{C_{12}+C_{21}}{2} \\ \frac{C_{21}+C_{12}}{2} & C_{22} \end{array} \right) + \left(\begin{array}{cc} 0 & \frac{C_{12}-C_{21}}{2} \\ \frac{C_{21}-C_{12}}{2} & 0 \end{array} \right) \right) \left(\begin{array}{c} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{array} \right) \geq 0. \tag{1.649}$$

Distributing the multiplication, we find

$$\begin{aligned} &\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \left(\begin{array}{cc} C_{11} & \frac{C_{12}+C_{21}}{2} \\ \frac{C_{21}+C_{12}}{2} & C_{22} \end{array} \right) \left(\begin{array}{c} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{array} \right) \\ &+ \underbrace{\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \left(\begin{array}{cc} 0 & \frac{C_{12}-C_{21}}{2} \\ \frac{C_{21}-C_{12}}{2} & 0 \end{array} \right) \left(\begin{array}{c} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{array} \right)}_{=0} \geq 0. \end{aligned} \tag{1.650}$$

The second term is identically zero for all values of temperature and velocity gradients. So what remains is the inequality involving only a symmetric matrix:

$$\left(-\frac{1}{T}\frac{\partial T}{\partial x} \quad \frac{1}{u}\frac{\partial u}{\partial x} \right) \left(\begin{array}{cc} C_{11} & \frac{C_{12}+C_{21}}{2} \\ \frac{C_{21}+C_{12}}{2} & C_{22} \end{array} \right) \left(\begin{array}{c} -\frac{1}{T}\frac{\partial T}{\partial x} \\ \frac{1}{u}\frac{\partial u}{\partial x} \end{array} \right) \geq 0. \tag{1.651}$$

Now in a well known result from linear algebra, a necessary and sufficient condition for satisfying the above inequality is that the new coefficient matrix be positive semi-definite. Further, the matrix will be positive semi-definite if it has positive semi-definite eigenvalues. The eigenvalues of the new coefficient matrix can be shown to be

$$\lambda = \frac{1}{2}\left((C_{11}+C_{22}) \pm \sqrt{(C_{11}-C_{22})^2 + (C_{12}+C_{21})^2} \right) \tag{1.652}$$

Since the terms inside the radical are positive semi-definite, the eigenvalues must be real. This is a consequence of the parent matrix being symmetric. Now we require two positive semi-definite eigenvalues. First, if $C_{11} + C_{22} < 0$, we obviously have at least one negative eigenvalue, so we demand that $C_{11} + C_{22} \geq 0$. We then must have

$$C_{11} + C_{22} \geq \sqrt{(C_{11}-C_{22})^2 + (C_{12}+C_{21})^2}. \tag{1.653}$$

This gives rise to

$$(C_{11} + C_{22})^2 \geq (C_{11}-C_{22})^2 + (C_{12}+C_{21})^2. \tag{1.654}$$

Expanding and simplifying, one gets

$$C_{11}C_{22} \geq \left(\frac{C_{12}+C_{21}}{2} \right)^2. \tag{1.655}$$

Now the right side is positive semi-definite, so the left side must be also. Thus

$$C_{11}C_{22} \geq 0. \tag{1.656}$$

The only way for the sum and product of C_{11} and C_{22} to be positive semi-definite is to demand that $C_{11} \geq 0$ and $C_{22} \geq 0$. Thus we arrive at the final set of conditions to satisfy the second law:

$$C_{11} \geq 0, \tag{1.657}$$

$$C_{22} \geq 0, \tag{1.658}$$

$$C_{11}C_{22} \geq \left(\frac{C_{12}+C_{21}}{2}\right)^2. \tag{1.659}$$

Now an important school of thought, founded by Onsager[32] in twentieth century thermodynamics takes an extra step and makes the *further* assertion that the original matrix C_{ij} itself must be symmetric. That is $C_{12} = C_{21}$. This remarkable assertion is independent of the second law, and is, for other scenarios, consistent with experimental results. Consequently, the second law in combination with Onsager's independent demand, requires that

$$C_{11} \geq 0, \tag{1.660}$$

$$C_{22} \geq 0, \tag{1.661}$$

$$C_{12} \leq \sqrt{C_{11}C_{22}}. \tag{1.662}$$

All this said, we must dismiss our hypothesis in this specific case on other physical grounds, namely that such a hypothesis results in an infinite shear stress for a fluid at rest! Note that in the special case in which $\partial T/\partial x = 0$, our hypothesis predicts $\tau = C_{22}(T/u^2)(\partial u/\partial x)$. Obviously this is inconsistent with any observation and so we reject this hypothesis. Additionally, this assumed form is not frame invariant because of the velocity dependency. So, why did we go to the trouble to do the above? First, we now have confidence that we should not expect to find heat flux to depend on deformation. Second, it illustrates some general techniques in continuum mechanics. Moreover, the techniques we used have actually been applied to other more complex phenomena which are physical, and of great practical importance.

1.5.2.1.2 Real physical effects. That such a matrix such as we studied in the previous section was asserted to be symmetric is a manifestation of what is known as a general *Onsager relation*, developed by Onsager in 1931 with a statistical mechanics basis for more general systems and for which he was awarded a Nobel Prize in chemistry in 1968. These actually describe a surprising variety of physical phenomena, and are described in detail many texts, including Fung and Woods. A well-known example is the Peltier[33] effect in which conduction of both heat and electrical charge is influenced by gradients of charge and temperature. This forms the basis of the operation of a thermocouple. Other relations exist are the Soret effect in which diffusive mass fluxes are induced by temperature gradients, the

[32]Lars Onsager, 1903-1976, Norwegian-born American physical chemist, earned Ph.D. and taught at Yale, developed a systematic theory for irreversible chemical processes.

[33]Jean Charles Athanase Peltier, 1785-1845, French clockmaker, retired at 30 to study science.

Dufour effect in which a diffusive energy flux is induced by a species concentration gradient, the Hall[34] effect for coupled electrical and magnetic effects (which explains the operation of an electric motor), the Seeback[35] effect in which electromotive forces are induced by different conducting elements at different temperatures, the Thomson[36] effect in which heat is transferred when electric current flows in a conductor in which there is a temperature gradient, and the principle of detailed balance for multi-species chemical reactions.

1.5.2.2 Strong form of the Clausius-Duhem inequality

A less general way to satisfy the second law is to take the sufficient (but not necessary!) condition that each individual term in the entropy inequality to be greater than or equal to zero:

$$-\frac{1}{T^2}q_i\partial_i T \geq 0, \qquad \text{and} \tag{1.663}$$

$$\frac{1}{T}\tau_{ij}\partial_{(i}v_{j)} \geq 0. \tag{1.664}$$

Once again, this is called the *strong form* of the entropy inequality (or the strong form of the Clausius-Duhem inequality), and is potentially overly restrictive.

1.5.3 Fourier's law

Let us examine the restriction on q_i from the strong form of the entropy inequality to infer the common constitutive relation known as *Fourier's law*.[37] The portion of the strong form of the entropy inequality with which we are concerned here is

$$-\frac{1}{T^2}q_i\partial_i T \geq 0. \tag{1.665}$$

Now *one* way to guarantee this inequality is satisfied is to specify the constitutive relation for the heat flux vector as

$$q_i = -k\partial_i T, \qquad \text{with} \qquad k \geq 0. \tag{1.666}$$

This is the well known Fourier's Law for an isotropic material, where k is the thermal conductivity. It has the proper behavior under Galilean transformations and rotations; more

[34]Edwin Herbert Hall, 1855-1938, Maine-born American physicist, educated at Johns Hopkins University where he discovered the Hall effect while working on his dissertation, taught at Harvard.

[35]Thomas Johann Seebeck, 1770-1831, German medical doctor who studied at Berlin and Göttingen.

[36]William Thomson (Lord Kelvin), 1824-1907, Belfast-born British mathematician and physicist, graduated and taught at Glasgow University, key figure in all of 19th century engineering science including mathematics, thermodynamics, and electrodynamics.

[37]Jean Baptiste Joseph Fourier, 1768-1830, French mathematician and Egyptologist who studied the transfer of heat and the representation of mathematical functions by infinite series summations of other functions. Son of a tailor.

importantly, it is consistent with macro-scale experiments for isotropic materials, and can be justified from an underlying micro-scale theory. Substitution of Fourier's law for an isotropic material into the entropy inequality yields

$$\frac{1}{T^2}k(\partial_i T)(\partial_i T) \geq 0, \tag{1.667}$$

which for $k \geq 0$ is a true statement. Note the second law allows other forms as well. The expression $q_i = -k((\partial_j T)(\partial_j T))\partial_i T$ is consistent with the second law. It does not match experiments well for most materials however.

Following Duhamel,[38] we can also generalize Fourier's law for an anisotropic material. Let us only consider anisotropic materials for which the conductivity in any given direction is a constant. For such materials, the thermal conductivity is a tensor k_{ij}, and Fourier's law generalizes to

$$q_i = -k_{ij}\partial_j T. \tag{1.668}$$

This effectively states that for a fixed temperature gradient, the heat flux depends on the orientation. This is characteristic of anisotropic substances such as layered materials. Substitution of the generalized Fourier's law into the entropy inequality (for $\tau_{ij} = 0$) gives now

$$\frac{1}{T^2}k_{ij}(\partial_j T)(\partial_i T) \geq 0, \tag{1.669}$$

$$\frac{1}{T^2}(\partial_i T)k_{ij}(\partial_j T) \geq 0, \tag{1.670}$$

$$\frac{1}{T^2}(\nabla T)^T \cdot \mathsf{K} \cdot \nabla T \geq 0. \tag{1.671}$$

Now $1/T^2 > 0$, so we must have $(\partial_i T)k_{ij}(\partial_j T) \geq 0$ for all possible values of ∇T. Now any possible anti-symmetric portion of k_{ij} cannot contribute to the inequality. We can see this by expanding k_{ij} in the entropy inequality to get

$$\partial_i T\left(\frac{1}{2}(k_{ij}+k_{ji}) + \frac{1}{2}(k_{ij}-k_{ji})\right)\partial_j T \geq 0, \tag{1.672}$$

$$\partial_i T\left(k_{(ij)} + k_{[ij]}\right)\partial_j T \geq 0, \tag{1.673}$$

$$(\partial_i T)k_{(ij)}(\partial_j T) + \underbrace{(\partial_i T)k_{[ij]}(\partial_j T)}_{=0} \geq 0, \tag{1.674}$$

$$(\partial_i T)k_{(ij)}(\partial_j T) \geq 0. \tag{1.675}$$

The anti-symmetric part of k_{ij} makes no contribution to the entropy generation because it involves the tensor inner product of a symmetric tensor with an anti-symmetric tensor, which is identically zero.

Next, we again use the well-known result from linear algebra that the entropy inequality is satisfied if $k_{(ij)}$ is a positive semi-definite tensor. This will be the case if all the eigenvalues

[38]Jean Marie Constant Duhamel, 1797-1872, highly regarded mathematics teacher at École Polytechnique in Paris who applied mathematics to problems in heat transfer, mechanics, and acoustics.

of $k_{(ij)}$ are non-negative. That this is sufficient to satisfy the entropy inequality is made plausible if we consider $\partial_j T$ to be an eigenvector, so that $k_{(ij)}\partial_j T = \lambda\delta_{ij}\partial_j T$ giving rise to an entropy inequality of

$$(\partial_i T)\lambda\delta_{ij}(\partial_j T) \geq 0, \tag{1.676}$$

$$\lambda(\partial_i T)(\partial_i T) \geq 0. \tag{1.677}$$

The inequality holds for all $\partial_i T$ as long as $\lambda \geq 0$.

Further now, when we consider the contribution of the heat flux vector to the energy equation, we see any possible anti-symmetric portion of the conductivity tensor will be inconsequential as well. This is seen by the following analysis, which considers only relevant terms in the energy equation

$$\begin{aligned}
\rho\frac{de}{dt} &= -\partial_i q_i + \ldots, && (1.678)\\
&= \partial_i\left(k_{ij}\partial_j T\right) + \ldots, && (1.679)\\
&= k_{ij}\partial_i\partial_j T + \ldots, && (1.680)\\
&= \left(k_{(ij)} + k_{[ij]}\right)\partial_i\partial_j T + \ldots, && (1.681)\\
&= k_{(ij)}\partial_i\partial_j T + \underbrace{k_{[ij]}\partial_i\partial_j T}_{=0} + \ldots, && (1.682)\\
&= k_{(ij)}\partial_i\partial_j T + \ldots. && (1.683)
\end{aligned}$$

So, it seems any possible anti-symmetric portion of k_{ij} will have no consequence as far as the first or second laws are concerned. However, an anti-symmetric portion of k_{ij} would induce a heat flux orthogonal to the direction of the temperature gradient. In a remarkable confirmation of Onsager's principle, experimental measurements on anisotropic crystalline materials demonstrate that there is no component of heat flux orthogonal to the temperature gradient, and thus, the conductivity matrix k_{ij} in fact has zero anti-symmetric part, and thus is symmetric, $k_{ij} = k_{ji}$. For our particular case with a tensorial conductivity, the competing effects are the heat fluxes in three directions, caused by temperature gradients in three directions:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = -\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}\begin{pmatrix} \partial_1 T \\ \partial_2 T \\ \partial_3 T \end{pmatrix}. \tag{1.684}$$

The symmetry condition, Onsager's principle, requires that $k_{12} = k_{21}$, $k_{13} = k_{31}$, and $k_{23} = k_{32}$. So, the experimentally verified Onsager principle further holds that the heat flux for an anisotropic material is given by

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = -\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{pmatrix}\begin{pmatrix} \partial_1 T \\ \partial_2 T \\ \partial_3 T \end{pmatrix}. \tag{1.685}$$

Now it is well known that the conductivity matrix k_{ij} will be positive semi-definite if all its eigenvalues are non-negative. The eigenvalues will be guaranteed real upon adopting

Onsager symmetry. The characteristic polynomial for the eigenvalues is given by

$$\lambda^3 - I_k^{(1)}\lambda^2 + I_k^{(2)}\lambda - I_k^{(3)} = 0, \tag{1.686}$$

where the invariants of the conductivity tensor k_{ij}, are given by the standard

$$I_k^{(1)} = k_{ii} = \text{tr}\ \mathsf{K}, \tag{1.687}$$

$$I_k^{(2)} = \frac{1}{2}(k_{ii}k_{jj} - k_{ij}k_{ji}) = (\det \mathsf{K})\left(\text{tr}\ \mathsf{K}^{-1}\right), \tag{1.688}$$

$$I_k^{(3)} = \epsilon_{ijk}k_{1j}k_{2j}k_{3j} = \det \mathsf{K}. \tag{1.689}$$

In a standard result from linear algebra, one can show that if all three invariants are positive semi-definite, then the eigenvalues are all positive semi-definite, and as a result, the matrix itself is positive semi-definite. Hence, in order for k_{ij} to be positive semi-definite we demand that

$$I_k^{(1)} \geq 0, \tag{1.690}$$

$$I_k^{(2)} \geq 0, \tag{1.691}$$

$$I_k^{(3)} \geq 0, \tag{1.692}$$

which is equivalent to demanding that

$$k_{11} + k_{22} + k_{33} \geq 0, \tag{1.693}$$

$$k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{12}^2 - k_{13}^2 - k_{23}^2 \geq 0, \tag{1.694}$$

$$k_{13}(k_{12}k_{23} - k_{22}k_{13}) + k_{23}(k_{12}k_{13} - k_{11}k_{23}) + k_{33}(k_{11}k_{22} - k_{12}k_{12}) \geq 0. \tag{1.695}$$

If $\det \mathsf{K} \neq 0$, the conditions reduce to

$$\text{tr}\ \mathsf{K} \geq 0, \tag{1.696}$$

$$\text{tr}\ \mathsf{K}^{-1} \geq 0, \tag{1.697}$$

$$\det \mathsf{K} > 0. \tag{1.698}$$

Now by considering $\partial_i T = (1,0,0)^T$, and demanding $(\partial_i T)k_{ij}(\partial_j T) \geq 0$, we conclude that $k_{11} \geq 0$. Similarly, by considering $\partial_i T = (0,1,0)^T$ and $\partial_i T = (0,0,1)^T$, we conclude that $k_{22} \geq 0$ and $k_{33} \geq 0$, respectively. Thus tr $\mathsf{K} \geq 0$ is automatically satisfied. In equation form, we then have

$$k_{11} \geq 0, \tag{1.699}$$

$$k_{22} \geq 0, \tag{1.700}$$

$$k_{33} \geq 0, \tag{1.701}$$

$$k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{12}^2 - k_{13}^2 - k_{23}^2 \geq 0, \tag{1.702}$$

$$k_{13}(k_{12}k_{23} - k_{22}k_{13}) + k_{23}(k_{12}k_{13} - k_{11}k_{23}) + k_{33}(k_{11}k_{22} - k_{12}k_{12}) \geq 0. \tag{1.703}$$

While by no means a proof, numerical experimentation gives strong indication that the remaining conditions can be satisfied if, loosely stated, $k_{11}, k_{22}, k_{33} >> |k_{12}|, |k_{23}|, |k_{13}|$. That is, for positive semi-definiteness,

- *each* diagonal element must be positive semi-definite,
- off-diagonal terms can be positive or negative, and
- diagonal terms must have amplitudes which are, loosely speaking, larger than the amplitudes of off-diagonal terms.

Example 1.11

Let us consider heat conduction in the limit of two dimensions and a constant anisotropic conductivity tensor, without imposing Onsager's conditions.

Let us take then

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = - \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix}. \tag{1.704}$$

The second law demands that

$$\begin{pmatrix} \partial_1 T & \partial_2 T \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} \geq 0. \tag{1.705}$$

This is expanded as

$$\begin{pmatrix} \partial_1 T & \partial_2 T \end{pmatrix} \left(\begin{pmatrix} k_{11} & \frac{k_{12}+k_{21}}{2} \\ \frac{k_{21}+k_{12}}{2} & k_{22} \end{pmatrix} + \begin{pmatrix} 0 & \frac{k_{12}-k_{21}}{2} \\ \frac{k_{21}-k_{12}}{2} & 0 \end{pmatrix} \right) \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} \geq 0. \tag{1.706}$$

As before, the anti-symmetric portion makes no contribution to the left hand side, giving rise to

$$\begin{pmatrix} \partial_1 T & \partial_2 T \end{pmatrix} \begin{pmatrix} k_{11} & \frac{k_{12}+k_{21}}{2} \\ \frac{k_{21}+k_{12}}{2} & k_{22} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} \geq 0. \tag{1.707}$$

And, demanding that the eigenvalues of the symmetric part of the conductivity tensor be positive gives rise to the conditions, identical to that of an earlier analysis, that

$$k_{11} \geq 0, \tag{1.708}$$

$$k_{22} \geq 0, \tag{1.709}$$

$$k_{11} k_{22} \geq \left(\frac{k_{12} + k_{21}}{2} \right)^2. \tag{1.710}$$

The energy equation becomes

$$\rho \frac{de}{dt} = -\partial_i q_i + \ldots, \tag{1.711}$$

$$= \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} + \ldots, \tag{1.712}$$

$$= \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} k_{11}\partial_1 T + k_{12}\partial_2 T \\ k_{21}\partial_1 T + k_{22}\partial_2 T \end{pmatrix} + \ldots, \tag{1.713}$$

$$= k_{11}\partial_1\partial_1 T + (k_{12} + k_{21})\partial_1\partial_2 T + k_{22}\partial_2\partial_2 T + \ldots, \tag{1.714}$$

$$= k_{11} \frac{\partial^2 T}{\partial x_1^2} + (k_{12} + k_{21}) \frac{\partial^2 T}{\partial x_1 \partial x_2} + k_{22} \frac{\partial^2 T}{\partial x_2^2} + \ldots. \tag{1.715}$$

One sees that the energy evolution depends only on the symmetric part of the conductivity tensor.

Imposition of Onsager's relations gives simply $k_{12} = k_{21}$, giving rise to second law restrictions of

$$k_{11} \geq 0, \tag{1.716}$$

$$k_{22} \geq 0, \tag{1.717}$$

$$k_{11}k_{22} \geq k_{12}^2, \tag{1.718}$$

and an energy equation of

$$\rho \frac{de}{dt} = k_{11}\frac{\partial^2 T}{\partial x_1^2} + 2k_{12}\frac{\partial^2 T}{\partial x_1 \partial x_2} + k_{22}\frac{\partial^2 T}{\partial x_2^2} + \dots. \tag{1.719}$$

Example 1.12

Consider the ramifications of a heat flux vector in violation of Onsager's principle: flux in which the anisotropic conductivity is purely anti-symmetric. For simplicity consider an incompressible solid with constant specific heat c. For the heat flux, we take

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = -\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}\begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix}. \tag{1.720}$$

This holds that heat flux in the 1 direction is induced only by temperature gradients in the 2 direction and heat flux in the 2 direction is induced only by temperature gradients in the 1 direction.

The second law demands that

$$(\partial_1 T \quad \partial_2 T) \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}\begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix} \geq 0, \tag{1.721}$$

$$(\partial_1 T \quad \partial_2 T) \begin{pmatrix} -\beta\partial_2 T \\ \beta\partial_1 T \end{pmatrix} \geq 0, \tag{1.722}$$

$$-\beta(\partial_1 T)(\partial_2 T) + \beta(\partial_1 T)(\partial_2 T) \geq 0, \tag{1.723}$$

$$0 \geq 0. \tag{1.724}$$

So, the second law holds.

For the incompressible solid with constant heat capacity, the velocity field is zero, and the energy equation reduces to the simple

$$\rho c \frac{\partial T}{\partial t} = -\partial_i q_i. \tag{1.725}$$

Imposing our unusual expression for heat flux, we get

$$\rho c \frac{\partial T}{\partial t} = (\partial_1 \quad \partial_2) \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}\begin{pmatrix} \partial_1 T \\ \partial_2 T \end{pmatrix}, \tag{1.726}$$

$$= (\partial_1 \quad \partial_2) \begin{pmatrix} -\beta\partial_2 T \\ \beta\partial_1 T \end{pmatrix}, \tag{1.727}$$

$$= -\beta\partial_1\partial_2 T + \beta\partial_1\partial_2 T, \tag{1.728}$$

$$= 0. \tag{1.729}$$

So, this unusual heat flux vector is one which induces no change in temperature. In terms of the first law of thermodynamics, a net energy flux into a control volume in the 1 direction is exactly counterbalanced by an net energy flux out of the same control volume in the 2 direction. Thus the first law holds as well.

Let us consider a temperature distribution for this unusual material. And let us consider it to apply to the domain $x \in [0,1]$, $y \in [0,1]$, $t \in [0,\infty]$. Take

$$T(x_1, x_2, t) = x_2. \tag{1.730}$$

Obviously this satisfies the first law as $\frac{\partial T}{\partial t} = 0$. Let us check the heat flux.

$$\begin{aligned} q_1 &= \beta \partial_2 T = \beta, & (1.731) \\ q_2 &= -\beta \partial_1 T = 0. & (1.732) \end{aligned}$$

Now the lower boundary at $x_2 = 0$ has $T = 0$. The upper boundary has $x_2 = 1$ so $T = 1$. And this constant temperature gradient in the 2 direction is inducing a constant heat flux in the 1 direction, $q_1 = -\beta$. The energy flux that enters at $x_1 = 0$ departs at $x_1 = 1$, maintaining energy conservation.

One can consider an equivalent problem in cylindrical coordinates. Taking

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \tag{1.733}$$

and applying the chain rule,

$$\begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial \theta}{\partial x_1} \\ \frac{\partial r}{\partial x_2} & \frac{\partial \theta}{\partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}, \tag{1.734}$$

one finds

$$\begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}. \tag{1.735}$$

So, transforming $q_1 = \beta\partial_2 T$, and $q_2 = -\beta\partial_1 T$ gives

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \beta \begin{pmatrix} \sin\theta & \frac{\cos\theta}{r} \\ -\cos\theta & \frac{\sin\theta}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial r} \\ \frac{\partial T}{\partial \theta} \end{pmatrix}. \tag{1.736}$$

Standard trigonometry gives

$$\begin{pmatrix} q_r \\ q_\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{rotation matrix}} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{1.737}$$

Applying the rotation matrix to both sides gives then

$$\begin{aligned} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \beta \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sin\theta & \frac{\cos\theta}{r} \\ -\cos\theta & \frac{\sin\theta}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial r} \\ \frac{\partial T}{\partial \theta} \end{pmatrix}, & (1.738) \\ \begin{pmatrix} q_r \\ q_\theta \end{pmatrix} &= \beta \begin{pmatrix} 0 & \frac{1}{r} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial r} \\ \frac{\partial T}{\partial \theta} \end{pmatrix}, & (1.739) \end{aligned}$$

or simply

$$\begin{aligned} q_r &= \frac{\beta}{r}\frac{\partial T}{\partial \theta}, & (1.740) \\ q_\theta &= -\beta\frac{\partial T}{\partial r}. & (1.741) \end{aligned}$$

Now the steady state temperature distribution in the annular region $1/2 < r < 1$, $T = r$, describes a domain with an inner boundary held at $T = 1/2$ and an outer boundary held at $T = 1$. Such a temperature distribution would induce a heat flux in the θ direction only, so that $q_r = 0$ and $q_\theta = -\beta$. That is, the heat goes round and round the domain, but never enters or exits at any boundary.

Now such a flux is counterintuitive precisely because it has never been observed or measured. It is for this reason that we can adopt Onsager's hypothesis and demand that, independent of the first and second laws of thermodynamics,

$$\beta = 0, \tag{1.742}$$

and the conductivity tenser is purely symmetric.

1.5.4 Stress-strain rate relation for a Newtonian fluid

We now seek to satisfy the second part of the strong form of the entropy inequality, namely (and recalling that $T > 0$)

$$\underbrace{\tau_{ij}\partial_{(i}v_{j)}}_{\Phi} \geq 0. \tag{1.743}$$

This form suggests that we seek a constitutive equation for the viscous stress tensor τ_{ij} which is a function of the deformation tensor $\partial_{(i}v_{j)}$. Fortunately, such a form exists, which moreover agrees with macro-scale experiments and micro-scale theories. Here we will focus on the simplest of such theories, for what is known as a *Newtonian fluid*, a fluid which is isotropic and whose viscous stress varies linearly with strain rate. In general, this is a discipline unto itself known as rheology.

1.5.4.1 Underlying experiments

We can pull a flat plate over a fluid and measure the force necessary to maintain a specified velocity. This situation and some expected results are sketched in Figure 1.26. We observe that

- At the upper and lower plate surfaces, the fluid has the same velocity of each plate. This is called the *no slip* condition.
- The faster the velocity V of the upper plate is, the higher the force necessary to pull the plate is. The increase can be linear or non-linear.
- When experiments are carried out with different plate area and different gap width, a single universal curve results when F/A is plotted against V/h.
- The velocity profile is linear with increasing x_2.

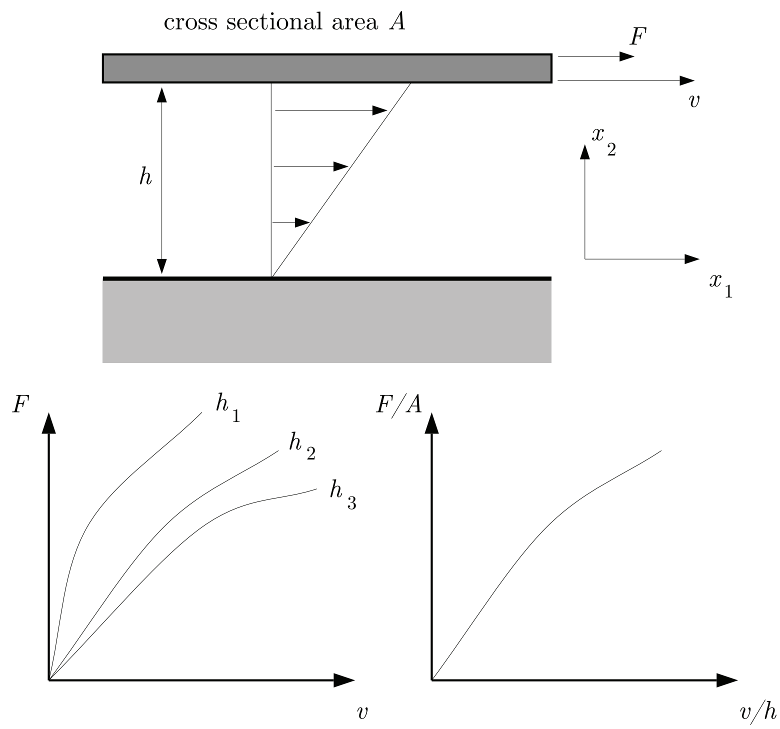

Figure 1.26: Sketch of simple Couette flow experiment with measurements of stress versus strain rate.

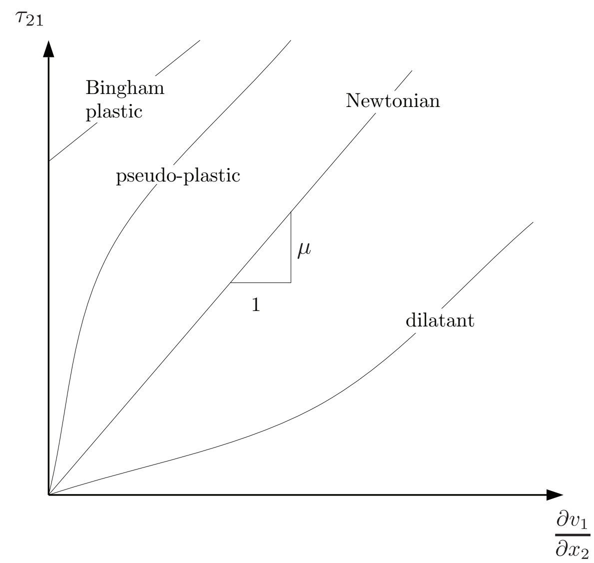

Figure 1.27: Variation of viscous stress with strain rate for typical fluids.

In a way similar on a molecular scale to energy diffusion, this experiment is describing a diffusion of momentum from the pulled plate into the fluid below it. The constitutive equation we develop for viscous stress, when combined with the governing axioms, will model momentum diffusion.

We can associate F/A with a shear stress: τ_{21}, recalling stress on the 2 face in the 1 direction. We can associate V/h with a velocity gradient, here $\partial_2 v_1$. We note that considering the velocity gradient is essentially equivalent to considering the deformation gradient, as far as the second law is concerned, and so we will be loose here in our use of the term. We define the *coefficient of viscosity* μ for this configuration as

$$\mu = \frac{\tau_{21}}{\partial_2 v_1} = \frac{\text{viscous stress}}{\text{strain rate}}. \tag{1.744}$$

The viscosity is the analog of Young's[39] modulus in solid mechanics, which is the ratio of stress to strain. In general μ is a thermodynamic property of a material. It is often a strong function of temperature, but can vary with pressure as well. A Newtonian fluid has a viscosity which does not depend on strain rate (but could depend on temperature and pressure). A non-Newtonian fluid has a viscosity which is strain rate dependent (and possible temperature and pressure). Some typical behavior is sketched in Figure 1.27. We shall focus here on fluids whose viscosity is not a function of strain rate. Much of our development will be valid for temperature and pressure dependent viscosity, while most actual examples will consider only constant viscosity.

[39]Thomas Young, 1773-1829, English physician and physicist whose experiments in interferometry revived the wave theory of light, Egyptologist who helped decipher the Rosetta stone, worked on surface tension in fluids, gave the word "energy" scientific significance, and developed Young's modulus in elasticity.

1.5.4.2 Analysis for isotropic Newtonian fluid

Here we shall outline the method described by Whitaker (p. 139-145) to describe the viscous stress as a function of strain rate for an isotropic fluid with constant viscosity. An isotropic fluid has no directional dependencies when subjected to a force. A fluid composed of aligned long chain polymers is an example of a fluid that is most likely not isotropic. Following Whitaker, we

- *postulate* that stress is a function of deformation rate (strain rate) only:[40]

$$\tau_{ij} = f_{ij}(\partial_{(k}v_{l)}). \tag{1.745}$$

Written out in more detail, we have postulated a relationship of the form

$$\tau_{11} = f_{11}(\partial_{(1}v_{1)}, \partial_{(2}v_{2)}, \partial_{(3}v_{3)}, \partial_{(1}v_{2)}, \partial_{(2}v_{3)}, \partial_{(3}v_{1)}\partial_{(2}v_{1)}, \partial_{(3}v_{2)}, \partial_{(1}v_{3)}), \tag{1.746}$$
$$\tau_{12} = f_{12}(\partial_{(1}v_{1)}, \partial_{(2}v_{2)}, \partial_{(3}v_{3)}, \partial_{(1}v_{2)}, \partial_{(2}v_{3)}, \partial_{(3}v_{1)}\partial_{(2}v_{1)}, \partial_{(3}v_{2)}, \partial_{(1}v_{3)}), \tag{1.747}$$
$$\vdots \tag{1.748}$$
$$\tau_{33} = f_{33}(\partial_{(1}v_{1)}, \partial_{(2}v_{2)}, \partial_{(3}v_{3)}, \partial_{(1}v_{2)}, \partial_{(2}v_{3)}, \partial_{(3}v_{1)}\partial_{(2}v_{1)}, \partial_{(3}v_{2)}, \partial_{(1}v_{3)}). \tag{1.749}$$

- require that $\tau_{ij} = 0$ if $\partial_{(i}v_{j)} = 0$, hence, no strain rate, no stress.
- require that stress is *linearly* related to strain rate:

$$\tau_{ij} = \hat{C}_{ijkl}\partial_{(k}v_{l)}. \tag{1.750}$$

This is the imposition of the assumption of a Newtonian fluid. Here $\hat{C}_{ijkl}$ is a fourth order tensor. Thus we have in matrix form

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \\ \tau_{21} \\ \tau_{32} \\ \tau_{13} \end{pmatrix} = \begin{pmatrix} \hat{C}_{1111} & \hat{C}_{1122} & \hat{C}_{1133} & \hat{C}_{1112} & \hat{C}_{1123} & \hat{C}_{1131} & \hat{C}_{1121} & \hat{C}_{1132} & \hat{C}_{1113} \\ \hat{C}_{2211} & \hat{C}_{2222} & \hat{C}_{2233} & \hat{C}_{2212} & \hat{C}_{2223} & \hat{C}_{2231} & \hat{C}_{2221} & \hat{C}_{2232} & \hat{C}_{2213} \\ \hat{C}_{3311} & \hat{C}_{3322} & \hat{C}_{3333} & \hat{C}_{3312} & \hat{C}_{3323} & \hat{C}_{3331} & \hat{C}_{3321} & \hat{C}_{3332} & \hat{C}_{3313} \\ \hat{C}_{1211} & \hat{C}_{1222} & \hat{C}_{1233} & \hat{C}_{1212} & \hat{C}_{1223} & \hat{C}_{1231} & \hat{C}_{1221} & \hat{C}_{1232} & \hat{C}_{1213} \\ \hat{C}_{2311} & \hat{C}_{2322} & \hat{C}_{2333} & \hat{C}_{2312} & \hat{C}_{2323} & \hat{C}_{2331} & \hat{C}_{2321} & \hat{C}_{2332} & \hat{C}_{2313} \\ \hat{C}_{3111} & \hat{C}_{3122} & \hat{C}_{3133} & \hat{C}_{3112} & \hat{C}_{3123} & \hat{C}_{3131} & \hat{C}_{3121} & \hat{C}_{3132} & \hat{C}_{3113} \\ \hat{C}_{2111} & \hat{C}_{2122} & \hat{C}_{2133} & \hat{C}_{2112} & \hat{C}_{2123} & \hat{C}_{2131} & \hat{C}_{2121} & \hat{C}_{2132} & \hat{C}_{2113} \\ \hat{C}_{3211} & \hat{C}_{3222} & \hat{C}_{3233} & \hat{C}_{3212} & \hat{C}_{3223} & \hat{C}_{3231} & \hat{C}_{3221} & \hat{C}_{3232} & \hat{C}_{3213} \\ \hat{C}_{1311} & \hat{C}_{1322} & \hat{C}_{1333} & \hat{C}_{1312} & \hat{C}_{1323} & \hat{C}_{1331} & \hat{C}_{1321} & \hat{C}_{1332} & \hat{C}_{1313} \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \\ \partial_{(2}v_{1)} \\ \partial_{(3}v_{2)} \\ \partial_{(1}v_{3)} \end{pmatrix} \tag{1.751}$$

There are $3^4 = 81$ unknown coefficients $\hat{C}_{ijkl}$. We found one of them in our simple experiment in which we found

$$\tau_{21} = \tau_{12} = \mu\partial_2 v_1 = \mu(2\partial_{(1}v_{2)}).$$

Hence in this special case $\hat{C}_{1212} = 2\mu$.

[40] Thus we are not allowing viscous stress to be a function of the rigid body rotation rate. While it seems intuitive that rigid body rotation should not induce viscous stress, Batchelor mentions that there is no rigorous proof for this; hence, we describe our statement as a postulate.

Now we could do eighty-one separate experiments, or we could take advantage of the assumption that the fluid has no directional dependency. We will take the following approach. Observer A conducts an experiment to measure the stress tensor in reference frame $\mathcal{A}$. The observer begins with the "viscosity matrix" $\hat{C}_{ijkl}$. The experiment is conducted by varying strain rate and measuring stress. With complete knowledge A feels confident this knowledge could be used to predict the stress in rotated frame $\mathcal{A}'$.

Consider observer A' who is oriented to frame $\mathcal{A}'$. Oblivious to observer A, A' conducts the same experiment to measure what for her or him is τ'_{ij} The value that A' measures must be the same that A predicts in order for the system to be isotropic. This places restrictions on the viscosity matrix $\hat{C}_{ijkl}$. We intend to show that if the fluid is isotropic only *two* of the eighty-one coefficients are distinct and non-zero.

We first use symmetry properties of the stress and strain rate tensor to reduce to thirty-six unknown coefficients. We note that in actuality there are only six independent components of stress and six independent components of deformation since both are symmetric tensors. Consequently, we can write our linear stress-strain rate relation as

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{pmatrix} = \begin{pmatrix} \hat{C}_{1111} & \hat{C}_{1122} & \hat{C}_{1133} & \hat{C}_{1112}+\hat{C}_{1121} & \hat{C}_{1123}+\hat{C}_{1132} & \hat{C}_{1131}+\hat{C}_{1113} \\ \hat{C}_{2211} & \hat{C}_{2222} & \hat{C}_{2233} & \hat{C}_{2212}+\hat{C}_{2221} & \hat{C}_{2223}+\hat{C}_{2232} & \hat{C}_{2231}+\hat{C}_{2213} \\ \hat{C}_{3311} & \hat{C}_{3322} & \hat{C}_{3333} & \hat{C}_{3312}+\hat{C}_{3321} & \hat{C}_{3323}+\hat{C}_{3332} & \hat{C}_{3331}+\hat{C}_{3313} \\ \hat{C}_{1211} & \hat{C}_{1222} & \hat{C}_{1233} & \hat{C}_{1212}+\hat{C}_{1221} & \hat{C}_{1223}+\hat{C}_{1232} & \hat{C}_{1231}+\hat{C}_{1213} \\ \hat{C}_{2311} & \hat{C}_{2322} & \hat{C}_{2333} & \hat{C}_{2312}+\hat{C}_{2321} & \hat{C}_{2323}+\hat{C}_{2332} & \hat{C}_{2331}+\hat{C}_{2313} \\ \hat{C}_{3111} & \hat{C}_{3122} & \hat{C}_{3133} & \hat{C}_{3112}+\hat{C}_{3121} & \hat{C}_{3123}+\hat{C}_{3132} & \hat{C}_{3131}+\hat{C}_{3113} \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \end{pmatrix}. \tag{1.752}$$

Now adopting Whitaker's notation for simplification, we define the above matrix of $\hat{C}$'s as a new matrix of C's. Here, now C itself is not a tensor, while $\hat{C}$ is a tensor. We take equivalently then

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \end{pmatrix}. \tag{1.753}$$

Next, recalling that for tensorial quantities

$$\tau'_{ij} = \ell_{ki}\ell_{lj}\tau_{kl}, \tag{1.754}$$

$$\partial'_{(i}v'_{j)} = \ell_{ki}\ell_{lj}\partial_{(k}v_{l)}, \tag{1.755}$$

let us subject our fluid to a battery of rotations and see what can be concluded by enforcing material indifference.

- 180° *rotation about x_3 axis*

  For this rotation, sketched in Figure 1.28. we have direction cosines

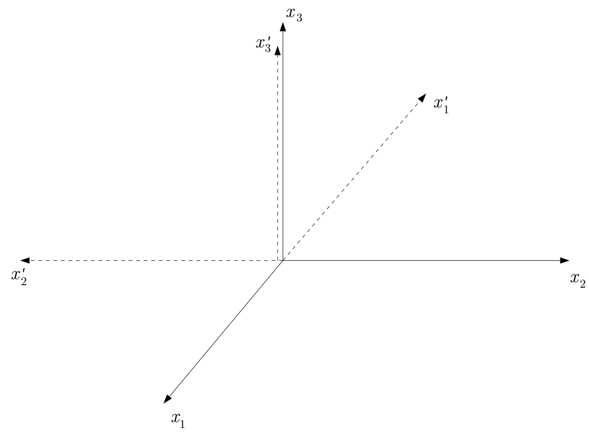

Figure 1.28: Rotation of 180° about x_3 axis.

$$\ell_{ki} = \begin{pmatrix} \ell_{11} = -1 & \ell_{12} = 0 & \ell_{13} = 0 \\ \ell_{21} = 0 & \ell_{22} = -1 & \ell_{23} = 0 \\ \ell_{31} = 0 & \ell_{32} = 0 & \ell_{33} = 1 \end{pmatrix}. \tag{1.756}$$

Applying the transformation rules to each term in the shear stress tensor, we get

$$\tau'_{11} = \ell_{k1}\ell_{l1}\tau_{kl} = (-1)^2\tau_{11} = \tau_{11}, \tag{1.757}$$
$$\tau'_{22} = \ell_{k2}\ell_{l2}\tau_{kl} = (-1)^2\tau_{22} = \tau_{22}, \tag{1.758}$$
$$\tau'_{33} = \ell_{k3}\ell_{l3}\tau_{kl} = (1)^2\tau_{33} = \tau_{33}, \tag{1.759}$$
$$\tau'_{12} = \ell_{k1}\ell_{l2}\tau_{kl} = (-1)^2\tau_{12} = \tau_{12}, \tag{1.760}$$
$$\tau'_{23} = \ell_{k2}\ell_{l3}\tau_{kl} = (-1)(1)\tau_{23} = -\tau_{23}, \tag{1.761}$$
$$\tau'_{31} = \ell_{k3}\ell_{l1}\tau_{kl} = (1)(-1)\tau_{31} = -\tau_{31}. \tag{1.762}$$

Likewise we find that

$$\partial'_{(1}v'_{1)} = \partial_{(1}v_{1)}, \tag{1.763}$$
$$\partial'_{(2}v'_{2)} = \partial_{(2}v_{2)}, \tag{1.764}$$
$$\partial'_{(3}v'_{3)} = \partial_{(3}v_{3)}, \tag{1.765}$$
$$\partial'_{(1}v'_{2)} = \partial_{(1}v_{2)}, \tag{1.766}$$
$$\partial'_{(2}v'_{3)} = -\partial_{(2}v_{3)}, \tag{1.767}$$
$$\partial'_{(3}v'_{1)} = -\partial_{(3}v_{1)}. \tag{1.768}$$

Now our observer A' who is in the rotated system would say, for instance that

$$\tau'_{11} = C_{11}\partial'_{(1}v'_{1)} + C_{12}\partial'_{(2}v'_{2)} + C_{13}\partial'_{(3}v'_{3)}C_{14}\partial'_{(1}v'_{2)} + C_{15}\partial'_{(2}v'_{3)} + C_{16}\partial'_{(3}v'_{1)}, \tag{1.769}$$

while our observer A who used tensor algebra to predict τ'_{11} would say

$$\tau'_{11} = C_{11}\partial'_{(1}v'_{1)} + C_{12}\partial'_{(2}v'_{2)} + C_{13}\partial'_{(3}v'_{3)}C_{14}\partial'_{(1}v'_{2)} - C_{15}\partial'_{(2}v'_{3)} - C_{16}\partial'_{(3}v'_{1)}, \tag{1.770}$$

Since we want both predictions to be the same, we must require that

$$C_{15} = C_{16} = 0. \tag{1.771}$$

In matrix form, our observer A would predict for the rotated frame that

$$\begin{pmatrix} \tau'_{11} \\ \tau'_{22} \\ \tau'_{33} \\ \tau'_{12} \\ -\tau'_{23} \\ -\tau'_{31} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \partial'_{(1}v'_{1)} \\ \partial'_{(2}v'_{2)} \\ \partial'_{(3}v'_{3)} \\ \partial'_{(1}v'_{2)} \\ -\partial'_{(2}v'_{3)} \\ -\partial'_{(3}v'_{1)} \end{pmatrix}. \tag{1.772}$$

To retain material difference between the predictions of our two observers, we thus require that $C_{15} = C_{16} = C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = C_{51} = C_{52} = C_{53} = C_{54} = C_{61} = C_{62} = C_{63} = C_{64} = 0$. This eliminates 16 coefficients and gives our viscosity matrix the form

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{21} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & C_{34} & 0 & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{65} & C_{66} \end{pmatrix}. \tag{1.773}$$

with only 20 independent coefficients.

- 180° *rotation about* x_1 *axis*

  This rotation is sketched in Figure 1.29.

  Leaving out the details of the previous section, this rotation has a set of direction cosines of

$$\ell_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.774}$$

  Application of this rotation leads to the conclusion that the viscosity matrix must be of the form

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}. \tag{1.775}$$

  with only 12 independent coefficients.

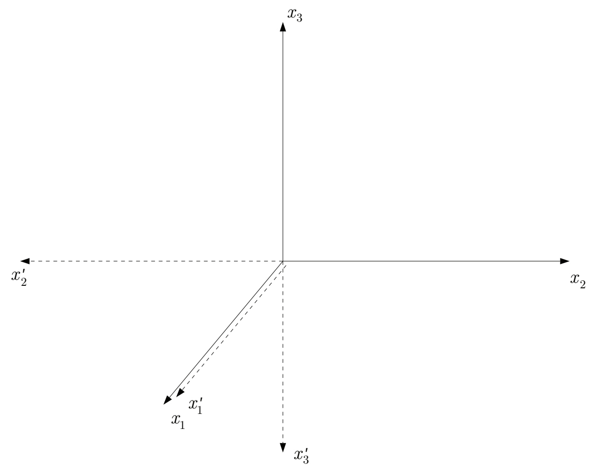

Figure 1.29: Rotation of 180° about x_1 axis.

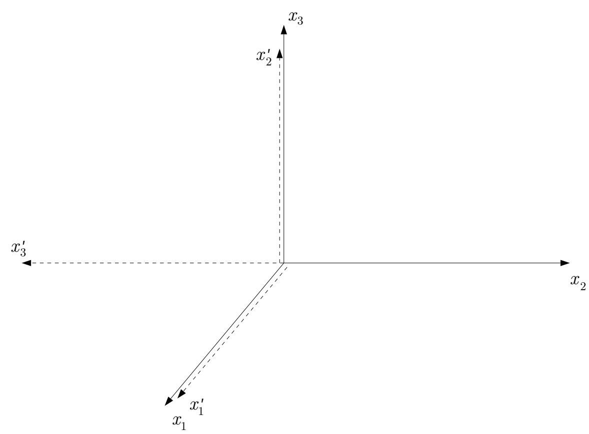

Figure 1.30: Rotation of 90° about x_1 axis.

- 180° *rotation about* x_2 *axis*

  One is tempted to perform this rotation as well, but nothing new is learned from it!

- 90° *rotation about* x_1 *axis*

  This rotation is sketched in Figure 1.30. This rotation has a set of direction cosines of

$$\ell_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.776}$$

  Application of this rotation leads to the conclusion that the viscosity matrix must be

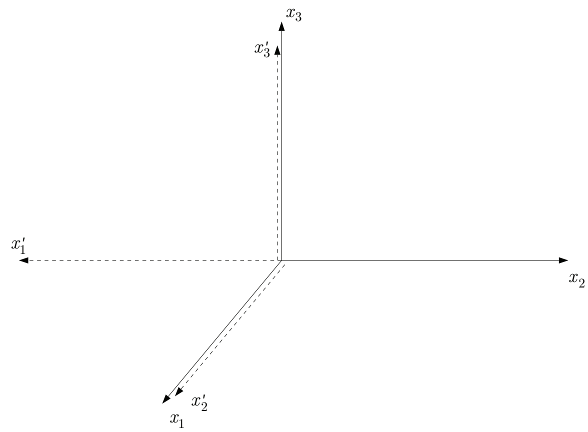

Figure 1.31: Rotation of 90° about x_3 axis.

of the form

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{21} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}. \tag{1.777}$$

with only 8 independent coefficients.

- 90° *rotation about* x_3 *axis*

  This rotation is sketched in Figure 1.31.

  This rotation has a set of direction cosines of

$$\ell_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.778}$$

  Application of this rotation leads to the conclusion that the viscosity matrix must be of the form

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}. \tag{1.779}$$

  with only 3 independent coefficients.

- 90° *rotation about* x_2 *axis*

  We learn nothing from this rotation.

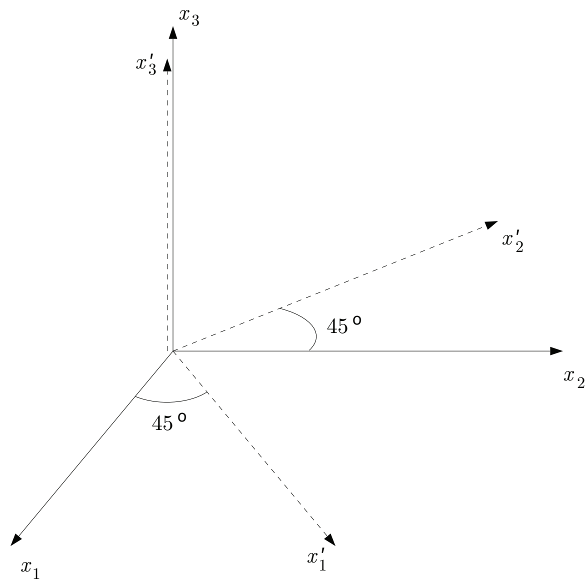

Figure 1.32: Rotation of 45° about x_3 axis.

- 45° *rotation about* x_3 *axis*

  This rotation is sketched in Figure 1.32.

  This rotation has a set of direction cosines of

$$\ell_{ij} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.780}$$

  After a lot of algebra, application of this rotation lead to the conclusion that the viscosity matrix must be of the form

$$\begin{pmatrix} C_{44}+C_{12} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{44}+C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{44}+C_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}. \tag{1.781}$$

  with only 2 independent coefficients.

Try as we might, we cannot reduce this any further with more rotations. It can be proved more rigorously, as shown in most books on tensor analysis, that this is the furthest reduction that can be made. So, for an isotropic Newtonian fluid, we can expect two independent coefficients to parameterize the relation between strain rate and viscous stress. The relation between stress and strain rate can be expressed in detail as

$$\tau_{11} = C_{44}\partial_{(1}v_{1)} + C_{12}\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)} + \partial_{(3}v_{3)}\right), \quad (1.782)$$

$$\tau_{22} = C_{44}\partial_{(2}v_{2)} + C_{12}\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)} + \partial_{(3}v_{3)}\right), \quad (1.783)$$

$$\tau_{33} = C_{44}\partial_{(3}v_{3)} + C_{12}\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)} + \partial_{(3}v_{3)}\right), \quad (1.784)$$

$$\tau_{12} = C_{44}\partial_{(1}v_{2)}, \quad (1.785)$$

$$\tau_{23} = C_{44}\partial_{(2}v_{3)}, \quad (1.786)$$

$$\tau_{31} = C_{44}\partial_{(3}v_{1)}. \quad (1.787)$$

Using traditional notation, we take

- $C_{44} \equiv 2\mu$, where μ is the first coefficient of viscosity, and
- $C_{12} \equiv \lambda$, where λ is the second coefficient of viscosity.
- A similar analysis in solid mechanics leads one to conclude for an isotropic material in which the stress tensor is linearly related to the strain (rather than the strain rate) gives rise to two independent coefficients, the elastic modulus and the shear modulus. In solids, these both can be measured, and they are independent.

In terms of our original fourth order tensor, we can write the linear relationship $\tau_{ij} = \hat{C}_{ijkl}\partial_{(i}v_{j)}$ as

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \\ \tau_{21} \\ \tau_{32} \\ \tau_{13} \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \\ \partial_{(2}v_{1)} \\ \partial_{(3}v_{2)} \\ \partial_{(1}v_{3)} \end{pmatrix}. \quad (1.788)$$

We note that because of the symmetry of $\partial_{(i}v_{j)}$ that the above representation is not unique in that the following, as well as other linear combinations, is an identically equivalent statement:

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \\ \tau_{21} \\ \tau_{32} \\ \tau_{13} \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & \mu \\ 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \\ \partial_{(2}v_{1)} \\ \partial_{(3}v_{2)} \\ \partial_{(1}v_{3)} \end{pmatrix}. \quad (1.789)$$

In shorthand Cartesian index and Gibbs notation, the viscous stress tensor is given by

$$\tau_{ij} = 2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k \delta_{ij}, \tag{1.790}$$

$$\boldsymbol{\tau} = 2\mu\left(\frac{\nabla\mathbf{v}^T + (\nabla\mathbf{v}^T)^T}{2}\right) + \lambda(\nabla^T\cdot\mathbf{v})\mathsf{I}. \tag{1.791}$$

By performing minor algebraic manipulations, the viscous stress tensor can be cast in a way which elucidates more of the physics of how strain rate influences stress. It is easily verified by direct expansion that the viscous stress tensor can be written as

$$\tau_{ij} = \underbrace{\left((2\mu+3\lambda)\underbrace{\frac{\partial_k v_k}{3}}_{\text{mean strain rate}}\delta_{ij}\right)}_{\text{mean viscous stress}} + \underbrace{2\mu\left(\underbrace{\partial_{(i}v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij}}_{\text{deviatoric strain rate}}\right)}_{\text{deviatoric viscous stress}}, \tag{1.792}$$

$$\boldsymbol{\tau} = (2\mu+3\lambda)\frac{\nabla^T\cdot\mathbf{v}}{3}\mathsf{I} + 2\mu\left(\frac{\nabla\mathbf{v}^T + (\nabla\mathbf{v}^T)^T}{2} - \frac{1}{3}\nabla^T\cdot\mathbf{v}\mathsf{I}\right). \tag{1.793}$$

Here it is seen that a mean strain rate, really a volumetric change, induces a mean viscous stress, as long as $\lambda \neq -(2/3)\mu$. If either $\lambda = -(2/3)\mu$ or $\partial_k v_k = 0$, all viscous stress is deviatoric. Further, for $\mu \neq 0$, a deviatoric strain rate induces a deviatoric viscous stress. We can form the mean viscous stress by contracting the viscous stress tensor:

$$\frac{1}{3}\tau_{ii} = \left(\frac{2}{3}\mu + \lambda\right)\partial_k v_k. \tag{1.794}$$

Note that the mean viscous stress is a scalar, and is thus independent of orientation; it is directly proportional to the first invariant of the viscous stress tensor. Obviously the mean viscous stress is zero if $\lambda = -(2/3)\mu$. Now the total stress tensor is given by

$$T_{ij} = -p\delta_{ij} + 2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k\delta_{ij}, \tag{1.795}$$

$$\mathsf{T} = -p\mathsf{I} + 2\mu\left(\frac{\nabla\mathbf{v}^T + (\nabla\mathbf{v}^T)^T}{2}\right) + \lambda(\nabla^T\cdot\mathbf{v})\mathsf{I}. \tag{1.796}$$

We notice the stress tensor has three components, 1) a uniform diagonal tensor with the hydrostatic pressure, 2) a tensor which is directly proportional to the strain rate tensor, and 3) a uniform diagonal tensor which is proportional to the first invariant of the strain rate tensor: $I_{\dot{\epsilon}}^{(1)} = \text{tr}\ (\partial_{(i}v_{k)}) = \partial_k v_k$. Consequently, the stress tensor can be written as

$$T_{ij} = \underbrace{\left(-p + \lambda I_{\dot{\epsilon}}^{(1)}\right)\delta_{ij}}_{\text{isotropic}} + \underbrace{2\mu\partial_{(i}v_{j)}}_{\text{linear in strain rate}}, \tag{1.797}$$

$$\mathsf{T} = \left(-p + \lambda I_{\dot{\epsilon}}^{(1)}\right)\mathsf{I} + 2\mu\left(\frac{\nabla\mathbf{v}^T + (\nabla\mathbf{v}^T)^T}{2}\right). \tag{1.798}$$

Recalling that $\delta_{ij} = \mathbf{I}$ as well as $I_{\dot{\epsilon}}^{(1)}$ are invariant under a rotation of coordinate axes, we deduce that the stress is related linearly to the strain rate. Moreover when the axes are rotated to be aligned with the principal axes of strain rate, the stress is purely normal stress and takes on its principal value.

Let us next consider two typical elements to aid in interpreting the relation between viscous stress and strain rate for a general Newtonian fluid.

1.5.4.2.1 Diagonal component Consider a typical diagonal component of the viscous stress tensor, say τ_{11}:

$$\tau_{11} = \underbrace{\left((2\mu + 3\lambda)\underbrace{\left(\frac{\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3}{3}\right)}_{\text{mean strain rate}}\right)}_{\text{mean viscous stress}} + \underbrace{2\mu\left(\underbrace{\partial_1 v_1 - \frac{1}{3}(\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3)}_{\text{deviatoric strain rate}}\right)}_{\text{deviatoric viscous stress}}. \quad (1.799)$$

Note that if we choose our axes to be the principal axes of the strain-rate tensor, then these terms will appear on the diagonal of the stress tensor and there will be no off-diagonal elements. Thus the fundamental physics of the stress-strain relationship are completely embodied in a natural way in the above expression.

1.5.4.2.2 Off-diagonal component If we are not aligned with the principal axes, then off-diagonal terms will be non-zero. A typical off-diagonal component of the viscous stress tensor, say τ_{12}, has the following form:

$$\tau_{12} = 2\mu\left(\partial_{(1}v_{2)} + \lambda\partial_k v_k \underbrace{\delta_{12}}_{=0}\right), \quad (1.800)$$

$$= 2\mu\partial_{(1}v_{2)}, \quad (1.801)$$

$$= \mu(\partial_1 v_2 + \partial_2 v_1). \quad (1.802)$$

Note this is associated with shear deformation for elements aligned with the 1 and 2 axes, and that it is independent of the value of λ, which is only associated with the mean strain rate.

1.5.4.3 Stokes' assumption

It is a straightforward matter to measure μ. It is not at all straightforward to measure λ. As discussed earlier, Stokes in the mid-nineteenth century suggested to require that the mechanical pressure (that is the average normal stress) be equal to the thermodynamic pressure. We have seen that the consequence of this is Eq. (1.466): $\tau_{ii} = 0$. If we enforce this on our expression for τ_{ij}, we get

$$\tau_{ii} = 0 = 2\mu\partial_{(i}v_{i)} + \lambda\partial_k v_k \delta_{ii}, \quad (1.803)$$
$$= 2\mu\partial_i v_i + 3\lambda\partial_k v_k, \quad (1.804)$$
$$= 2\mu\partial_i v_i + 3\lambda\partial_i v_i, \quad (1.805)$$
$$= (2\mu + 3\lambda)\partial_i v_i. \quad (1.806)$$

Since in general $\partial_i v_i \neq 0$, Stokes' assumption implies that

$$\lambda = -\frac{2}{3}\mu. \quad (1.807)$$

So, a Newtonian fluid satisfying Stokes' assumption has the following constitutive equation for viscous stress

$$\tau_{ij} = \underbrace{2\mu \underbrace{\left(\partial_{(i}v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij}\right)}_{\text{deviatoric strain rate}}}_{\text{deviatoric viscous stress}}, \quad (1.808)$$

$$\boldsymbol{\tau} = 2\mu\left(\frac{(\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T)}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v})\mathbf{I}\right). \quad (1.809)$$

Note that incompressible flows have $\partial_i v_i = 0$; thus, λ plays no role in determining the viscous stress in such flows. For the fluid that obeys Stokes' assumption, the viscous stress is entirely deviatoric and is induced only by a deviatoric strain rate.

1.5.4.4 Second law restrictions

Recall that in order that the constitutive equation for viscous stress be consistent with second law of thermodynamics, that it is sufficient (but perhaps overly restrictive) to require that

$$\frac{1}{T}\tau_{ij}\partial_{(i}v_{j)} \geq 0. \quad (1.810)$$

Invoking our constitutive equation for viscous stress, and realizing that the absolute temperature $T > 0$, we have then that

$$\Phi = (2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k \delta_{ij})(\partial_{(i}v_{j)}) \geq 0. \quad (1.811)$$

This reduces to the sum of two squares:

$$\Phi = 2\mu\partial_{(i}v_{j)}\partial_{(i}v_{j)} + \lambda\partial_k v_k \partial_i v_i \geq 0. \quad (1.812)$$

We then seek restrictions on μ and λ such that this is true. Obviously requiring $\mu \geq 0$ and $\lambda \geq 0$ guarantees satisfaction of the second law. However, Stokes' assumption of $\lambda = -\frac{2}{3}\mu$ does not meet this criterion, and so we are motivated to check more carefully to see if we actually need to be that restrictive.

1.5.4.4.1 One dimensional systems Let us first check the criterion for a strictly one-dimensional system. For such a system, our second law restriction reduces to

$$2\mu\partial_{(1}v_{1)}\partial_{(1}v_{1)} + \lambda\partial_1 v_1\partial_1 v_1 \geq 0, \tag{1.813}$$

$$(2\mu+\lambda)\partial_1 v_1\partial_1 v_1 \geq 0, \tag{1.814}$$

$$2\mu+\lambda \geq 0, \tag{1.815}$$

$$\lambda \geq -2\mu. \tag{1.816}$$

Obviously if $\mu > 0$ and $\lambda = -\frac{2}{3}\mu$, the entropy inequality is satisfied. We also could satisfy the inequality for negative μ with sufficiently large positive λ.

1.5.4.4.2 Two dimensional systems Extending this to a two dimensional system is more complicated. For such systems, expansion of our second law condition gives

$$2\mu\partial_{(1}v_{1)}\partial_{(1}v_{1)} + 2\mu\partial_{(1}v_{2)}\partial_{(1}v_{2)} + 2\mu\partial_{(2}v_{1)}\partial_{(2}v_{1)} + 2\mu\partial_{(2}v_{2)}\partial_{(2}v_{2)} + \lambda\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)}\right)\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)}\right) \geq 0. \tag{1.817}$$

Taking advantage of symmetry of the deformation tensor, we can say

$$2\mu\partial_{(1}v_{1)}\partial_{(1}v_{1)} + 4\mu\partial_{(1}v_{2)}\partial_{(1}v_{2)} + 2\mu\partial_{(2}v_{2)}\partial_{(2}v_{2)} + \lambda\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)}\right)\left(\partial_{(1}v_{1)} + \partial_{(2}v_{2)}\right) \geq 0. \tag{1.818}$$

Expanding the product and regrouping gives

$$(2\mu+\lambda)\partial_{(1}v_{1)}\partial_{(1}v_{1)} + 4\mu\partial_{(1}v_{2)}\partial_{(1}v_{2)} + (2\mu+\lambda)\partial_{(2}v_{2)}\partial_{(2}v_{2)} + 2\lambda\partial_{(1}v_{1)}\partial_{(2}v_{2)} \geq 0. \tag{1.819}$$

In matrix form, we can write this inequality in the form known from linear algebra as a quadratic form:

$$\Phi = \left(\partial_{(1}v_{1)} \quad \partial_{(2}v_{2)} \quad \partial_{(1}v_{2)} \right) \begin{pmatrix} (2\mu+\lambda) & \lambda & 0 \\ \lambda & (2\mu+\lambda) & 0 \\ 0 & 0 & 4\mu \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(1}v_{2)} \end{pmatrix} \geq 0. \tag{1.820}$$

As we have discussed before, the condition that this hold for all values of the deformation is that the symmetric part of the coefficient matrix have eigenvalues which are greater than or equal to zero. In fact, here the coefficient matrix is purely symmetric. Let us find the eigenvalues κ of the coefficient matrix. The eigenvalues are found by evaluating the following equation

$$\begin{vmatrix} (2\mu+\lambda)-\kappa & \lambda & 0 \\ \lambda & (2\mu+\lambda)-\kappa & 0 \\ 0 & 0 & 4\mu-\kappa \end{vmatrix} = 0. \tag{1.821}$$

We get the characteristic polynomial

$$(4\mu-\kappa)\left((2\mu+\lambda-\kappa)^2 - \lambda^2\right) = 0. \tag{1.822}$$

This has roots

$$\kappa = 4\mu, \tag{1.823}$$
$$\kappa = 2\mu, \tag{1.824}$$
$$\kappa = 2(\mu+\lambda). \tag{1.825}$$

For the two-dimensional system, we see now formally that we must satisfy both

$$\mu \geq 0, \tag{1.826}$$
$$\lambda \geq -\mu. \tag{1.827}$$

This is more restrictive than for the one-dimensional system, but we see that a fluid obeying Stokes' assumption $\lambda = -\frac{2}{3}\mu$ still satisfies this inequality.

1.5.4.4.3 Three dimensional systems For a full three dimensional variation, the entropy inequality $(2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k \delta_{ij})(\partial_{(i}v_{j)}) \geq 0$, when expanded, is equivalent to the following quadratic form

$$\Phi = \begin{pmatrix} \partial_{(1}v_{1)} & \partial_{(2}v_{2)} & \partial_{(3}v_{3)} & \partial_{(1}v_{2)} & \partial_{(2}v_{3)} & \partial_{(3}v_{1)} \end{pmatrix} \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\mu \end{pmatrix} \begin{pmatrix} \partial_{(1}v_{1)} \\ \partial_{(2}v_{2)} \\ \partial_{(3}v_{3)} \\ \partial_{(1}v_{2)} \\ \partial_{(2}v_{3)} \\ \partial_{(3}v_{1)} \end{pmatrix} \geq 0. \tag{1.828}$$

Again this must hold for arbitrary values of the deformation, so we must require that the eigenvalues κ of the interior matrix be greater than or equal to zero to satisfy the entropy inequality. It is easy to show that the six eigenvalues for the interior matrix are

$$\kappa = 2\mu, \tag{1.829}$$
$$\kappa = 2\mu, \tag{1.830}$$
$$\kappa = 4\mu, \tag{1.831}$$
$$\kappa = 4\mu, \tag{1.832}$$
$$\kappa = 4\mu, \tag{1.833}$$
$$\kappa = 3\lambda + 2\mu. \tag{1.834}$$

Two of the eigenvalues are degenerate, but this is not a particular problem. We need now that $\kappa \geq 0$, so the entropy inequality requires that

$$\mu \geq 0, \tag{1.835}$$
$$\lambda \geq -\frac{2}{3}\mu. \tag{1.836}$$

Obviously a fluid which satisfies Stokes' assumption does not violate the entropy inequality, but it does give rise to a minimum level of satisfaction. This does not mean the fluid is isentropic! It simply means one of the six eigenvalues is zero.

Now using standard techniques from linear algebra for quadratic forms, the entropy inequality can, after much effort, be manipulated into the form

$$\Phi = \frac{2}{3}\mu\left((\partial_{(1}v_{1)} - \partial_{(2}v_{2)})^2 + (\partial_{(2}v_{2)} - \partial_{(3}v_{3)})^2 + (\partial_{(3}v_{3)} - \partial_{(1}v_{1)})^2\right) + \left(\lambda + \frac{2}{3}\mu\right)(\partial_{(1}v_{1)} + \partial_{(2}v_{2)} + \partial_{(3}v_{3)})^2 + 4\mu((\partial_{(1}v_{2)})^2 + (\partial_{(2}v_{3)})^2 + (\partial_{(3}v_{1)})^2) \geq 0. \quad (1.837)$$

Obviously, this is a sum of perfect squares, and holds for all values of the strain rate tensor. It can be verified by direct expansion that this term is identical to the strong form of the entropy inequality for viscous stress. It can further be verified by direct expansion that the entropy inequality can also be written more compactly as

$$\Phi = 2\mu \underbrace{\left(\partial_{(i}v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij}\right)}_{\text{deviatoric strain rate}} \underbrace{\left(\partial_{(i}v_{j)} - \frac{1}{3}\partial_m v_m \delta_{ij}\right)}_{\text{deviatoric strain rate}} + \left(\lambda + \frac{2}{3}\mu\right) \underbrace{(\partial_i v_i)(\partial_j v_j)}_{(\text{mean strain rate})^2} \geq 0. \quad (1.838)$$

So, we see that for a Newtonian fluid that the increase in entropy due to viscous dissipation is attributable to two effects: deviatoric strain rate and mean strain rate. The terms involving both are perfect squares, so as long as $\mu \geq 0$ and $\lambda \geq -\frac{2}{3}\mu$, the second law is not violated by viscous effects.

We can also write the strong form of the entropy inequality for a Newtonian fluid $(2\mu\partial_{(i}v_{j)} + \lambda\partial_k v_k \delta_{ij})(\partial_{(i}v_{j)}) \geq 0$, in terms of the principal invariants of strain rate. Leaving out details, which can be verified by direct expansion of all terms, we find the following form

$$\Phi = 2\mu\left(\frac{2}{3}\left(I_{\dot{\epsilon}}^{(1)}\right)^2 - 2I_{\dot{\epsilon}}^{(2)}\right) + \left(\lambda + \frac{2}{3}\mu\right)\left(I_{\dot{\epsilon}}^{(1)}\right)^2 \geq 0. \quad (1.839)$$

Because this is in terms of the invariants, we are assured that it is independent of the orientation of the coordinate system.

It is, however, not obvious that this form is positive semi-definite. We can use the definitions of the invariants of strain rate to rewrite the inequality as

$$\Phi = 2\mu\left(\partial_{(i}v_{j)}\partial_{(j}v_{i)} - \frac{1}{3}(\partial_i v_i)(\partial_j v_j)\right) + \left(\lambda + \frac{2}{3}\mu\right)(\partial_i v_i)(\partial_j v_j) \geq 0. \quad (1.840)$$

In terms of the eigenvalues of the strain rate tensor, κ_1, κ_2, and κ_3, this becomes

$$\Phi = 2\mu\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 - \frac{1}{3}(\kappa_1 + \kappa_2 + \kappa_3)^2\right) + \left(\lambda + \frac{2}{3}\mu\right)(\kappa_1 + \kappa_2 + \kappa_3)^2 \geq 0. \quad (1.841)$$

This then reduces to a positive semi-definite form:

$$\Phi = \frac{2}{3}\mu\left((\kappa_1 - \kappa_2)^2 + (\kappa_1 - \kappa_3)^2 + (\kappa_2 - \kappa_3)^2\right) + \left(\lambda + \frac{2}{3}\mu\right)(\kappa_1 + \kappa_2 + \kappa_3)^2 \geq 0. \quad (1.842)$$

Since the eigenvalues are invariant under rotation, this form is invariant.

We summarize by noting relations between mean and deviatoric stress and strain rates for Newtonian fluids. The influence of each on each has been seen or is easily shown to be as follows:

- A mean strain rate will induce a *time rate of change* in the mean thermodynamic stress via traditional thermodynamic relations[41] and will induce an additional mean viscous stress for fluids that do not obey Stokes' assumption.
- A deviatoric strain rate will not directly induce a mean stress.
- A deviatoric strain rate will directly induce a deviatoric stress.
- A mean strain rate will induce entropy production only for a fluid that does not obey Stokes' assumption.
- A deviatoric strain rate will always induce entropy production in a viscous fluid.

1.5.5 Equations of state

Thermodynamic equations of state provide algebraic relations between variables such as pressure, temperature, energy, and entropy. They do not involve velocity. They are formally valid for materials at rest. As long as the times scales of equilibration of the thermodynamic variables are much faster than the finest time scales of fluid dynamics, it is a valid assumption to use an ordinary equations of state. Such assumptions can be violated in very high speed flows in which vibrational and rotational modes of oscillation become excited. They may also be invalid in highly rarefied flows such as might occur in the upper atmosphere.

Typically, we will require two types of equations, a *thermal equation of state* which gives the pressure as a function of two independent thermodynamic variables, e.g.

$$p = p(\rho, T), \tag{1.843}$$

and a *caloric equation of state* which gives the internal energy as a function of two independent thermodynamic variables, e.g.

$$e = e(\rho, T). \tag{1.844}$$

There are additional conditions regarding internal consistency of the equations of state; that is, just any stray functional forms will not do.

We outline here a method for generating equations of state with internal consistency based on satisfying the entropy inequality. First let us define a new thermodynamic variable, a, the Helmholtz[42] free energy:

$$a = e - Ts. \tag{1.845}$$

[41] e.g. for an isothermal ideal gas $dp/dt = RT(d\rho/dt) = -\rho RT\partial_i v_i$

[42] Hermann von Helmholtz, 1821-1894, Potsdam-born German physicist and philosopher, descendant of William Penn, the founder of Pennsylvania, empiricist and refuter of the notion that scientific conclusions could be drawn from philosophical ideas, graduated from medical school, wrote convincingly on the science and physiology of music, developed theories of vortex motion as well as thermodynamics and electrodynamics.

We can take the material time derivative of Eq. (1.845) to get

$$\frac{da}{dt} = \frac{de}{dt} - T\frac{ds}{dt} - s\frac{dT}{dt}. \tag{1.846}$$

It is shown in thermodynamics texts that there are a set of natural, "canonical," variables for describing a which are T and ρ. That is, we take $a = a(T, \rho)$. Taking the time derivative of this form of a and using the chain rule tells us another form for da/dt:

$$\frac{da}{dt} = \left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} + \left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt}. \tag{1.847}$$

Now we also have the energy equation and entropy inequality:

$$\rho\frac{de}{dt} = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j, \tag{1.848}$$

$$\rho\frac{ds}{dt} \geq -\partial_i \left(\frac{q_i}{T}\right). \tag{1.849}$$

Using Eq. (1.846) to eliminate de/dt in favor of da/dt in the energy equation, Eq. (1.848), gives a modified energy equation:

$$\rho\left(\frac{da}{dt} + T\frac{ds}{dt} + s\frac{dT}{dt}\right) = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j. \tag{1.850}$$

Next, we use Eq. (1.847) to eliminate da/dt in Eq. (1.850) to get

$$\rho\left(\left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} + \left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt} + T\frac{ds}{dt} + s\frac{dT}{dt}\right) = -\partial_i q_i - p\partial_i v_i + \tau_{ij}\partial_i v_j. \tag{1.851}$$

Now in this modified energy equation, we solve for $\rho\ ds/dt$ to get

$$\rho\frac{ds}{dt} = -\frac{1}{T}\partial_i q_i - \frac{p}{T}\partial_i v_i + \frac{1}{T}\tau_{ij}\partial_i v_j - \frac{\rho}{T}\left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} - \frac{\rho}{T}\left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt} - \frac{\rho s}{T}\frac{dT}{dt}. \tag{1.852}$$

Substituting this version of the energy conservation equation into the second law, Eq. (1.849), gives

$$-\frac{1}{T}\partial_i q_i - \frac{p}{T}\partial_i v_i + \frac{1}{T}\tau_{ij}\partial_i v_j - \frac{\rho}{T}\left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} - \frac{\rho}{T}\left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt} - \frac{\rho s}{T}\frac{dT}{dt} \geq -\partial_i \left(\frac{q_i}{T}\right). \tag{1.853}$$

Rearranging and using the mass conservation relation to eliminate $\partial_i v_i$, we get

$$-\frac{q_i}{T^2}\partial_i T - \frac{p}{T}\left(-\frac{1}{\rho}\frac{d\rho}{dt}\right) + \frac{1}{T}\tau_{ij}\partial_i v_j - \frac{\rho}{T}\left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} - \frac{\rho}{T}\left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt} - \frac{\rho s}{T}\frac{dT}{dt} \geq 0, \tag{1.854}$$

$$-\frac{q_i}{T}\partial_i T + \frac{p}{\rho}\frac{d\rho}{dt} + \tau_{ij}\partial_i v_j - \rho\left.\frac{\partial a}{\partial T}\right|_{\rho} \frac{dT}{dt} - \rho\left.\frac{\partial a}{\partial \rho}\right|_{T} \frac{d\rho}{dt} - \rho s\frac{dT}{dt} \geq 0, \tag{1.855}$$

$$-\frac{q_i}{T}\partial_i T + \tau_{ij}\partial_i v_j + \frac{1}{\rho}\frac{d\rho}{dt}\left(p - \rho^2 \left.\frac{\partial a}{\partial \rho}\right|_{T}\right) - \rho\frac{dT}{dt}\left(s + \left.\frac{\partial a}{\partial T}\right|_{\rho}\right) \geq 0. \tag{1.856}$$

Now in our discussion of the strong form of the energy inequality, we have already found forms for q_i and τ_{ij} for which the terms involving these phenomena are positive semi-definite. We can guarantee the remaining two terms are consistent with the second law, and are associated with reversible processes by requiring that

$$p = \rho^2 \left.\frac{\partial a}{\partial \rho}\right|_T, \tag{1.857}$$

$$s = -\left.\frac{\partial a}{\partial T}\right|_\rho. \tag{1.858}$$

For example, if we take the non-obvious, but experimentally defensible choice for a of

$$a = c_v(T - T_o) - c_v T \ln\left(\frac{T}{T_o}\right) + RT\ln\left(\frac{\rho}{\rho_o}\right), \tag{1.859}$$

then we get for pressure

$$p = \rho^2 \left.\frac{\partial a}{\partial \rho}\right|_T = \rho^2\left(\frac{RT}{\rho}\right) = \rho R T. \tag{1.860}$$

The above equation for pressure a thermal equation of state for an *ideal gas*, and R is known as the gas constant. It is the ratio of the universal gas constant and the molecular mass of the particular gas.

Solving for entropy s, we get

$$s = -\left.\frac{\partial a}{\partial T}\right|_\rho = c_v \ln\left(\frac{T}{T_o}\right) - R\ln\left(\frac{\rho}{\rho_o}\right). \tag{1.861}$$

Then, we get for e

$$e = a + Ts = c_v(T - T_o). \tag{1.862}$$

We call the above equation for energy a caloric equation of state for *calorically perfect* gas. It is calorically perfect because the specific heat at constant volume c_v is assumed a true constant here. In general for ideal gases, it can be shown to be at most a function of temperature.

1.6 Boundary and interface conditions

At fluid solid interfaces, it is observed in the continuum regime that the fluid sticks to the solid boundary, so that we can safely take the fluid and solid velocities to be identical at the interface. This is called the *no slip* condition. As one approaches the molecular level, this breaks down.

At the interface of two distinct, immiscible fluids, one requires that stress be continuous across the interface, that the energy flux be continuous across the interface. Density need not be continuous in the absence of mass diffusion. Were mass diffusion present, the fluids would not be immiscible, and density would be a continuous variable. Additionally the effect of surface tension may need to be accounted for. We shall not consider surface tension in this course, but many texts give a complete treatment.

1.7 Complete set of compressible Navier-Stokes equations

Here we pause once more to write a complete set of equations, the compressible Navier [43] -Stokes equations, written here for a fluid which satisfies Stokes' assumption, but for which the viscosity μ (as well as thermal conductivity k) may be variable. They are given in a form similar to that done in an earlier section.

1.7.0.1 Conservative form

1.7.0.1.1 Cartesian index form

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{1.863}$$

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) = \rho f_i - \partial_i p + \partial_j \left(2\mu \left(\partial_{(j} v_{i)} - \frac{1}{3}\partial_k v_k \delta_{ji} \right) \right), \tag{1.864}$$

$$\partial_o \left(\rho \left(e + \frac{1}{2} v_j v_j \right) \right) + \partial_i \left(\rho v_i \left(e + \frac{1}{2} v_j v_j \right) \right) = \rho v_i f_i - \partial_i(p v_i) + \partial_i (k \partial_i T) + \partial_i \left(2\mu \left(\partial_{(i} v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij} \right) v_j \right), \tag{1.865}$$

$$p = p(\rho, T), \tag{1.866}$$

$$e = e(\rho, T), \tag{1.867}$$

$$\mu = \mu(\rho, T), \tag{1.868}$$

$$k = k(\rho, T). \tag{1.869}$$

1.7.0.1.2 Gibbs form

$$\frac{\partial \rho}{\partial t} + \nabla^T \cdot (\rho \mathbf{v}) = 0, \tag{1.870}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \left(\nabla^T \cdot (\rho \mathbf{v}\mathbf{v}^T) \right)^T = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \left(2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v}) \mathbf{I} \right) \right) \right)^T, \tag{1.871}$$

$$\frac{\partial}{\partial t} \left(\rho \left(e + \frac{1}{2}\mathbf{v}^T \cdot \mathbf{v} \right) \right) + \nabla^T \cdot \left(\rho \mathbf{v} \left(e + \frac{1}{2}\mathbf{v}^T \cdot \mathbf{v} \right) \right) = \rho \mathbf{v}^T \cdot \mathbf{f} - \nabla^T \cdot (p\mathbf{v}) + \nabla^T \cdot (k \nabla T)$$

[43] Claude Louis Marie Henri Navier, 1785-1836, Dijon-born French civil engineer and mathematician, studied under Fourier, taught applied mechanics at École des Ponts et Chaussées, replaced Cauchy as professor at École Polytechnique, specialist in road and bridge building, did not fully understand shear stress in a fluid and used faulty logic in arriving at his equations.

$$+\nabla^T \cdot \left(\left(2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v})\mathbf{I} \right) \right) \cdot \mathbf{v} \right), \quad (1.872)$$

$$p = p(\rho, T), \quad (1.873)$$

$$e = e(\rho, T), \quad (1.874)$$

$$\mu = \mu(\rho, T), \quad (1.875)$$

$$k = k(\rho, T). \quad (1.876)$$

1.7.0.2 Non-conservative form

1.7.0.2.1 Cartesian index form

$$\frac{d\rho}{dt} = -\rho \partial_i v_i, \quad (1.877)$$

$$\rho \frac{dv_i}{dt} = \rho f_i - \partial_i p + \partial_j \left(2\mu \left(\partial_{(j} v_{i)} - \frac{1}{3} \partial_k v_k \delta_{ji} \right) \right), \quad (1.878)$$

$$\rho \frac{de}{dt} = -p \partial_i v_i + \partial_i (k \partial_i T) + 2\mu \left(\partial_{(i} v_{j)} - \frac{1}{3} \partial_k v_k \delta_{ij} \right) \partial_i v_j, \quad (1.879)$$

$$p = p(\rho, T), \quad (1.880)$$

$$e = e(\rho, T), \quad (1.881)$$

$$\mu = \mu(\rho, T), \quad (1.882)$$

$$k = k(\rho, T) \quad (1.883)$$

1.7.0.2.2 Gibbs form

$$\frac{d\rho}{dt} = -\rho \nabla^T \cdot \mathbf{v}, \quad (1.884)$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \left(\nabla^T \cdot \left(2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v})^T}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v})\mathbf{I} \right) \right) \right)^T, \quad (1.885)$$

$$\rho \frac{de}{dt} = -p \nabla^T \cdot \mathbf{v} + \nabla^T \cdot (k \nabla T) + 2\mu \left(\frac{\nabla \mathbf{v}^T + (\nabla \mathbf{v}^T)^T}{2} - \frac{1}{3}(\nabla^T \cdot \mathbf{v})\mathbf{I} \right) : \nabla \mathbf{v}^T, \quad (1.886)$$

$$p = p(\rho, T), \quad (1.887)$$

$$e = e(\rho, T), \quad (1.888)$$

$$\mu = \mu(\rho, T), \quad (1.889)$$

$$k = k(\rho, T). \quad (1.890)$$

We take μ, and k to be thermodynamic properties of temperature and density. In practice, both dependencies are often weak, especially the dependency of μ and k on density. We also assume we know the form of the external body force per unit mass f_i. We also no longer formally require the angular momentum principle, as it has been absorbed into our constitutive equation for viscous stress. We also need not write the second law, as we can guarantee its satisfaction as long as $\mu \geq 0, k \geq 0$.

In summary, we have nine unknowns, ρ,v_i (3), p, e, T, μ, and k, and nine equations, mass, linear momenta (3), energy, thermal state, caloric state, and thermodynamic relations for viscosity and thermal conductivity. When coupled with initial, interface, and boundary conditions, all dependent variables can, in principle, be expressed as functions of position x_i and time t, and this knowledge utilized to design devices of practical importance.

1.8 Incompressible Navier-Stokes equations with constant properties

If we make the assumption, which can be justified in the limit when fluid particle velocities are small relative to the velocity of sound waves in the fluid, that density changes following a particle are negligible (that is $\frac{d\rho}{dt} \to 0$), the Navier-Stokes equations simplify considerably. Note that this does not imply the density is constant everywhere in the flow. Our assumption allows for stratified flows, for which the density of individual particles still can remain constant. We shall also assume viscosity μ, and thermal conductivity k are constants, though this is not necessary.

Let us examine the mass, linear momenta, and energy equations in this limit.

1.8.1 Mass

Expanding the mass equation

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{1.891}$$

we get

$$\underbrace{\partial_o \rho + v_i \partial_i \rho}_{\frac{d\rho}{dt} \to 0} + \rho \partial_i v_i = 0. \tag{1.892}$$

We are assuming the first two terms in the above expression, which form $d\rho/dt$, go to zero; hence the mass equation becomes $\rho \partial_i v_i = 0$. Since $\rho > 0$, we can say

$$\partial_i v_i = 0. \tag{1.893}$$

So, for an incompressible fluid, the relative expansion rate for a fluid particle is zero.

1.8.2 Linear momenta

Let us first consider the viscous term:

$$\partial_j \left(2\mu \left(\partial_{(j} v_{i)} - \frac{1}{3} \underbrace{\partial_k v_k}_{=0} \delta_{ij} \right) \right), \tag{1.894}$$

$$\partial_j \left(2\mu \left(\partial_{(j} v_{i)} \right) \right), \tag{1.895}$$

$$\partial_j \left(\mu \left(\partial_i v_j + \partial_j v_i\right)\right), \tag{1.896}$$

$$\text{since } \mu \text{ is constant here} \tag{1.897}$$

$$\mu \left(\partial_j \partial_i v_j + \partial_j \partial_j v_i\right), \tag{1.898}$$

$$\mu \left(\partial_i \underbrace{\partial_j v_j}_{=0} + \partial_j \partial_j v_i\right), \tag{1.899}$$

$$\mu \partial_j \partial_j v_i. \tag{1.900}$$

Everything else in the linear momenta equation is unchanged; hence we get

$$\rho \partial_o v_i + \rho v_j \partial_j v_i = \rho f_i - \partial_i p + \mu \partial_j \partial_j v_i. \tag{1.901}$$

Note that in the incompressible constant viscosity limit, the mass and linear momenta equations form a complete set of four equations in four unknowns: p,v_i. We will see that in this limit the energy equation is coupled to mass, and linear momenta, but it is only a one-way coupling.

1.8.3 Energy

Let us also choose our material to be a liquid, for which the specific heat at constant pressure, c_p is nearly identical to the specific heat at constant volume c_v as long as the ratio $T\alpha_p^2/\kappa_T/\rho/c_p << 1$. Here α_p is the coefficient of isobaric expansion, and κ_T is the coefficient of isothermal compressibility. As long as the liquid is well away from the vaporization point, this is a good assumption for most materials. We will thus take for the liquid $c_p = c_v = c$. For an incompressible gas there are some subtleties to this analysis, involving the low Mach number limit which makes the results not obvious. We will not address that problem in this course; many texts do, but many also shove the problem under the rug! For a compressible gas there are no such problems. For an incompressible liquid whose specific heat is a constant, we have $e = cT + e_o$. The compressible energy equation in full generality is

$$\rho \frac{de}{dt} = -p \partial_i v_i - \partial_i q_i + \tau_{ij} \partial_i v_j. \tag{1.902}$$

Imposing our constitutive equations and assumption of incompressibility onto this, we get

$$\rho \frac{d}{dt}\left(cT + e_o\right) = -p \underbrace{\partial_i v_i}_{=0} - \partial_i(-k\partial_i T) + 2\mu \left(\partial_{(i} v_{j)} - \frac{1}{3} \underbrace{\partial_k v_k}_{=0} \delta_{ij}\right) \partial_i v_j, \tag{1.903}$$

$$\rho c \frac{dT}{dt} = k \partial_i \partial_i T + 2\mu \partial_{(i} v_{j)} \partial_i v_j, \tag{1.904}$$

$$= k \partial_i \partial_i T + 2\mu \underbrace{\partial_{(i} v_{j)}}_{\text{sym.}} \left(\underbrace{\partial_{(i} v_{j)}}_{\text{sym.}} + \underbrace{\partial_{[i} v_{j]}}_{\text{antisym.}}\right), \tag{1.905}$$

$$= k \partial_i \partial_i T + \underbrace{2\mu \partial_{(i} v_{j)} \partial_{(i} v_{j)}}_{\Phi}. \tag{1.906}$$

For incompressible flows with constant properties, the viscous dissipation function Φ reduces to

$$\Phi = 2\mu \partial_{(i}v_{j)}\partial_{(i}v_{j)}. \tag{1.907}$$

It is a scalar function and obviously positive for $\mu > 0$ since it is a tensor inner product of a tensor with itself.

1.8.4 Summary of incompressible constant property equations

The incompressible constant property equations for a liquid are summarized below in Gibbs notation:

$$\nabla^T \cdot \mathbf{v} = 0, \tag{1.908}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \mu \nabla^2 \mathbf{v}, \tag{1.909}$$

$$\rho c \frac{dT}{dt} = k \nabla^2 T + \Phi. \tag{1.910}$$

For an ideal gas, it turns out that we should replace c in the above equation by c_p. The alternative, c_v would seem to be the proper choice, but careful analysis in the limit of low Mach number shows this to be incorrect.

1.8.5 Limits for one-dimensional diffusion

Note for a static fluid ($v_i = 0$), we have $d/dt = \partial/\partial t$ and $\Phi = 0$; hence the energy equation can be written in a familiar form

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T. \tag{1.911}$$

Here $\alpha = k/(\rho c)$ is defined as the thermal diffusivity. For one dimensional cases where all variation is in the x_2 direction, we get

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x_2^2}. \tag{1.912}$$

Compare this to the momentum equation for a very specific form of the velocity field, namely, $v_i(x_i) = v_1(x_2, t)$. When we also have no pressure gradient and no body force, the linear momenta principle reduces to

$$\frac{\partial v_1}{\partial t} = \nu \frac{\partial^2 v_1}{\partial x_2^2}. \tag{1.913}$$

Here $\nu = \mu/\rho$ is the momentum diffusivity. This equation has an identical form to that for one-dimensional energy diffusion. In fact the physical mechanism governing both, random molecular collisions, is the same.

1.9 Dimensionless compressible Navier-Stokes equations

Here we discuss how to scale the Navier-Stokes equations into a set of dimensionless equations. Panton gives a general background for scaling. White's *Viscous Flow* has a detailed discussion of the dimensionless form of the Navier-Stokes equations.

Consider the Navier-Stokes equations for a calorically perfect ideal gas which has Newtonian behavior, satisfies Stokes' assumption, and has constant viscosity, thermal conductivity, and specific heat:

$$\partial_o \rho + \partial_i(\rho v_i) = 0, \tag{1.914}$$

$$\partial_o(\rho v_i) + \partial_j(\rho v_j v_i) = \rho f_i - \partial_i p + \mu \partial_j \left(2 \left(\partial_{(j} v_{i)} - \frac{1}{3} \partial_k v_k \delta_{ji} \right) \right), \tag{1.915}$$

$$\partial_o \left(\rho \left(e + \frac{1}{2} v_j v_j \right) \right) + \partial_i \left(\rho v_i \left(e + \frac{1}{2} v_j v_j \right) \right) = \rho v_i f_i - \partial_i(p v_i) + k \partial_i \partial_i T + \mu \partial_i \left(2 \left(\partial_{(i} v_{j)} - \frac{1}{3} \partial_k v_k \delta_{ij} \right) v_j \right), \tag{1.916}$$

$$p = \rho R T, \tag{1.917}$$

$$e = c_v T + \hat{e}. \tag{1.918}$$

Here R is the gas constant for the particular gas we are considering, which is the ratio of the universal gas constant $\Re$ and the gas's molecular mass $\mathcal{M}$: $R = \Re/\mathcal{M}$. Also $\hat{e}$ is a constant.

Now solutions to the above equations, which may be of the form, for example, of $p(x_1, x_2, x_3, t)$, are necessarily parameterized by the constants from constitutive laws such as c_v, R, μ, k, f_i, in addition to parameters from initial and boundary conditions. That is our solutions will really be of the form

$$p(x_1, x_2, x_3, t; c_v, R, \mu, k, f_i, \ldots). \tag{1.919}$$

It is desirable for many reasons to reduce the number of parametric dependencies of these solutions. Some of these reasons include

- identification of groups of terms that truly govern the features of the flow,
- efficiency of presentation of results, and
- efficiency of design of experiments.

The Navier-Stokes equations (and nearly all sets of physically motivated equations) can be reduced in complexity by considering *scaled* versions of the same equations.

For a given problem, the proper scales are *non-unique*, though some choices will be more helpful than others. One generally uses the following rules of thumb in choosing scales:

- reduce variables so that their scaled value is near unity,

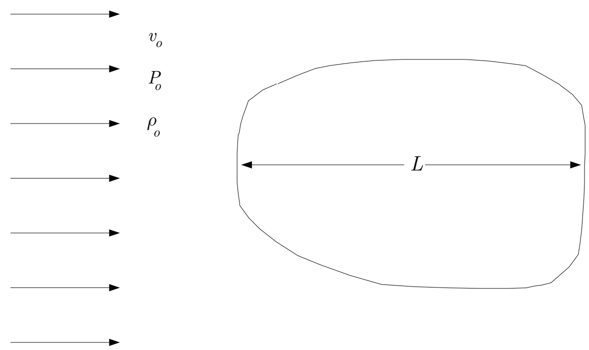

Figure 1.33: Figure of known flow from infinity approaching body with characteristic length L.

- demonstrate that certain physical mechanisms may be negligible relative to other physical mechanisms, and
- simplify initial and boundary conditions.

In forming dimensionless equations, one must usually look for

- characteristic length scale L, and
- characteristic time scale t_c.

Often an ambient velocity or sound speed exists which can be used to form either a length or time scale, for example

- given $v_o, L \longrightarrow t_c = \frac{L}{v_o}$,
- given $v_o, t_c \longrightarrow L = v_o t_c$.

If for example our physical problem involves the flow over a body of length L (and whose other dimensions are of the same order as L), and free-stream conditions are known to be $p = p_o$, $v_i = (v_o, 0, 0)^T$, $\rho = \rho_o$, as sketched in Figure 1.33, Knowledge of free-stream pressure and density fixes all other free-stream thermodynamic variables, e.g. e, T, via the thermodynamic relations. For this problem, let the $*$ subscript represent a dimensionless variable. Define the following scaled dependent variables:

$$\rho_* = \frac{\rho}{\rho_o}, \qquad p_* = \frac{p}{p_o}, \qquad v_{*i} = \frac{v_i}{v_o}, \qquad T_* = \frac{\rho_o R}{p_o} T, \qquad e_* = \frac{\rho_o}{p_o} e. \tag{1.920}$$

Define the following scaled independent variables:

$$x_{*i} = \frac{x_i}{L}, \qquad t_* = \frac{v_o}{L} t. \tag{1.921}$$

With these definitions, the operators must also be scaled, that is,

$$\begin{aligned} \partial_o = \frac{\partial}{\partial t} = \frac{dt_*}{dt}\frac{\partial}{\partial t_*} = \frac{v_o}{L}\frac{\partial}{\partial t_*} = \frac{v_o}{L}\partial_{*o}, \\ \partial_{*o} = \frac{L}{v_o}\partial_o. \\ \partial_i = \frac{\partial}{\partial x_i} = \frac{dx_{*i}}{dx_i}\frac{\partial}{\partial x_{*i}} = \frac{1}{L}\frac{\partial}{\partial x_{*i}} = \frac{1}{L}\partial_{*i}, \\ \partial_{*i} = L\partial_i. \end{aligned} \quad (1.922)$$

1.9.1 Mass

Let us make these substitutions into the mass equation:

$$\partial_o\rho + \partial_i(\rho v_i) = 0, \quad (1.923)$$

$$\frac{v_o}{L}\partial_{*o}(\rho_o\rho_*) + \frac{1}{L}\partial_{*i}(\rho_o\rho_* v_o v_{*i}) = 0, \quad (1.924)$$

$$\frac{\rho_o v_o}{L}\left(\partial_{*o}\rho_* + \partial_{*i}(\rho_* v_{*i})\right) = 0, \quad (1.925)$$

$$\partial_{*o}\rho_* + \partial_{*i}(\rho_* v_{*i}) = 0. \quad (1.926)$$

The mass equation is unchanged in form when we transform to a dimensionless version.

1.9.2 Linear momenta

We have a similar analysis for the linear momenta equation.

$$\begin{aligned} \partial_o(\rho v_i) \\ +\partial_j(\rho v_j v_i) &= \rho f_i - \partial_i p \\ &\quad +\mu\partial_j\left(2\left(\partial_{(j}v_{i)} - \frac{1}{3}\partial_k v_k\delta_{ji}\right)\right), \end{aligned} \quad (1.927)$$

$$\begin{aligned} \frac{v_o}{L}\partial_{*o}(\rho_o v_o\rho_* v_{*i}) \\ +\frac{1}{L}\partial_{*j}(\rho_o\rho_* v_o v_{*j} v_o v_{*i}) &= \rho_o\rho_* f_i - \frac{1}{L}\partial_{*i}(p_o p_*) \\ &\quad +\frac{\mu}{L}\partial_{*j}\left(\frac{2}{L}\left(\partial_{(*j}v_o v_{*i)} - \frac{1}{3L}\partial_{*k}v_o v_{*k}\delta_{ji}\right)\right), \end{aligned} \quad (1.928)$$

$$\begin{aligned} \frac{\rho_o v_o^2}{L}\partial_{*o}(\rho_* v_{*i}) \\ +\frac{\rho_o v_o^2}{L}\partial_{*j}(\rho_* v_{*j} v_{*i}) &= \rho_o\rho_* f_i - \frac{p_o}{L}\partial_{*i}(p_*) \end{aligned}$$

$$+\frac{\mu v_o}{L^2}\partial_{*j}\left(2\left(\partial_{(*j}v_{*i)}-\frac{1}{3}\partial_{*k}v_{*k}\delta_{ji}\right)\right), \tag{1.929}$$

$$\partial_{*o}(\rho_* v_{*i})+\partial_{*j}(\rho_* v_{*j}v_{*i}) = \frac{f_i L}{v_o^2}\rho_* - \frac{p_o}{\rho_o v_o^2}\partial_{*i}(p_*)$$
$$+\frac{2\mu}{\rho_o v_o L}\partial_{*j}\left(\partial_{(*j}v_{*i)}-\frac{1}{3}\partial_{*k}v_{*k}\delta_{ji}\right). \tag{1.930}$$

With this scaling, we have generated three distinct dimensionless groups of terms which drive the linear momenta equation:

$$\frac{f_i L}{v_o^2}, \qquad \frac{p_o}{\rho_o v_o^2}, \qquad \text{and} \qquad \frac{\mu}{\rho_o v_o L}. \tag{1.931}$$

These groups are closely related to the following groups of terms, which have the associated interpretations indicated:

- *Froude number Fr*:[44] With the body force per unit mass $f_i = g\hat{g}_i$, where $g > 0$ is the gravitational acceleration magnitude and $\hat{g}_i$ is a unit vector pointing in the direction of gravitational acceleration,

$$Fr^2 \equiv \frac{v_o^2}{gL} = \frac{\text{flow kinetic energy}}{\text{gravitational potential energy}}. \tag{1.932}$$

- *Mach number M_o*:[45] With the Mach number M_o defined as the ratio of the ambient velocity to the ambient sound speed, and recalling that for a calorically perfect ideal gas that the square of the ambient sound speed, a_o^2 is $a_o^2 = \gamma\frac{p_o}{\rho_o}$, where γ is the ratio of specific heats $\gamma = \frac{c_p}{c_v} = (1 + R/c_v)$, we have

$$M_o^2 \equiv \frac{v_o^2}{a_o^2} = \frac{v_o^2}{\gamma\frac{p_o}{\rho_o}} = \frac{\rho_o v_o^2}{\gamma p_o} = \frac{v_o^2}{\gamma R T_o} = \frac{\text{flow kinetic energy}}{\text{thermal energy}}. \tag{1.933}$$

  Here we have taken $T_o = p_o/\rho_o/R$.

- *Reynolds number Re*: We have

$$Re \equiv \frac{\rho_o v_o L}{\mu} = \frac{\rho_o v_o^2}{\mu\frac{v_o}{L}} = \frac{\text{dynamic pressure}}{\text{viscous stress}}. \tag{1.934}$$

[44] William Froude, 1810-1879, English engineer and naval architect, Oxford educated.

[45] Ernst Mach, 1838-1926, Viennese physicist and philosopher who worked in optics, mechanics, and wave dynamics, received doctorate at University of Vienna and taught mathematics at University of Graz and physics at Charles University of Prague, developed fundamental ideas of inertia which influenced Einstein.

With these definitions, we get

$$\partial_{*o}(\rho_* v_{*i}) + \partial_{*j}(\rho_* v_{*j} v_{*i}) = \frac{1}{Fr^2}\hat{g}_i \rho_* - \frac{1}{\gamma}\frac{1}{M_o^2}\partial_{*i}(p_*) + \frac{2}{Re}\partial_{*j}\left(\partial_{(*j} v_{*i)} - \frac{1}{3}\partial_{*k} v_{*k}\delta_{ji}\right). \quad (1.935)$$

The relative magnitudes of Fr, M_o, and Re play a crucial role in determining which physical mechanisms are most influential in changing the fluid's linear momenta.

1.9.3 Energy

The analysis is of the exact same form, but more tedious, for the energy equation.

$$\partial_o\left(\rho\left(e + \frac{1}{2}v_j v_j\right)\right) + \partial_i\left(\rho v_i\left(e + \frac{1}{2}v_j v_j\right)\right) = k\partial_i\partial_i T - \partial_i(p v_i) + \mu\partial_i\left(2\left(\partial_{(i} v_{j)} - \frac{1}{3}\partial_k v_k \delta_{ij}\right) v_j\right) + \rho v_i f_i, \quad (1.936)$$

$$\frac{v_o}{L}\partial_{*o}\left(\rho_o\rho_*\left(\frac{p_o}{\rho_o}e_* + \frac{1}{2}v_o^2 v_{*j}v_{*j}\right)\right) + \frac{1}{L}\partial_{i*}\left(\rho_o\rho_* v_o v_{i*}\left(\frac{p_o}{\rho_o}e_* + \frac{1}{2}v_o^2 v_{*j}v_{*j}\right)\right) = \frac{k}{L^2}\partial_{*i}\partial_{*i}\frac{p_o}{\rho_o R}T_* - \frac{1}{L}\partial_{*i}(p_o p_* v_o v_{*i}) + \frac{\mu}{L}\partial_{*i}\left(\frac{2}{L}\left(\partial_{(*i} v_o v_{*j)} - \frac{1}{3}\partial_{*k} v_o v_{*k}\delta_{ij}\right) v_o v_{*j}\right) + \rho_o\rho_* v_o v_{*i} f_i, \quad (1.937)$$

$$\frac{\rho_o v_o}{L}\frac{p_o}{\rho_o}\partial_{*o}\left(\rho_*\left(e_* + \frac{1}{2}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}}v_{*j}v_{*j}\right)\right) + \frac{\rho_o v_o}{L}\frac{p_o}{\rho_o}\partial_{*i}\left(\rho_* v_{*i}\left(e_* + \frac{1}{2}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}}v_{*j}v_{*j}\right)\right) = \frac{k}{L^2}\frac{p_o}{\rho_o}\frac{1}{R}\partial_{*i}\partial_{*i}T_* - \frac{p_o v_o}{L}\partial_{*i}(p_* v_{*i}) + \frac{2\mu v_o^2}{L^2}\partial_{*i}\left(\left(\partial_{(*i} v_{*j)} - \frac{1}{3}\partial_{*k} v_{*k}\delta_{ij}\right) v_{*j}\right) + \rho_o v_o f_i \rho_* v_{*i}, \quad (1.938)$$

$$\partial_{*o}\left(\rho_*\left(e_* + \frac{1}{2}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}}v_{*j}v_{*j}\right)\right)$$

$$
\begin{aligned}
+\partial_{*i}\left(\rho_* v_{*i}\left(e_* + \frac{1}{2}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}} v_{*j}v_{*j}\right)\right) \quad = \quad & \frac{k}{LR\rho_o v_o}\partial_{*i}\partial_{*i}T_* \\
& -\partial_{*i}(p_* v_{*i}) \\
& +\frac{2\mu v_o^2}{L^2}\frac{L}{\rho_o v_o}\frac{1}{\frac{p_o}{\rho_o}}\partial_{*i}\left(\left(\partial_{(*i}v_{*j)} - \frac{1}{3}\partial_{*k}v_{*k}\delta_{ij}\right)v_{*j}\right) \\
& +\frac{f_i L}{\frac{p_o}{\rho_o}}\rho_* v_{*i}. \qquad (1.939)
\end{aligned}
$$

Now examining the dimensionless groups, we see that

$$
\frac{k}{LR\rho_o v_o} = \frac{k}{c_p}\frac{c_p}{R}\frac{1}{L\rho_o v_o} = \frac{k}{\mu c_p}\frac{c_p}{c_p - c_v}\frac{\mu}{\rho_o v_o L} = \frac{1}{Pr}\frac{\gamma}{\gamma - 1}\frac{1}{Re}. \qquad (1.940)
$$

Here we have a new dimensionless group, the Prandtl[46] number, Pr, where

$$
Pr \equiv \frac{\mu c_p}{k} = \frac{\frac{\mu}{\rho_o}}{\frac{k}{\rho_o c_p}} = \frac{\text{momentum diffusivity}}{\text{energy diffusivity}} = \frac{\nu}{\alpha}. \qquad (1.941)
$$

We also see that

$$
\frac{f_i L}{\frac{p_o}{\rho_o}} = \frac{\gamma g L \hat{g}_i}{\gamma\frac{p_o}{\rho_o}} = \frac{v_o^2}{\gamma\frac{p_o}{\rho_o}}\gamma\frac{gL}{v_o^2}\hat{g}_i = \gamma\frac{M_o^2}{Fr^2}\hat{g}_i, \qquad (1.942)
$$

$$
\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}} = \gamma M_o^2, \qquad (1.943)
$$

$$
\frac{2\mu v_o^2}{L^2}\frac{L}{\rho_o v_o}\frac{1}{\frac{p_o}{\rho_o}} = \frac{2\mu}{\rho_o v_o L}\frac{\gamma v_o^2}{\gamma\frac{p_o}{\rho_o}} = 2\frac{1}{Re}\gamma M_o^2. \qquad (1.944)
$$

So, the dimensionless energy equation becomes

$$
\begin{aligned}
& \partial_{*o}\left(\rho_*\left(e_* + \frac{1}{2}\gamma M_o^2 v_{*j}v_{*j}\right)\right) \\
& +\partial_{*i}\left(\rho_* v_{*i}\left(e_* + \frac{1}{2}\gamma M_o^2 v_{*j}v_{*j}\right)\right) \quad = \quad \frac{\gamma}{\gamma - 1}\frac{1}{Pr}\frac{1}{Re}\partial_{*i}\partial_{*i}T_* \\
& \qquad -\partial_{*i}(p_* v_{*i}) \\
& \qquad +2\gamma\frac{M_o^2}{Re}\partial_{*i}\left(\left(\partial_{(*i}v_{*j)} - \frac{1}{3}\partial_{*k}v_{*k}\delta_{ij}\right)v_{*j}\right) \\
& \qquad +\frac{\gamma M_o^2}{Fr^2}\hat{g}_i\rho_* v_{*i}. \qquad (1.945)
\end{aligned}
$$

[46]Ludwig Prandtl, 1875-1953, German mechanician and father of aerodynamics, primarily worked at University of Göttingen, discoverer of the boundary layer, pioneer of dirigibles, and advocate of monoplanes.

1.9.4 Thermal state equation

$$p_o p_* = \rho_o \rho_* R \left(\frac{p_o}{\rho_o R} \right) T_*, \tag{1.946}$$

$$p_* = \rho_* T_*. \tag{1.947}$$

1.9.5 Caloric state equation

$$\frac{p_o}{\rho_o} e_* = c_v \left(\frac{p_o}{\rho_o R} \right) T_* + \hat{e}, \tag{1.948}$$

$$e_* = \frac{c_v}{R} T_* + \frac{\rho_o \hat{e}}{p_o}, \tag{1.949}$$

$$e_* = \frac{1}{\gamma - 1} T_* + \underbrace{\frac{\rho_o \hat{e}}{p_o}}_{\text{unimportant}} \tag{1.950}$$

For completeness, we retain the term $\frac{\rho_o \hat{e}}{p_o}$. It actually plays no role in this non-reactive flow since energy only enters via its derivatives. When flows with chemical reactions are modeled, this term may be important.

1.9.6 Upstream conditions

Scaling the upstream conditions, we get

$$p_* = 1, \qquad \rho_* = 1, \qquad v_{*i} = (1, 0, 0)^T. \tag{1.951}$$

With this we then get secondary relationships

$$T_* = 1, \qquad e_* = \frac{1}{\gamma - 1} + \frac{\rho_o \hat{e}}{p_o}. \tag{1.952}$$

1.9.7 Reduction in parameters

We lastly note that our original system had the following ten independent parameters:

$$\rho_o,\ p_o,\ c_v,\ R,\ L,\ v_o,\ \mu,\ k,\ f_i,\ \hat{e}. \tag{1.953}$$

Our scaled system however has only *six* independent parameters:

$$Re,\ Pr,\ M_o,\ Fr,\ \gamma,\ \frac{\rho_o \hat{e}}{p_o}. \tag{1.954}$$

Note we have lost no information, nor made any approximations, and we have a system with fewer dependencies.

1.10 First integrals of linear momentum

Under special circumstances, we can integrate the linear momentum principle to obtain a simplified equation. We will consider two cases here, what is known as Bernoulli's[47] equation and Crocco's[48] equation. In a later chapter on rotational flows, we will also consider the Helmholtz equation and Kelvin's theorem, which are also first integrals in special cases.

1.10.1 Bernoulli's equation

What we commonly call Bernoulli's equation is really a first integral of the linear momenta principle. Under different assumptions, we can get different flavors of Bernoulli's equation. A first integral of the linear momenta principle exists under the following conditions:

- viscous stresses are negligible relative to other terms, $\tau_{ij} \sim 0$,
- the fluid is barotropic, $p = p(\rho)$ or $\rho = \rho(p)$.
- body forces are conservative, so we can write $f_i = -\partial_i \hat{\phi}$, where $\hat{\phi}$ is a known potential function, and
- either
  - the flow is irrotational, $\omega_k = \epsilon_{kij}\partial_i v_j = 0$, or
  - the flow is steady, $\partial_o = 0$.

First consider a version of the general linear momenta equation in non-conservative form, Eq. (1.480) scaled by ρ:

$$\partial_o v_i + v_j \partial_j v_i = -\frac{1}{\rho}\partial_i p + f_i + \frac{1}{\rho}\partial_j \tau_{ji}. \tag{1.955}$$

Now use our vector identity, Eq. (1.178), to rewrite the advective term, and impose our assumptions above to arrive at

$$\partial_o v_i + \partial_i \left(\frac{1}{2} v_j v_j\right) - \epsilon_{ijk} v_j \omega_k = -\frac{1}{\rho}\partial_i p - \partial_i \hat{\phi}. \tag{1.956}$$

Now let us define, just for this particular analysis, a new function Υ. We will take Υ to be a function of pressure p, and thus implicitly, a function of x_i and t. For the barotropic fluid, we define Υ as

$$\Upsilon(p(x_i, t)) \equiv \int_{p_o}^{p(x_i,t)} \frac{d\hat{p}}{\rho(\hat{p})}. \tag{1.957}$$

[47]Daniel Bernoulli, 1700-1782, Dutch-born Swiss mathematician of the prolific and mathematical Bernoulli family, son of Johann Bernoulli, studied at Heidelberg, Strasbourg, and Basel, receiving M.D. degree, served in St. Petersburg and lectured at the University of Basel, put forth his fluid mechanical principle in the 1738 *Hydrodynamica*, in competition with his father's 1738 *Hydraulica*.

[48]Luigi Crocco, 1909-1986, Sicilian-born, Italian applied mathematician and theoretical aerodynamicist and rocket engineer, taught at University of Rome, Princeton, and Paris.

Note that in the special case of incompressible flow that $\Upsilon = p/\rho$. Recalling Leibniz's rule,

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\ dx = \int_{x=a(t)}^{x=b(t)} \frac{\partial f}{\partial t}\ dx + \frac{db}{dt}f(b(t),t) - \frac{da}{dt}f(a(t),t), \tag{1.958}$$

we let $\partial/\partial x_i$ play the role of d/dt to get

$$\frac{\partial}{\partial x_i}\Upsilon = \frac{\partial}{\partial x_i}\int_{p_o}^{p(x_i,t)} \frac{d\hat{p}}{\rho(\hat{p})} = \frac{1}{\rho(p(x_i,t))}\frac{\partial p}{\partial x_i} - \frac{1}{\rho(p_o)}\underbrace{\frac{\partial p_o}{\partial x_i}}_{=0} + \int_{p_o}^{p(x_i,t)} \underbrace{\frac{\partial}{\partial x_i}\left(\frac{1}{\rho(\hat{p})}\right)}_{=0} d\hat{p}. \tag{1.959}$$

As p_o is constant, and the integrand has no *explicit* dependency on x_i, we get

$$\frac{\partial}{\partial x_i}\Upsilon = \frac{1}{\rho(p(x_i,t))}\frac{\partial p}{\partial x_i}. \tag{1.960}$$

So, our linear momenta principle reduces to

$$\partial_o v_i + \partial_i\left(\frac{1}{2}v_j v_j\right) - \epsilon_{ijk}v_j\omega_k = -\partial_i\Upsilon - \partial_i\hat{\phi}. \tag{1.961}$$

Consider now some special cases:

1.10.1.1 Irrotational case

If the fluid is irrotational, we have $\omega_k = \epsilon_{klm}\partial_l v_m = 0$. Consequently, we can write the velocity vector as the gradient of a potential function ϕ, known as the *velocity potential*:

$$\partial_m\phi = v_m. \tag{1.962}$$

Note that if the velocity takes this form, then the vorticity is

$$\omega_k = \epsilon_{klm}\partial_l\partial_m\phi. \tag{1.963}$$

Since ϵ_{klm} is anti-symmetric and $\partial_l\partial_m$ is symmetric, their tensor inner product must be zero; hence, such a flow is irrotational: $\omega_k = \epsilon_{klm}\partial_l\partial_m\phi = 0$. So, the linear momenta principle, Eq. (1.961), reduces to

$$\partial_o\partial_i\phi + \partial_i\left(\frac{1}{2}(\partial_j\phi)(\partial_j\phi)\right) = -\partial_i\Upsilon - \partial_i\hat{\phi}, \tag{1.964}$$

$$\partial_i\left(\partial_o\phi + \frac{1}{2}(\partial_j\phi)(\partial_j\phi) + \Upsilon + \hat{\phi}\right) = 0, \tag{1.965}$$

$$\partial_o\phi + \frac{1}{2}(\partial_j\phi)(\partial_j\phi) + \Upsilon + \hat{\phi} = f(t). \tag{1.966}$$

Here $f(t)$ is an arbitrary function of time, which can be chosen to match conditions in a given problem.

1.10.1.2 Steady case

1.10.1.2.1 Streamline integration Here we take $\partial_o = 0$, but $\omega_k \neq 0$. Rearranging the steady version of the linear momenta equation, Eq. (1.961), we get

$$\partial_i \left(\frac{1}{2} v_j v_j\right) + \partial_i \Upsilon + \partial_i \hat{\phi} = \epsilon_{ijk} v_j \omega_k, \tag{1.967}$$

$$\partial_i \left(\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi}\right) = \epsilon_{ijk} v_j \omega_k. \tag{1.968}$$

Taking the inner product of both sides with v_i, we get

$$v_i \partial_i \left(\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi}\right) = v_i \epsilon_{ijk} v_j \omega_k, \tag{1.969}$$

$$= \underbrace{\epsilon_{ijk} v_i v_j}_{=0} \omega_k, \tag{1.970}$$

$$= 0. \tag{1.971}$$

The term on the right hand side is zero because it is the tensor inner product of a symmetric and anti-symmetric tensor.

For a local coordinate system which has component s aligned with the velocity vector v_i, and the other two directions n, and b, mutually orthogonal, we have $v_i = (v_s, 0, 0)^T$. Our linear momenta principle then reduces to

$$(v_s, 0, 0) \begin{pmatrix} \partial_s [] \\ \partial_n [] \\ \partial_b [] \end{pmatrix} = 0. \tag{1.972}$$

Forming this dot product yields

$$v_s \frac{\partial}{\partial s} \left(\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi}\right) = 0. \tag{1.973}$$

For $v_s \neq 0$, we get that

$$\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi} = C(n, b). \tag{1.974}$$

On a particular streamline, the function $C(n, b)$ will be a constant.

1.10.1.2.2 Lamb surfaces We can extend the idea of integration along a streamline to describe what are known as *Lamb surfaces*[49] by again considering the steady, inviscid linear momentum principle with conservative body forces, Eq. (1.968):

$$\partial_i \left(\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi}\right) = \epsilon_{ijk} v_j \omega_k. \tag{1.975}$$

[49]Sir Horace Lamb, 1849-1934, English fluid mechanician, first studied at Owens College Manchester followed by mathematics at Cambridge, taught at Adelaide, Australia, then returned to the University of Manchester, prolific writer of textbooks.

Now taking the quantity $\mathcal{B}$ to be

$$\mathcal{B} \equiv \frac{1}{2} v_j v_j + \Upsilon + \hat{\phi}, \tag{1.976}$$

the linear momentum principle, Eq. (1.968), becomes

$$\partial_i \mathcal{B} = \epsilon_{ijk} v_j \omega_k \tag{1.977}$$

Now the vector $\epsilon_{ijk} v_j \omega_k$ is orthogonal to both velocity v_j and vorticity ω_k because of the nature of the cross product. Also the vector $\partial_i \mathcal{B}$ is orthogonal to a surface on which $\mathcal{B}$ is constant. Consequently, the surface on which $\mathcal{B}$ is constant must be tangent to both the velocity and vorticity vectors. Surfaces of constant $\mathcal{B}$ thus are composed of families of streamlines on which the Bernoulli constant has the same value. In addition they contain families of vortex lines. These are the Lamb surfaces of the flow, named after Sir Horace Lamb, the British fluid mechanician of the late 19th and early 20th century.

1.10.1.3 Irrotational, steady, incompressible case

In this case, we recover the form most commonly used (and misused) of Bernoulli's equation, namely,

$$\frac{1}{2} v_j v_j + \Upsilon + \hat{\phi} = C. \tag{1.978}$$

The constant is truly constant throughout the flow field. With $\Upsilon = p/\rho$ here and $\hat{\phi} = g_z z$ (with $g_z > 0$, and rising z corresponding to rising distance from the earth's surface, we get $\mathbf{f} = -\nabla \hat{\phi} = -g_z \mathbf{k}$) for a constant gravitational field, and v the magnitude of the velocity vector, we get

$$\frac{1}{2} v^2 + \frac{p}{\rho} + g_z z = C. \tag{1.979}$$

1.10.2 Crocco's theorem

It is common, especially in texts on compressible flow, to present what is known as *Crocco's theorem*. The many different versions presented in many standard texts are non-uniform and often of unclear validity. Its utility is confined mainly to providing an alternative way of expressing the linear momentum principle which provides some insight into the factors which influence fluid motion. In special cases, it can be integrated to form a more useful relationship, similar to Bernoulli's equation, between fundamental fluid variables. The heredity of this theorem is not always clear, though, as we shall see it is nothing more than a combination of the linear momentum principle coupled with some definitions from thermodynamics. Its derivation is often confined to inviscid flows. Here we will first present a result valid for general viscous flows for the evolution of stagnation enthalpy, which is closely related to Crocco's theorem. Next we will show how one of the restrictions can be relaxed so as to obtain what we call the extended Crocco's theorem. We then show how this reduces to a form which is similar to a form presented in many texts.

1.10.2.1 Stagnation enthalpy variation

First again consider the general linear momenta equation, Eq. (1.955):

$$\partial_o v_i + v_j \partial_j v_i = -\frac{1}{\rho}\partial_i p + f_i + \frac{1}{\rho}\partial_j \tau_{ji}. \tag{1.980}$$

Now, as before in the development of Bernoulli's equation, use our vector identity, Eq. (1.178), to rewrite the advective term, but retain the viscous terms to get

$$\partial_o v_i + \partial_i \left(\frac{1}{2} v_j v_j\right) - \epsilon_{ijk} v_j \omega_k = -\frac{1}{\rho}\partial_i p + f_i + \frac{1}{\rho}\partial_j \tau_{ji}. \tag{1.981}$$

Taking the dot product with v_i, and rearranging, we get

$$\partial_o \left(\frac{1}{2} v_i v_i\right) + v_i \partial_i \left(\frac{1}{2} v_j v_j\right) = \underbrace{\epsilon_{ijk} v_i v_j}_{=0} \omega_k - \frac{1}{\rho} v_i \partial_i p + v_i f_i + \frac{1}{\rho} v_i \partial_j \tau_{ji}. \tag{1.982}$$

Again, since ϵ_{ijk} is anti-symmetric and $v_i v_j$ is symmetric, their tensor inner product is zero, so we get

$$\partial_o \left(\frac{1}{2} v_i v_i\right) + v_i \partial_i \left(\frac{1}{2} v_j v_j\right) = -\frac{1}{\rho} v_i \partial_i p + v_i f_i + \frac{1}{\rho} v_i \partial_j \tau_{ji}. \tag{1.983}$$

Now recall the Gibbs relation from thermodynamics, Eq. (1.541):

$$T\ ds = de - \frac{p}{\rho^2}\ d\rho. \tag{1.984}$$

Also recall the definition of enthalpy h, Eq. (1.531):

$$h = e + \frac{p}{\rho}. \tag{1.985}$$

Differentiating the equation for enthalpy, we recover Eq. (1.534):

$$dh = de + \frac{1}{\rho}\ dp - \frac{p}{\rho^2}\ d\rho. \tag{1.986}$$

Eliminating de in favor of dh in the Gibbs equation gives

$$T\ ds = dh - \frac{1}{\rho}\ dp. \tag{1.987}$$

If we choose to apply this relation to the motion following a fluid particle, we can say then that

$$T\frac{ds}{dt} = \frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt}. \tag{1.988}$$

Expanding, we get

$$T(\partial_o s + v_i \partial_i s) = \partial_o h + v_i \partial_i h - \frac{1}{\rho}(\partial_o p + v_i \partial_i p). \tag{1.989}$$

Rearranging, we get

$$T(\partial_o s + v_i \partial_i s) - (\partial_o h + v_i \partial_i h) + \frac{1}{\rho}\partial_o p = -\frac{1}{\rho} v_i \partial_i p. \tag{1.990}$$

We then use the above identity to eliminate the pressure gradient term from the linear momentum equation in favor of enthalpy, entropy, and unsteady pressure terms:

$$\partial_o \left(\frac{1}{2} v_i v_i\right) + v_i \partial_i \left(\frac{1}{2} v_j v_j\right) = T(\partial_o s + v_i \partial_i s) - (\partial_o h + v_i \partial_i h) + \frac{1}{\rho}\partial_o p + v_i f_i + \frac{1}{\rho} v_i \partial_j \tau_{ji}. \tag{1.991}$$

Rearranging slightly, noting that $v_i v_i = v_j v_j$, and assuming the body force is conservative so that $f_i = -\partial_i \hat{\phi}$, we get

$$\partial_o \left(h + \frac{1}{2} v_j v_j + \hat{\phi}\right) + v_i \partial_i \left(h + \frac{1}{2} v_j v_j + \hat{\phi}\right) = T\left(\partial_o s + v_i \partial_i s\right) + \frac{1}{\rho}\partial_o p + \frac{1}{\rho} v_i \partial_j \tau_{ji}. \tag{1.992}$$

Note that here we have made the common assumption that the body force potential $\hat{\phi}$ is independent of time, which allows us to absorb it within the time derivative. If we define, as is common, the total enthalpy h_o as

$$h_o = h + \frac{1}{2} v_j v_j + \hat{\phi}, \tag{1.993}$$

we can then state

$$\partial_o h_o + v_i \partial_i h_o = T\left(\partial_o s + v_i \partial_i s\right) + \frac{1}{\rho}\partial_o p + \frac{1}{\rho} v_i \partial_j \tau_{ji}, \tag{1.994}$$

$$\frac{dh_o}{dt} = T\frac{ds}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}\mathbf{v}^T \cdot \left(\nabla^T \cdot \boldsymbol{\tau}\right)^T \tag{1.995}$$

We can use the first law of thermodynamics written in terms of entropy, Eq. (1.545), $\rho(ds/dt) = -(1/T)\partial_i q_i + (1/T)\tau_{ij}\partial_i v_j$, to eliminate the entropy derivative in favor of those terms which generate entropy to arrive at

$$\rho\frac{dh_o}{dt} = \partial_i(\tau_{ij} v_j - q_i) + \partial_o p. \tag{1.996}$$

Thus, we see that the total enthalpy of a fluid particle is influenced by energy and momentum diffusion as well as an unsteady pressure field.

1.10.2.2 Extended Crocco's theorem

With a slight modification of the preceding analysis, we can arrive at the *extended Crocco's theorem*. Begin once more with an earlier version of the linear momenta principle:

$$\partial_o v_i + \partial_i \left(\frac{1}{2} v_j v_j \right) - \epsilon_{ijk} v_j \omega_k = -\frac{1}{\rho} \partial_i p + f_i + \frac{1}{\rho} \partial_j \tau_{ji}. \tag{1.997}$$

Now assume we have a functional representation of enthalpy in the form

$$h = h(s, p). \tag{1.998}$$

Then we get

$$dh = \left. \frac{\partial h}{\partial s} \right|_p ds + \left. \frac{\partial h}{\partial p} \right|_s dp. \tag{1.999}$$

We also thus deduce from the Gibbs relation $dh = T ds + (1/\rho) dp$ that

$$\left. \frac{\partial h}{\partial s} \right|_p = T, \qquad \left. \frac{\partial h}{\partial p} \right|_s = \frac{1}{\rho}. \tag{1.1000}$$

Now, since we have $h = h(s, p)$, we can take its derivative with respect to each and all of the coordinate directions to obtain

$$\frac{\partial h}{\partial x_i} = \left. \frac{\partial h}{\partial s} \right|_p \frac{\partial s}{\partial x_i} + \left. \frac{\partial h}{\partial p} \right|_s \frac{\partial p}{\partial x_i}. \tag{1.1001}$$

or

$$\partial_i h = \left. \frac{\partial h}{\partial s} \right|_p \partial_i s + \left. \frac{\partial h}{\partial p} \right|_s \partial_i p. \tag{1.1002}$$

Substituting known values for the thermodynamic derivatives, we get

$$\partial_i h = T \partial_i s + \frac{1}{\rho} \partial_i p. \tag{1.1003}$$

We can use this to eliminate directly the pressure gradient term from the linear momentum equation to obtain then

$$\partial_o v_i + \partial_i \left(\frac{1}{2} v_j v_j \right) - \epsilon_{ijk} v_j \omega_k = T \partial_i s - \partial_i h + f_i + \frac{1}{\rho} \partial_j \tau_{ji}. \tag{1.1004}$$

Rearranging slightly, and again assuming the body force is conservative so that $f_i = -\partial_i \hat{\phi}$, we get the extended Crocco's theorem:

$$\partial_o v_i + \partial_i \left(h + \frac{1}{2} v_j v_j + \hat{\phi} \right) = T \partial_i s + \epsilon_{ijk} v_j \omega_k + \frac{1}{\rho} \partial_j \tau_{ji}. \tag{1.1005}$$

Again, employing the total enthalpy, $h_o = h + \frac{1}{2} v_j v_j + \hat{\phi}$, we write the extended Crocco's theorem as

$$\partial_o v_i + \partial_i h_o = T \partial_i s + \epsilon_{ijk} v_j \omega_k + \frac{1}{\rho} \partial_j \tau_{ji}. \tag{1.1006}$$

1.10.2.3 Traditional Crocco's theorem

For a steady, inviscid flow, the extended Crocco's theorem reduces to what is usually called Crocco's theorem:

$$\partial_i h_o = T\partial_i s + \epsilon_{ijk} v_j \omega_k, \quad (1.1007)$$

$$\nabla h_o = T\nabla s + \mathbf{v} \times \boldsymbol{\omega}. \quad (1.1008)$$

If the flow is further required to be homeoentropic, we get

$$\partial_i h_o = \epsilon_{ijk} v_j \omega_k. \quad (1.1009)$$

Similar to Lamb surfaces, we find that surfaces on which h_o is constant are parallel to both the velocity and vorticity vector fields. Taking the dot product with v_i, we get

$$v_i \partial_i h_o = v_i \epsilon_{ijk} v_j \omega_k, \quad (1.1010)$$

$$= \epsilon_{ijk} v_i v_j \omega_k, \quad (1.1011)$$

$$= 0. \quad (1.1012)$$

Integrating this along a streamline, as for Bernoulli's equation, we find

$$h_o = C(n, b), \quad (1.1013)$$

$$h + \frac{1}{2} v_j v_j + \hat{\phi} = C(n, b), \quad (1.1014)$$

so we see that the stagnation enthalpy is constant along a streamline and varies from streamline to streamline. If the flow is steady, homeoentropic, and irrotational, the total enthalpy will be constant throughout the flow-field:

$$h + \frac{1}{2} v_j v_j + \hat{\phi} = C. \quad (1.1015)$$

Chapter 2

Vortex dynamics

see Panton, Chapter 13,
see Yih, Chapter 2.

In this chapter we will consider in detail the kinematics and dynamics of rotating fluids, sometimes called vortex dynamics. The two most common quantities which are used to characterize rotating fluids are

- the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, and
- the circulation $\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{r}$.

Both will be important in this chapter.

Although it is entirely possible to use Cartesian index notation to describe a rotating fluid, some of the ideas are better conveyed in a non-Cartesian system, such as the cylindrical coordinate system. For that reason, and for the sake of giving the student more experience with the other common notation, the Gibbs notation will often be used in the chapter.

2.1 Transformations to cylindrical coordinates

The rotation of a fluid about an axis induces an acceleration in that a fluid particle's velocity vector is certainly changing with respect to time. Such a motion is most easily described with a set of cylindrical coordinates. The transformation and inverse transformation to and from cylindrical $(r, \theta, \hat{z})$ coordinates to Cartesian (x, y, z) is given by the familiar

$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2}, \tag{2.1}$$

$$y = r\sin\theta, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right), \tag{2.2}$$

$$z = \hat{z}, \qquad \hat{z} = z. \tag{2.3}$$

Most of the basic distinctions between the two systems can be understood by considering two-dimensional geometries. The representation of an arbitrary point in both two-dimensional

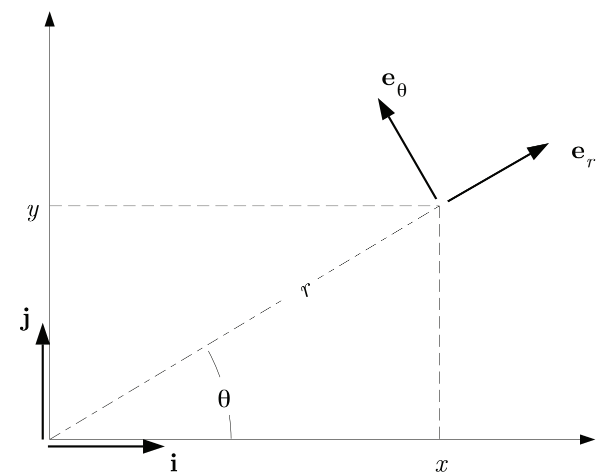

Figure 2.1: Representation of a point in Cartesian and cylindrical coordinates along with unit vectors for both systems.

(x, y) Cartesian and two-dimensional (r, θ) cylindrical coordinate systems along with the unit basis vectors for both systems, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{e}_r$, $\mathbf{e}_\theta$, is sketched in Figure 2.1,

2.1.1 Centripetal and Coriolis acceleration

The fact that a point in motion is accompanied by changes in the basis vectors with respect to time in the cylindrical representation, but not for Cartesian basis vectors, accounts for the most striking differences in the formulations of the governing equations, namely the appearance of

- centripetal acceleration, and
- Coriolis[1] acceleration

in the cylindrical representation.

Consider the representations of the velocity vector $\mathbf{v}$ in both coordinate systems:

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j}, \qquad \text{or} \tag{2.4}$$

$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta. \tag{2.5}$$

Now the unsteady (as opposed to the convective) part of the acceleration vector of a particle is simply the partial derivative of the velocity vector with respect to time. Now formally, we must allow for variations of the unit basis vectors as well as the components themselves so

[1]Gaspard Gustave de Coriolis, 1792-1843, Paris-born mathematician, taught with Navier, introduced the terms "work" and "kinetic energy" with modern scientific meaning, wrote on the mathematical theory of billiards.

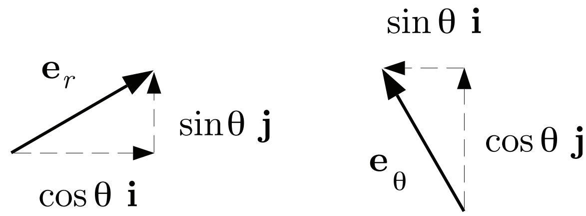

Figure 2.2: Geometrical representation of cylindrical unit vectors in terms of Cartesian unit vectors.

that

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial u}{\partial t}\mathbf{i} + u\underbrace{\frac{\partial \mathbf{i}}{\partial t}}_{=0} + \frac{\partial v}{\partial t}\mathbf{j} + v\underbrace{\frac{\partial \mathbf{j}}{\partial t}}_{=0}, \tag{2.6}$$

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial v_r}{\partial t}\mathbf{e}_r + v_r\frac{\partial \mathbf{e}_r}{\partial t} + \frac{\partial v_\theta}{\partial t}\mathbf{e}_\theta + v_\theta\frac{\partial \mathbf{e}_\theta}{\partial t}. \tag{2.7}$$

Now the time derivatives of the Cartesian basis vectors is zero, as they are defined not to change with the position of the particle. Hence for a Cartesian representation, we have for the unsteady component of acceleration the familiar:

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial u}{\partial t}\mathbf{i} + \frac{\partial v}{\partial t}\mathbf{j}. \tag{2.8}$$

However the time derivative of the cylindrical basis vectors does change with time for particles in motion! To see this, let us first relate $\mathbf{e}_r$ and $\mathbf{e}_\theta$ to $\mathbf{i}$ and $\mathbf{j}$. From the sketch of Figure 2.2, it is clear that

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \tag{2.9}$$

$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}. \tag{2.10}$$

This is a linear system of equations. We can use Cramer's rule to invert to find

$$\mathbf{i} = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta, \tag{2.11}$$

$$\mathbf{j} = \sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta. \tag{2.12}$$

Now, examining time derivatives of the unit vectors, we see that

$$\frac{\partial \mathbf{e}_r}{\partial t} = -\sin\theta\frac{\partial\theta}{\partial t}\mathbf{i} + \cos\theta\frac{\partial\theta}{\partial t}\mathbf{j}, \tag{2.13}$$

$$= \frac{\partial\theta}{\partial t}\mathbf{e}_\theta, \tag{2.14}$$

and

$$\frac{\partial \mathbf{e}_\theta}{\partial t} = -\cos\theta\frac{\partial\theta}{\partial t}\mathbf{i} - \sin\theta\frac{\partial\theta}{\partial t}\mathbf{j}, \tag{2.15}$$

$$= -\frac{\partial\theta}{\partial t}\mathbf{e}_r. \tag{2.16}$$

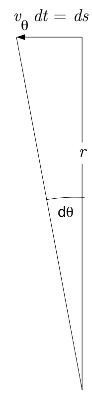

Figure 2.3: Sketch of relation of differential distance ds to velocity in angular direction v_θ.

so there is a formal variation of the unit vectors with respect to time as long as the angular velocity $\frac{\partial \theta}{\partial t} \neq 0$. So the acceleration vector is

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial v_r}{\partial t}\mathbf{e}_r + v_r \frac{\partial \theta}{\partial t}\mathbf{e}_\theta + \frac{\partial v_\theta}{\partial t}\mathbf{e}_\theta - v_\theta \frac{\partial \theta}{\partial t}\mathbf{e}_r, \tag{2.17}$$

$$= \left(\frac{\partial v_r}{\partial t} - v_\theta \frac{\partial \theta}{\partial t}\right)\mathbf{e}_r + \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial \theta}{\partial t}\right)\mathbf{e}_\theta. \tag{2.18}$$

Now from basic geometry, as sketched in Figure 2.3, we have

$$ds = r\ d\theta, \tag{2.19}$$

$$v_\theta dt = r\ d\theta, \tag{2.20}$$

$$\frac{v_\theta}{r} = \frac{\partial \theta}{\partial t}. \tag{2.21}$$

Consequently, we can write the unsteady component of acceleration as

$$\frac{\partial \mathbf{v}}{\partial t} = \left(\frac{\partial v_r}{\partial t} - \underbrace{\frac{v_\theta^2}{r}}_{\text{centripetal}}\right)\mathbf{e}_r + \left(\frac{\partial v_\theta}{\partial t} + \underbrace{\frac{v_r v_\theta}{r}}_{\text{Coriolis}}\right)\mathbf{e}_\theta. \tag{2.22}$$

Two, apparently *new*, accelerations have appeared as a consequence of the transformation: centripetal acceleration, $\frac{v_\theta^2}{r}$, directed towards the center, and Coriolis acceleration, $\frac{v_r v_\theta}{r}$, directed in the direction of increasing θ. These terms do not have explicit dependency on time derivatives of velocity. And yet when the equations are constructed in this coordinate system, they represent real accelerations, and are consequences of forces. As can be seen by considering the general theory of non-orthogonal coordinate transformations, terms like the centripetal and Coriolis acceleration are associated with the Christoffel symbols of the transformation.

Such terms perhaps contributed to the development of Einstein's theory of general relativity as well. Refusing to accept that our typical expression of a body force, mg, was fundamental, Einstein instead postulated that it was a term which was a relic of a coordinate transformation. He held that we in fact exist in a more complex geometry than classically considered. He constructed his theory of general relativity such that no gravitational force exists, but when coordinate transformations are employed to give us a classical view of the non-relativistic universe, the term mg appears in much the same way as centripetal and Coriolis accelerations appear when we transform to cylindrical coordinates.

2.1.2 Grad and div for cylindrical systems

We can use the chain rule to develop expressions for grad and div in cylindrical coordinate systems. Consider the Cartesian

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}. \tag{2.23}$$

The chain rule gives us

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \hat{z}}{\partial x}\frac{\partial}{\partial \hat{z}}, \tag{2.24}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \hat{z}}{\partial y}\frac{\partial}{\partial \hat{z}}, \tag{2.25}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \hat{z}}{\partial z}\frac{\partial}{\partial \hat{z}}, \tag{2.26}$$

$$\tag{2.27}$$

Now, we have

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta, \tag{2.28}$$

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta, \tag{2.29}$$

$$\frac{\partial r}{\partial z} = 0, \tag{2.30}$$

and

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}, \tag{2.31}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}, \tag{2.32}$$

$$\frac{\partial \theta}{\partial z} = 0, \tag{2.33}$$

and

$$\frac{\partial \hat{z}}{\partial x} = 0, \tag{2.34}$$
$$\frac{\partial \hat{z}}{\partial y} = 0, \tag{2.35}$$
$$\frac{\partial \hat{z}}{\partial z} = 1, \tag{2.36}$$

so

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \tag{2.37}$$
$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}, \tag{2.38}$$
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \hat{z}}. \tag{2.39}$$

2.1.2.1 Grad

So now we are prepared to write an explicit form for ∇ in cylindrical coordinates:

$$\begin{aligned}\nabla &= \underbrace{\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)}_{\frac{\partial}{\partial x}}\underbrace{(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta)}_{\mathbf{i}} \\ &\quad + \underbrace{\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)}_{\frac{\partial}{\partial y}}\underbrace{(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta)}_{\mathbf{j}} + \frac{\partial}{\partial \hat{z}}\mathbf{e}_{\hat{z}},\end{aligned} \tag{2.40}$$

$$\begin{aligned}\nabla &= \left(\left(\cos^2\theta + \sin^2\theta\right)\frac{\partial}{\partial r} + \left(-\frac{\sin\theta\cos\theta}{r} + \frac{\sin\theta\cos\theta}{r}\right)\frac{\partial}{\partial \theta}\right)\mathbf{e}_r \\ &\quad + \left((-\sin\theta\cos\theta + \sin\theta\cos\theta)\frac{\partial}{\partial r} + \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\frac{\partial}{\partial \theta}\right)\mathbf{e}_\theta \\ &\quad + \frac{\partial}{\partial \hat{z}}\mathbf{e}_{\hat{z}},\end{aligned} \tag{2.41}$$

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{\partial}{\partial \hat{z}}\mathbf{e}_{\hat{z}}. \tag{2.42}$$

We can now write a simple expression for the convective component, $\mathbf{v}^T \cdot \nabla$, of the acceleration vector:

$$\mathbf{v}^T \cdot \nabla = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_{\hat{z}}\frac{\partial}{\partial \hat{z}}. \tag{2.43}$$

2.1.2.2 Div

The divergence is straightforward. In Cartesian coordinates we have

$$\nabla^T \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{2.44}$$

In cylindrical, we replace derivatives with respect to x, y, z with those with respect to $r, \theta, \hat{z}$, so

$$\nabla^T \cdot \mathbf{v} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} + \sin\theta \frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial \hat{z}}. \tag{2.45}$$

Now u, v and w transform in the same way as x, y, and z, so

$$u = v_r \cos\theta - v_\theta \sin\theta, \tag{2.46}$$

$$v = v_r \sin\theta + v_\theta \cos\theta, \tag{2.47}$$

$$w = v_{\hat{z}}. \tag{2.48}$$

Substituting and taking partials, we find that

$$\begin{aligned} \nabla^T \cdot \mathbf{v} = {} & \cos\theta \left(\cos\theta \frac{\partial v_r}{\partial r} - \underbrace{\sin\theta \frac{\partial v_\theta}{\partial r}}_{\text{A}} \right) - \frac{\sin\theta}{r} \left(\underbrace{\cos\theta \frac{\partial v_r}{\partial \theta}}_{\text{B}} - \sin\theta v_r - \sin\theta \frac{\partial v_\theta}{\partial \theta} - \underbrace{\cos\theta v_\theta}_{\text{C}} \right) \\ & + \sin\theta \left(\sin\theta \frac{\partial v_r}{\partial r} + \underbrace{\cos\theta \frac{\partial v_\theta}{\partial r}}_{\text{A}} \right) + \frac{\cos\theta}{r} \left(\underbrace{\sin\theta \frac{\partial v_r}{\partial \theta}}_{\text{B}} + \cos\theta v_r + \cos\theta \frac{\partial v_\theta}{\partial \theta} - \underbrace{\sin\theta v_\theta}_{\text{C}} \right) \\ & + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}. \end{aligned} \tag{2.49}$$

When expanded, the terms labeled A, B, and C cancel in the above expression. Then using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we arrive at the simple form

$$\nabla^T \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}, \tag{2.50}$$

which is often rewritten as

$$\nabla^T \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}. \tag{2.51}$$

Using the same procedure, we can show that the Laplacian operator transforms to

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \hat{z}^2}. \tag{2.52}$$

2.1.3 Incompressible Navier-Stokes equations in cylindrical coordinates

Leaving out some additional details of the transformations, we find that the incompressible Navier-Stokes equations for a Newtonian fluid with constant viscosity and body force confined to the $-\hat{z}$ direction are

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_{\hat{z}}}{\partial \hat{z}}, \tag{2.53}$$

$$\left(\frac{\partial v_r}{\partial t} - \frac{v_\theta^2}{r}\right) + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_{\hat{z}}\frac{\partial v_r}{\partial \hat{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r - \frac{v_r^2}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right), \tag{2.54}$$

$$\left(\frac{\partial v_\theta}{\partial t} + \frac{v_r v_\theta}{r}\right) + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_{\hat{z}}\frac{\partial v_\theta}{\partial \hat{z}} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right), \tag{2.55}$$

$$\frac{\partial v_{\hat{z}}}{\partial t} + v_r\frac{\partial v_{\hat{z}}}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_{\hat{z}}}{\partial \theta} + v_{\hat{z}}\frac{\partial v_{\hat{z}}}{\partial \hat{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial \hat{z}} + \nu\nabla^2 v_{\hat{z}} - g_{\hat{z}}. \tag{2.56}$$

Note that in the acceleration terms, strictly unsteady terms, convective terms as well as centripetal and Coriolis terms appear. Also note that the viscous terms have additional complications that we have not considered in detail but arise because we must transform $\nabla^2\mathbf{v}$, and there are many non-intuitive terms which arise here when expanded in full.

2.2 Ideal rotational vortex

Let us consider the kinematics and dynamics of an ideal rotational vortex, which we define to be a fluid rotating as a solid body. Let us assume incompressible flow, so $\nabla^T \cdot \mathbf{v} = 0$, assume a simple velocity field, and ask what forces could have given rise to that velocity field. We will simply use z for the azimuthal coordinate instead of $\hat{z}$ here. Take

$$v_r = 0, \qquad v_\theta = \frac{\omega r}{2}, \qquad v_z = 0. \tag{2.57}$$

The kinematics of this flow are simple and sketched in Figure 2.4, Here ω is now defined as a constant. The velocity is zero at the origin and grows in amplitude with linear distance from the origin. The flow is steady, and the streamlines are circles centered about the origin. Obviously, as $r \to \infty$, the theory of relativity would suggest that such a flow would break down as the velocity approached the speed of light. In fact, one would find as well that as the velocities approached the sound speed that compressibility effects would become important far before relativistic effects.

Whatever the case, does this assumed velocity field satisfy incompressible mass conservation?

$$\frac{1}{r}\frac{\partial}{\partial r}(r(0)) + \frac{1}{r}\underbrace{\frac{\partial}{\partial \theta}\left(\frac{\omega r}{2}\right)}_{=0} + \frac{\partial}{\partial z}(0) \overbrace{=}^{?} 0. \tag{2.58}$$

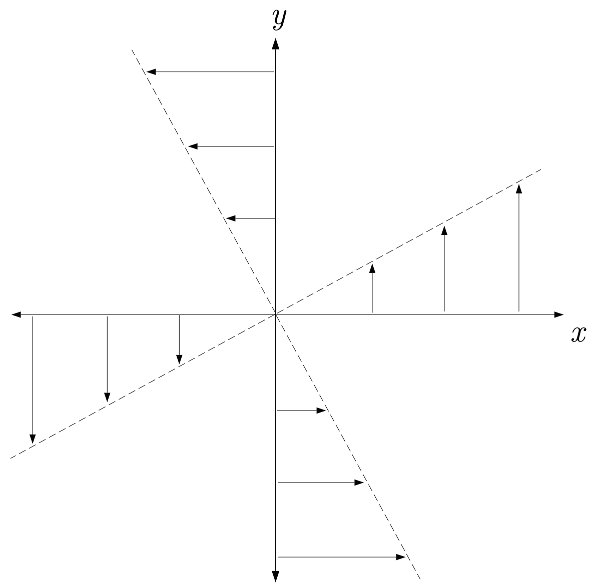

Figure 2.4: Sketch of a fluid rotating as a pure solid body.

Obviously it does.

Next let us consider the acceleration of an element of fluid and the forces which could give rise to that acceleration. First consider the material derivative for this flow

$$\frac{d}{dt} = \underbrace{\frac{\partial}{\partial t}}_{=0} + \underbrace{v_r}_{=0}\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + \underbrace{v_z}_{=0}\frac{\partial}{\partial z} = \frac{v_\theta}{r}\frac{\partial}{\partial \theta}. \tag{2.59}$$

But the only non-zero component of velocity, v_θ, has no dependency on θ, so the material derivative of velocity $\frac{d\mathbf{v}}{dt} = 0$.

Consider now the viscous terms for this flow. We recall for an incompressible Newtonian fluid that

$$\begin{aligned}
\tau_{ij} &= 2\mu\partial_{(i}v_{j)} + \lambda\underbrace{\partial_k v_k}_{=0}\delta_{ij}, && (2.60)\\
&= \mu\left(\partial_i v_j + \partial_j v_i\right), && (2.61)\\
\partial_j\tau_{ij} &= \mu\left(\partial_j\partial_i v_j + \partial_j\partial_j v_i\right), && (2.62)\\
&= \mu\left(\partial_i\underbrace{\partial_j v_j}_{=0} + \partial_j\partial_j v_i\right), && (2.63)\\
&= \mu\nabla^2\mathbf{v} && (2.64)
\end{aligned}$$

We also note that

$$\nabla\times\boldsymbol{\omega} = \epsilon_{ijk}\partial_j\omega_k = \epsilon_{ijk}\partial_j\epsilon_{kmn}\partial_m v_n, \tag{2.65}$$

$$
\begin{aligned}
&= \epsilon_{kij}\epsilon_{kmn}\partial_j\partial_m v_n, && (2.66)\\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\,\partial_j\partial_m v_n, && (2.67)\\
&= \partial_j\partial_i v_j - \partial_j\partial_j v_i, && (2.68)\\
&= \partial_i \underbrace{\partial_j v_j}_{=0} - \partial_j\partial_j v_i, && (2.69)\\
&= -\partial_j\partial_j v_i. && (2.70)
\end{aligned}
$$

Comparing, we see that for this incompressible flow,

$$\left(\nabla^T \cdot \boldsymbol{\tau}^T\right)^T = -\mu(\nabla \times \boldsymbol{\omega}). \tag{2.71}$$

Now, using relations that can be developed for the curl in cylindrical coordinates, we have for this flow that

$$
\begin{aligned}
\omega_r &= \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = 0, && (2.72)\\
\omega_\theta &= \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0, && (2.73)\\
\omega_z &= \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}, && (2.74)\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\omega r}{2}\right), && (2.75)\\
&= \omega. && (2.76)
\end{aligned}
$$

So the flow has a constant rotation rate, ω. Since it is constant, its curl is zero, and we have for this flow that $\left(\nabla^T \cdot \boldsymbol{\tau}^T\right)^T = 0$. We could just as well show for this flow that $\boldsymbol{\tau} = 0$. That is because the kinematics are those of pure rotation as a solid body with no deformation. No deformation implies no viscous stress.

Hence, the three linear momenta equations in the cylindrical coordinate system reduce to the following:

$$
\begin{aligned}
-\frac{v_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}, && (2.77)\\
0 &= -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}, && (2.78)\\
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_z. && (2.79)
\end{aligned}
$$

The r momentum equation strikes a balance between centripetal inertia and radial pressure gradients. The θ momentum equation shows that as there is no acceleration in this direction, there can be no net pressure force to induce it. The z momentum equation enforces a balance between pressure forces and gravitational body forces.

If we take $p = p(r, \theta, z)$ and $p(r_o, \theta, z_o) = p_o$, then

$$\begin{aligned} dp &= \frac{\partial p}{\partial r}\, dr + \frac{\partial p}{\partial \theta}\, d\theta + \frac{\partial p}{\partial z}\, dz, && (2.80) \\ &= \frac{\rho v_\theta^2}{r}\, dr + 0\ d\theta - \rho g_z\, dz, && (2.81) \\ &= \frac{\rho \omega^2 r^2}{4r}\, dr - \rho g_z\, dz, && (2.82) \\ &= \frac{\rho \omega^2 r}{4}\, dr - \rho g_z\, dz, && (2.83) \\ p - p_o &= \frac{\rho\omega^2}{8}(r^2 - r_o^2) - \rho g_z (z - z_o), && (2.84) \\ p(r, z) &= p_o + \frac{\rho\omega^2}{8}(r^2 - r_o^2) - \rho g_z (z - z_o). && (2.85) \end{aligned}$$

Now on a surface of constant pressure we have $p(r, z) = \hat{p}$. So

$$\begin{aligned} \hat{p} &= p_o + \frac{\rho\omega^2}{8}(r^2 - r_o^2) - \rho g_z (z - z_o), && (2.86) \\ \rho g_z (z - z_o) &= p_o - \hat{p} + \frac{\rho\omega^2}{8}(r^2 - r_o^2), && (2.87) \\ z &= z_o + \frac{p_o - \hat{p}}{\rho g_z} + \frac{\omega^2}{8 g_z}(r^2 - r_o^2). && (2.88) \end{aligned}$$

So a surface of constant pressure is a parabola in r with a minimum at $r = 0$. This is consistent with what one observes upon spinning a bucket of water.

Now let's rearrange our general equation for the pressure field and eliminate ω using $v_\theta = \frac{\omega r}{2}$ and defining $v_{\theta o} = \frac{\omega r_o}{2}$:

$$p - \frac{1}{2}\rho v_\theta^2 + \rho g_z z = p_o - \frac{1}{2}\rho v_{\theta o}^2 + \rho g_z z_o = C. \qquad (2.89)$$

This looks very similar to the steady *irrotational* incompressible Bernoulli equation in which $p + \frac{1}{2}\rho v^2 + \rho g_z z = K$. But there is a difference in the sign on one of the terms. Now add ρv_θ^2 to both sides of the equation to get

$$p + \frac{1}{2}\rho v_\theta^2 + \rho g_z z = C + \rho v_\theta^2. \qquad (2.90)$$

Now since $v_\theta = \frac{\omega r}{2}, v_r = 0$, we have lines of constant r as streamlines, and v_θ is constant on those streamlines, so that we get

$$p + \frac{1}{2}\rho v_\theta^2 + \rho g_z z = C', \qquad \text{on a streamline.} \qquad (2.91)$$

Here C' varies from streamline to streamline.

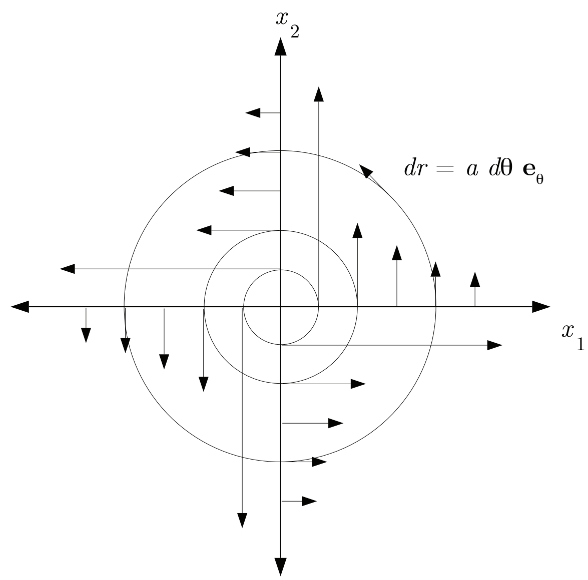

Figure 2.5: Sketch of an ideal irrotational point vortex.

We lastly note that the circulation for this system depends on position. If we choose our contour integral to be a circle of radius a about the origin we find

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{r}, \tag{2.92}$$

$$= \int_0^{2\pi} \left(\frac{1}{2}\omega a\right)(a\ d\theta), \tag{2.93}$$

$$= \pi a^2 \omega. \tag{2.94}$$

2.3 Ideal irrotational vortex

Now let us perform a similar analysis for the following velocity field:

$$v_r = 0, \qquad v_\theta = \frac{\Gamma_o}{2\pi r}, \qquad v_z = 0. \tag{2.95}$$

The kinematics of this flow are also simple and sketched in Figure 2.5, We see once again that the streamlines are circles about the origin. But here, as opposed to the ideal rotational vortex, $v_\theta \to 0$ as $r \to \infty$ and $v_\theta \to \infty$ as $r \to 0$. The vorticity vector of this flow is

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = 0, \tag{2.96}$$

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0, \tag{2.97}$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}, \tag{2.98}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\Gamma_o}{2\pi r}\right), \tag{2.99}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\Gamma_o}{2\pi}\right) = 0! \tag{2.100}$$

This flow field, which seems the epitome of a rotating flow, is formally irrotational as it has zero vorticity. What is happening is that a fluid element not at the origin is actually undergoing severe deformation as it rotates about the origin; however, it does not rotate about its own center of mass. Therefore, the vorticity vector is zero, except at the origin, where it is undefined.

The circulation for this flow about a circle of radius a is

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{r}, \tag{2.101}$$

$$= \int_0^{2\pi} v_\theta (a\ d\theta), \tag{2.102}$$

$$= \int_0^{2\pi} \frac{\Gamma_o}{2\pi a} a\ d\theta, \tag{2.103}$$

$$= \Gamma_o. \tag{2.104}$$

So the circulation is independent of the radius of the closed contour. In fact it can be shown that as long as the closed contour includes the origin in its interior that any closed contour will have this same circulation. We call Γ_o the ideal irrotational vortex strength, in that it is proportional to the magnitude of the velocity at any radius.

Let us once again consider the forces which could induce the motion of this vortex if the flow happens to be incompressible with constant properties and in a potential field where the gravitational body force per unit mass is $-g_z\mathbf{k}$. Recall again that $\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T = -\mu(\nabla \times \boldsymbol{\omega})$, and that since $\boldsymbol{\omega} = 0$ that $\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T = 0$ for this flow. Note also that because there is deformation here, that $\boldsymbol{\tau}$ itself is not zero, its divergence is. For example, if we consider one component of viscous stress $\tau_{r\theta}$ and use standard relations which can be derived for incompressible Newtonian fluids, we find that

$$\tau_{r\theta} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) = \mu r\frac{\partial}{\partial r}\left(\frac{\Gamma_o}{2\pi r^2}\right) = -\frac{\mu\Gamma_o}{\pi r^2}. \tag{2.105}$$

The equations of motion reduce to the same ones as for the ideal rotational vortex:

$$-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.106}$$

$$0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{2.107}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_z. \tag{2.108}$$

Once more we can deduce a pressure field which is consistent with these and the same set of conditions at $r = r_o$, $z = z_o$, with $p = p_o$:

$$
\begin{aligned}
dp &= \frac{\partial p}{\partial r}dr + \underbrace{\frac{\partial p}{\partial \theta}}_{=0} d\theta + \frac{\partial p}{\partial z}\, dz, && (2.109)\\
&= \frac{\rho v_\theta^2}{r}dr - \rho g_z\, dz, && (2.110)\\
&= \frac{\rho \Gamma_o^2}{4\pi^2}\frac{dr}{r^3} - \rho g_z\, dz, && (2.111)\\
p - p_o &= -\frac{\rho\Gamma_o^2}{8\pi^2}\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right) - \rho g_z(z - z_o), && (2.112)\\
p + \frac{\rho\Gamma_o^2}{8\pi^2}\frac{1}{r^2} + \rho g_z z &= p_o + \frac{\rho\Gamma_o^2}{8\pi^2}\frac{1}{r_o^2} + \rho g_z z_o, && (2.113)\\
p + \frac{1}{2}\rho v_\theta^2 + \rho g_z z &= p_o + \frac{1}{2}\rho v_{\theta o}^2 + \rho g_z z_o = C && (2.114)\\
& && (2.115)
\end{aligned}
$$

This is once again Bernoulli's equation. Here it is for an irrotational flow field that is also time-independent, so the Bernoulli constant C is truly constant for the entire flow field and not just along a streamline.

On isobars we have $p = \hat{p}$ which gives us

$$
\begin{aligned}
\hat{p} - p_o &= -\frac{\rho\Gamma_o^2}{8\pi^2}\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right) - \rho g_z(z - z_o), && (2.116)\\
z &= z_o + \frac{p_o - \hat{p}}{\rho g_z} + \frac{\Gamma_o^2}{8\pi^2 g_z}\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right) && (2.117)
\end{aligned}
$$

Note that the pressure goes to negative infinity at the origin. One can show that actual forces, obtained by integrating pressure over area, are in fact bounded.

2.4 Helmholtz vorticity transport equation

Here we will take the curl of the linear momenta principle to obtain a relationship, the Helmholtz vorticity transport equation, which shows how the vorticity field evolves in a general fluid.

2.4.1 General development

First, we recall some useful vector identities:

$$
(\mathbf{v}^T \cdot \nabla)\mathbf{v} = \nabla\left(\frac{\mathbf{v}^T\cdot\mathbf{v}}{2}\right) + \boldsymbol{\omega}\times\mathbf{v}, \qquad (2.118)
$$

$$\begin{aligned} \nabla \times (\mathbf{a} \times \mathbf{b}) &= (\mathbf{b}^T \cdot \nabla)\mathbf{a} - (\mathbf{a}^T \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla^T \cdot \mathbf{b}) - \mathbf{b}(\nabla^T \cdot \mathbf{a}), && (2.119)\\ \nabla \times (\nabla \phi) &= 0, && (2.120)\\ \nabla^T \cdot (\nabla \times \mathbf{v}) &= \nabla^T \cdot \boldsymbol{\omega} = 0. && (2.121) \end{aligned}$$

The first is Eq. (1.178); the others are easily proved.

We start now with the linear momenta principle for a general fluid:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}^T \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T. \quad (2.122)$$

We expand the term $(\mathbf{v}^T \cdot \nabla)\mathbf{v}$ and then apply the curl operator to both sides to get

$$\nabla \times \left(\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^T \cdot \mathbf{v}}{2}\right) + \boldsymbol{\omega} \times \mathbf{v}\right) = \nabla \times \left(\mathbf{f} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T\right). \quad (2.123)$$

This becomes, via the linearity of the various operators,

$$\frac{\partial}{\partial t}(\underbrace{\nabla \times \mathbf{v}}_{\boldsymbol{\omega}}) + \underbrace{\nabla \times \left(\nabla\left(\frac{\mathbf{v}^T \cdot \mathbf{v}}{2}\right)\right)}_{=0} + \nabla \times \boldsymbol{\omega} \times \mathbf{v} = \nabla \times \mathbf{f} - \nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times \left(\frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T\right). \quad (2.124)$$

Using our vector identity for the term with two cross products we get

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}^T \cdot \nabla)\boldsymbol{\omega}}_{=\frac{d\boldsymbol{\omega}}{dt}} - (\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v} + \boldsymbol{\omega}(\underbrace{\nabla^T \cdot \mathbf{v}}_{=-\frac{1}{\rho}\frac{d\rho}{dt}}) - \mathbf{v}(\underbrace{\nabla^T \cdot \boldsymbol{\omega}}_{=0}) = \nabla \times \mathbf{f} - \nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times \left(\frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T\right). \quad (2.125)$$

Rearranging, we have

$$\begin{aligned} \frac{d\boldsymbol{\omega}}{dt} - \frac{\boldsymbol{\omega}}{\rho}\frac{d\rho}{dt} &= (\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v} + \nabla \times \mathbf{f} - \nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times \left(\frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T\right), && (2.126)\\ \frac{1}{\rho}\frac{d\boldsymbol{\omega}}{dt} - \frac{\boldsymbol{\omega}}{\rho^2}\frac{d\rho}{dt} &= \left(\frac{\boldsymbol{\omega}^T}{\rho} \cdot \nabla\right)\mathbf{v} + \frac{1}{\rho}\nabla \times \mathbf{f} - \frac{1}{\rho}\nabla \times \left(\frac{1}{\rho}\nabla p\right) + \frac{1}{\rho}\nabla \times \left(\frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T\right), && (2.127)\\ \frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) &= \left(\frac{\boldsymbol{\omega}^T}{\rho} \cdot \nabla\right)\mathbf{v} + \frac{1}{\rho}\nabla \times \mathbf{f} - \frac{1}{\rho}\nabla \times \left(\frac{1}{\rho}\nabla p\right) + \frac{1}{\rho}\nabla \times \left(\frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T\right), && (2.128) \end{aligned}$$

Now consider the term $\nabla \times \left(\frac{1}{\rho}\nabla p\right)$. In Einstein notation, we have

$$\begin{aligned} \epsilon_{ijk}\partial_j\left(\frac{1}{\rho}\partial_k p\right) &= \epsilon_{ijk}\left(\frac{1}{\rho}\partial_j\partial_k p - \frac{1}{\rho^2}(\partial_j \rho)(\partial_k p)\right), && (2.129)\\ &= \frac{1}{\rho}\underbrace{\epsilon_{ijk}\partial_j\partial_k p}_{=0} - \frac{1}{\rho^2}\epsilon_{ijk}(\partial_j \rho)(\partial_k p), && (2.130)\\ &= -\frac{1}{\rho^2}\nabla\rho \times \nabla p. && (2.131) \end{aligned}$$

Multiplying both sides by ρ, we write the final general form of the vorticity transport equation as

$$\rho\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \underbrace{\left(\boldsymbol{\omega}^T\cdot\nabla\right)\mathbf{v}}_{A} + \underbrace{\nabla\times\mathbf{f}}_{B} + \underbrace{\frac{1}{\rho^2}\nabla\rho\times\nabla p}_{C} + \underbrace{\nabla\times\left(\frac{1}{\rho}\left(\nabla^T\cdot\boldsymbol{\tau}\right)^T\right)}_{D}. \tag{2.132}$$

Here we see the evolution of the vorticity scaled by the density is affected by four physical processes, which we describe in greater detail directly, namely

- A: bending and stretching of vortex tubes,
- B: non-conservative body forces (if $\mathbf{f} = -\nabla\hat{\phi}$, then $\mathbf{f}$ is conservative, and $\nabla\times\mathbf{f} = -\nabla\times\nabla\hat{\phi} = 0$. For example $\mathbf{f} = -g_z\mathbf{k}$ gives $\hat{\phi} = g_z z$),
- C: non-barotropic, also known as baroclinic, effects (if a fluid is barotropic, then $p = p(\rho)$ and $\nabla p = (dp/d\rho)\nabla\rho$ thus $\nabla\rho\times\nabla p = \nabla\rho\times(dp/d\rho)\,\nabla\rho = 0$.), and
- D: viscous effects.

2.4.2 Incompressible conservative body force limit

The Helmholtz vorticity transport equation (2.132) reduces significantly in special limiting cases involving incompressible flow in the limit of a conservative body force. In this limit Eq. (2.132) reduces to the following

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}^T\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla\times(\nabla^T\cdot\boldsymbol{\tau})^T. \tag{2.133}$$

2.4.2.1 Isotropic, Newtonian, constant viscosity

Now if we further require that the fluid be isotropic and Newtonian with constant viscosity, the viscous term can be written as

$$\begin{aligned}
\nabla\times(\nabla^T\cdot\boldsymbol{\tau})^T &= \epsilon_{ijk}\partial_j\partial_m(2\mu(\partial_{(m}v_{k)} - (1/3)\underbrace{\partial_l v_l}_{=0}\delta_{mk})), && (2.134)\\
&= \mu\epsilon_{ijk}\partial_j\partial_m(\partial_m v_k + \partial_k v_m), && (2.135)\\
&= \mu\epsilon_{ijk}\partial_j(\partial_m\partial_m v_k + \partial_m\partial_k v_m), && (2.136)\\
&= \mu\epsilon_{ijk}\partial_j(\partial_m\partial_m v_k + \partial_k\underbrace{\partial_m v_m}_{=0}), && (2.137)\\
&= \mu\partial_m\partial_m\underbrace{\epsilon_{ijk}\partial_j v_k}_{\boldsymbol{\omega}}, && (2.138)\\
&= \mu\nabla^2\boldsymbol{\omega}. && (2.139)
\end{aligned}$$

So we get, recalling that $\nu = \mu/\rho$,

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}^T\cdot\nabla)\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}. \tag{2.140}$$

2.4.2.2 Two-dimensional, isotropic, Newtonian, constant viscosity

If we further require two-dimensionality, then we have $\boldsymbol{\omega} = (0, 0, \omega_3(x_1, x_2))^T$, and $\nabla = (\partial_1, \partial_2, 0)^T$, so $\boldsymbol{\omega}^T \cdot \nabla = 0$. Thus, we get the very simple

$$\frac{d\omega_3}{dt} = \nu \nabla^2 \omega_3 = \nu \left(\frac{\partial^2 \omega_3}{\partial x_1^2} + \frac{\partial^2 \omega_3}{\partial x_2^2} \right). \tag{2.141}$$

If the flow is further inviscid $\nu = 0$, we get

$$\frac{d\omega_3}{dt} = 0, \tag{2.142}$$

and we find that there is no tendency for vorticity to change along a streamline. If we further have an initially irrotational state, then we get $\boldsymbol{\omega} = 0$ for all space and time.

2.4.3 Physical interpretations

Let us consider how two of the terms in Eq. (2.132) contribute to the generation of vorticity.

2.4.3.1 Baroclinic (non-barotropic) effects

If a fluid is barotropic then we can write $p = p(\rho)$, or $\rho = \rho(p)$. An isentropic calorically perfect ideal gas has $p/p_o = (\rho/\rho_o)^\gamma$, where γ is the ratio of specific heats, and the o subscript indicates a constant value. Such a gas is barotropic. for such a fluid, we must have by the chain rule that $\partial_i p = (dp/d\rho)\partial_i \rho$. Hence ∇p and $\nabla \rho$ are vectors which point in the same direction. Moreover, isobars (lines of constant pressure) must be parallel to isochores (lines of constant density). If, as sketched in Figure 2.6. we calculate the resultant vector from the net pressure force, as well as the center of mass for a finite fluid volume, we would see that the resultant force had no lever arm with the center of gravity. Hence it would generate no torque, and no tendency for the fluid element to rotate about its center of mass, hence no vorticity would be generated by this force.

For a baroclinic fluid, we do not have $p = p(\rho)$; hence, we must expect that ∇p points in a different direction than $\nabla \rho$. If we examine this scenario, as sketched in Figure 2.7, we discover that the resultant force from the pressure has a non-zero lever arm with the center of mass of the fluid element. Hence, it generates a torque, a tendency to rotate the fluid about G, and vorticity.

2.4.3.2 Bending and stretching of vortex tubes

Now let us consider generation of vorticity by three-dimensional effects. Such effects are commonly characterized as the bending and stretching of what is known as vortex tubes. Here we focus on just the following inviscid equation:

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v}. \tag{2.143}$$

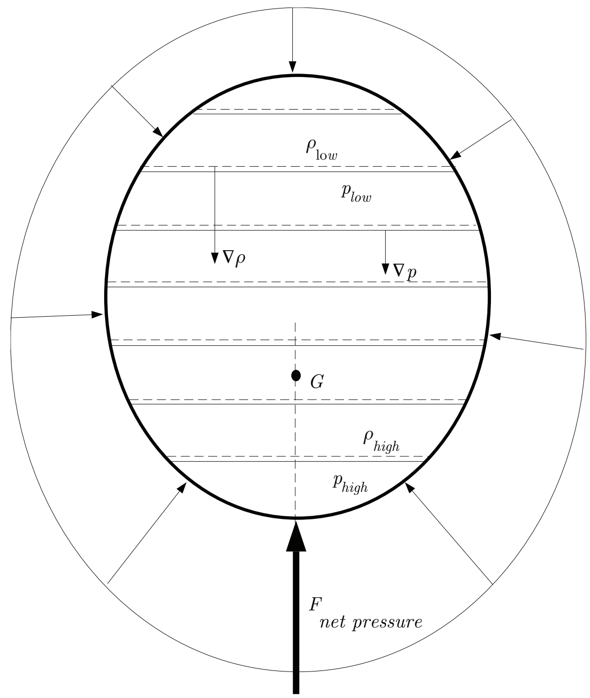

Figure 2.6: Isobars and isochores, center of mass G, and center of pressure for barotropic fluid.

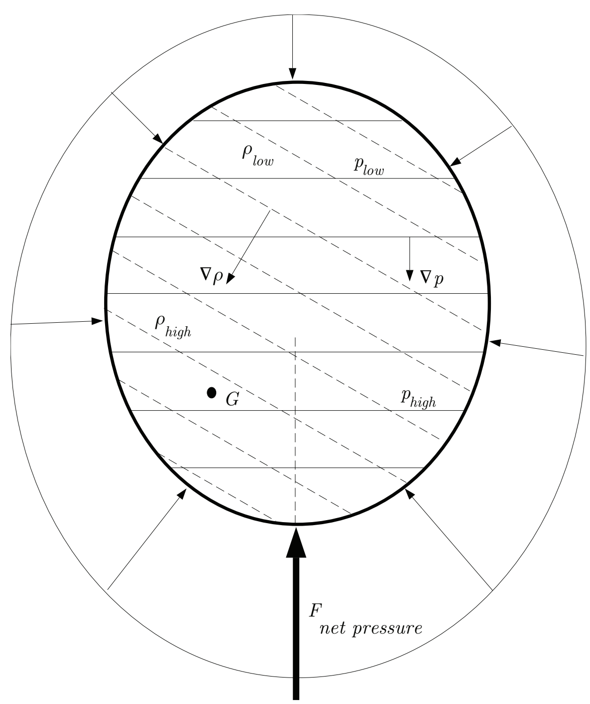

Figure 2.7: Isobars and isochores, center of mass G, and center of pressure for baroclinic fluid.

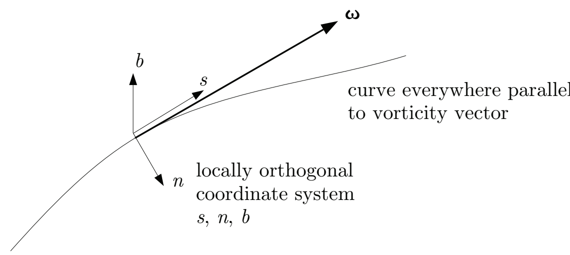

Figure 2.8: Local orthogonal intrinsic coordinate system oriented with local vorticity field.

If we consider a coordinate system which is oriented with the vorticity field as sketched in Figure 2.8, we will get many simplifications. We take the following directions

- s: the streamwise direction parallel to the vorticity vector,
- n: the principal normal direction, pointing towards the center of curvature,
- b: the biorthogonal direction, orthogonal to s and n.

With this system, we can say that

$$(\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v} = \left(\begin{pmatrix} \omega_s & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial n} \\ \frac{\partial}{\partial b} \end{pmatrix} \right) \mathbf{v}, \tag{2.144}$$

$$= \omega_s \frac{\partial \mathbf{v}}{\partial s}. \tag{2.145}$$

So for the inviscid flow we have

$$\frac{d\boldsymbol{\omega}}{dt} = \omega_s \frac{\partial \mathbf{v}}{\partial s}. \tag{2.146}$$

We have in terms of components

$$\frac{d\omega_s}{dt} = \omega_s \frac{\partial v_s}{\partial s}, \tag{2.147}$$

$$\frac{d\omega_n}{dt} = \omega_s \frac{\partial v_n}{\partial s}, \tag{2.148}$$

$$\frac{d\omega_b}{dt} = \omega_s \frac{\partial v_b}{\partial s}. \tag{2.149}$$

The term $\frac{\partial v_s}{\partial s}$ we know from kinematics represents a local stretching or extension. Just as a rotating figure skater increases his or her angular velocity by concentrating his or her mass about a vertical axis, so does a rotating fluid. The first of these expressions says that the component of rotation aligned with the present increases if there is stretching in that direction. This is sketched in Figure 2.9,

The second and third terms enforce that if v_n or v_b are changing in the s direction, when accompanied by non-zero ω_s, that changes in the non-aligned components of $\boldsymbol{\omega}$ are induced.

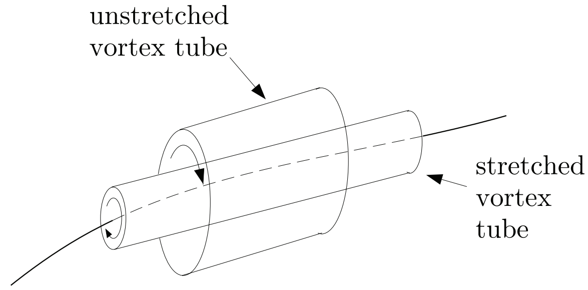

Figure 2.9: Increase in vorticity due to stretching of vortex tube.

Hence the previously zero components ω_n, ω_b acquired non-zero values, and the lines parallel to the vorticity vector bend. Hence, we have the term, bending of vortex tubes.

It is generally accepted that the bending and stretching of vortex tubes is an important mechanism in the transition from laminar to turbulent flow.

2.5 Kelvin's circulation theorem

Kelvin's circulation theorem describes how the circulation of a material region in a fluid changes with time. We first recall the definition of circulation Γ:

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{x}, \tag{2.150}$$

where C is a closed contour. We next take the material derivative of Γ to get

$$\begin{aligned}
\frac{d\Gamma}{dt} &= \frac{d}{dt}\oint_C \mathbf{v}^T \cdot d\mathbf{x}, & (2.151)\\
&= \oint_C \frac{d\mathbf{v}^T}{dt}\cdot d\mathbf{x} + \oint_C \mathbf{v}^T \cdot \frac{d}{dt} d\mathbf{x}, & (2.152)\\
&= \oint_C \frac{d\mathbf{v}^T}{dt}\cdot d\mathbf{x} + \oint_C \mathbf{v}^T \cdot d\left(\frac{d\mathbf{x}}{dt}\right), & (2.153)\\
&= \oint_C \frac{d\mathbf{v}^T}{dt}\cdot d\mathbf{x} + \oint_C \mathbf{v}^T \cdot d\mathbf{v}, & (2.154)\\
&= \oint_C \frac{d\mathbf{v}^T}{dt}\cdot d\mathbf{x} + \underbrace{\oint_C d\left(\frac{1}{2}\mathbf{v}^T\cdot\mathbf{v}\right)}_{=0}, & (2.155)\\
&= \oint_C \left(\frac{d\mathbf{v}}{dt}\right)^T \cdot d\mathbf{x}. & (2.156)
\end{aligned}$$

Here we note that because we have chosen a material region for our closed contour that $\frac{d\mathbf{x}}{dt}$ must be the fluid particle velocity. This then allows us to write the second term as a perfect differential, which integrates over the closed contour to be zero. We continue now by using the linear momentum principle to replace the particle acceleration with density-scaled forces to arrive at

$$\frac{d\Gamma}{dt} = \oint_C \left(\mathbf{f} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\left(\nabla^T \cdot \boldsymbol{\tau}\right)^T \right)^T \cdot d\mathbf{x}. \tag{2.157}$$

If now the fluid is inviscid ($\boldsymbol{\tau} = 0$), the body force is conservative ($\mathbf{f} = -\nabla\hat{\phi}$), and the fluid is barotropic ($(1/\rho)\nabla p = \nabla\Upsilon$), then we have

$$\frac{d\Gamma}{dt} = \oint_C \left(-\nabla\hat{\phi} - \nabla\Upsilon\right)^T \cdot d\mathbf{x}, \tag{2.158}$$

$$= -\oint_C \nabla^T \left(\hat{\phi} + \Upsilon\right) \cdot d\mathbf{x}, \tag{2.159}$$

$$= -\underbrace{\oint_C d\left(\hat{\phi} + \Upsilon\right)}_{=0}. \tag{2.160}$$

The integral on the right hand side is zero because the contour is closed; hence, the integral is path independent. Consequently, we arrive at the common version of Kelvin's circulation theorem which holds that for a fluid which is inviscid, barotropic, and subjected to conservative body forces, the circulation following a material region does not change with time:

$$\frac{d\Gamma}{dt} = 0. \tag{2.161}$$

Note that this is very similar to the Helmholtz equation, which, when we make the additional stipulation of two-dimensionality and incompressibility, gives $d\boldsymbol{\omega}/dt = 0$. This is not surprising as the vorticity is closely linked to the circulation via Stokes' theorem, which states

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{x} = \int_A (\nabla \times \mathbf{v})^T \cdot \mathbf{n}\, dA = \int_A \boldsymbol{\omega}^T \cdot \mathbf{n}\, dA. \tag{2.162}$$

2.6 Potential flow of ideal point vortices

Consider the fluid motion induced by the simultaneous interaction of a family of ideal *irrotational* point vortices in an incompressible flow field. Since the flow is irrotational and incompressible, we have the following useful results:

- Since $\nabla \times \mathbf{v} = 0$, we can write the velocity vector as the gradient of a scalar potential ϕ:

$$\mathbf{v} = \nabla\phi, \qquad \text{if irrotational.} \tag{2.163}$$

  We call ϕ the *velocity potential*.

- Since $\nabla^T \cdot \mathbf{v} = 0$, we have

$$\nabla^T \cdot \nabla\phi = \nabla^2\phi = 0, \tag{2.164}$$

or expanding, we have

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.165}$$

- We notice that the equation for ϕ is linear; hence the method of superposition is valid here for the velocity potential. That is, we can add an arbitrary number of velocity potentials together and get a viable flow field.

- The irrotational unsteady Bernoulli equation gives us the time and space dependent pressure field. This equation is not linear, so we do not expect pressures from elementary solutions to add to form total pressures.

Recalling that the incompressible, three dimensional constant viscosity Helmholtz equation can be written as

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega}^T \cdot \nabla)\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}, \tag{2.166}$$

we see that a flow which is initially irrotational everywhere in an unbounded fluid will always be irrotational, as $\frac{d\boldsymbol{\omega}}{dt} = 0$. There is no mechanism to change the vorticity from its uniform initial value of zero. This even holds for a viscous flow. However, in a bounded medium, the no-slip boundary condition almost always tends to diffuse vorticity into the flow as we shall see.

Further from Kelvin's circulation theorem, we also note that the circulation Γ has no tendency to change following a particle; that is Γ convects along particle pathlines.

2.6.1 Two interacting ideal vortices

Let us apply this notion to two ideal counterrotating vortices 1 and 2, with respective strengths, Γ_1 and Γ_2, as shown in Figure 2.10, Were it isolated, vortex 1 would have no tendency to move itself, but would induce a velocity at a distance h away from its center of $\frac{\Gamma_1}{2\pi h}$. This induced velocity in fact convects vortex 2, to satisfy Kelvin's circulation theorem. Similarly, vortex 2 induces a velocity of vortex 1 of $\frac{\Gamma_2}{2\pi h}$.

The center of rotation G is the point along the 1-2 axis for which the induced velocity is zero, as is illustrated in Figure 2.11, To calculate it we equate the induced velocities of each vortex

$$\frac{\Gamma_1}{2\pi h_G} = \frac{\Gamma_2}{2\pi(h - h_G)}, \tag{2.167}$$

$$(h - h_G)\Gamma_1 = h_G\Gamma_2, \tag{2.168}$$

$$h\Gamma_1 = h_G(\Gamma_1 + \Gamma_2), \tag{2.169}$$

$$h_G = h\frac{\Gamma_1}{\Gamma_1 + \Gamma_2}. \tag{2.170}$$

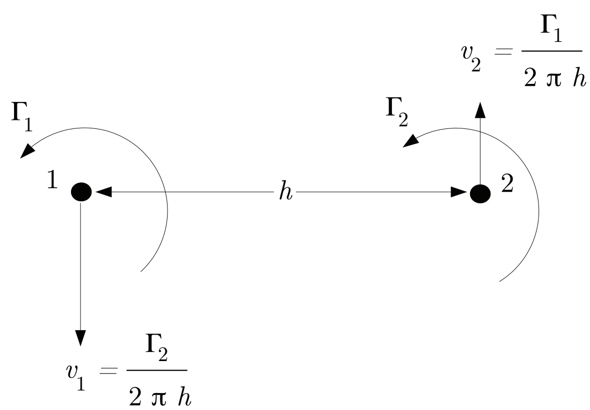

Figure 2.10: Sketch of the mutual influence of two ideal point vortices on each other.

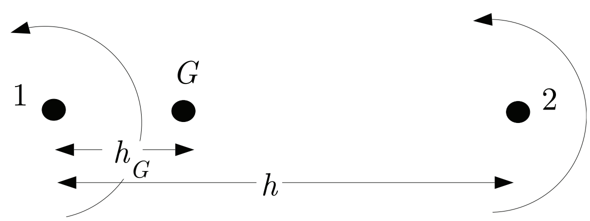

Figure 2.11: Sketch showing the center of rotation G.

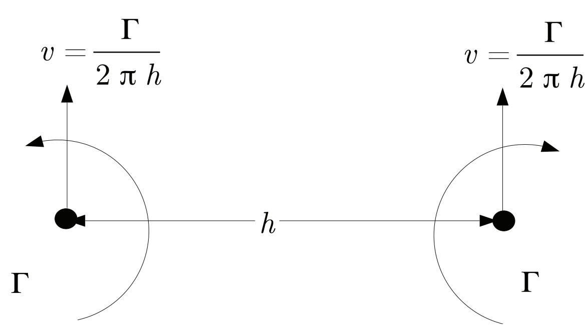

Figure 2.12: Sketch showing a pair of counterrotating vortices of equal strength

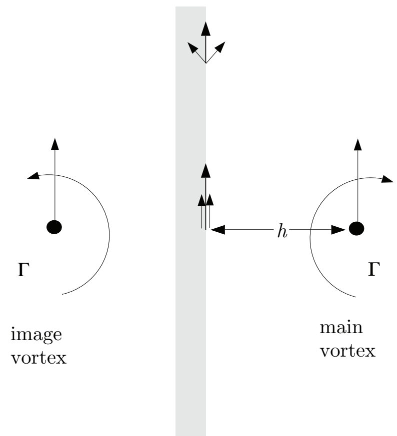

Figure 2.13: Sketch showing a vortex and its image to simulate an inviscid wall.

A pair of equal strength counterrotating vortices is illustrated in Figure 2.12. Such vortices induce the same velocity in each other, so they will propagate as a pair at a fixed distance from one another.

2.6.2 Image vortex

If we choose to model the fluid as inviscid, then there is no viscous stress, and we can no longer enforce the no slip condition at a wall. However at a slip wall, we must require that the velocity vector be parallel to the wall. We can model the motion of an ideal vortex separated by a distance h from an inviscid slip wall by placing a so-called *image vortex* on the other side of the wall. The image vortex will induce a velocity which when superposed with the original vortex, renders the resultant velocity to be parallel to the wall. A vortex and its image vortex, which generates a straight streamline at a wall, is sketched in Figure 2.13,

2.6.3 Vortex sheets

We can model the slip line between two inviscid fluids moving at different velocities by what is known as a *vortex sheet*. A vortex sheet is sketched in Figure 2.14. Here we have a distribution of small vortices, each of strength $d\Gamma$, on the x axis. Each of these vortices induces a small velocity $d\mathbf{v}$ at an arbitrary point $(\tilde{x}, \tilde{y})$. The influence of the point vortex at $(x, 0)$ is sketched in the figure. It generates a small velocity with magnitude

$$d|\mathbf{v}| = \frac{d\Gamma}{2\pi h} = \frac{d\Gamma}{2\pi\sqrt{(\tilde{x}-x)^2+\tilde{y}^2}} \qquad (2.171)$$

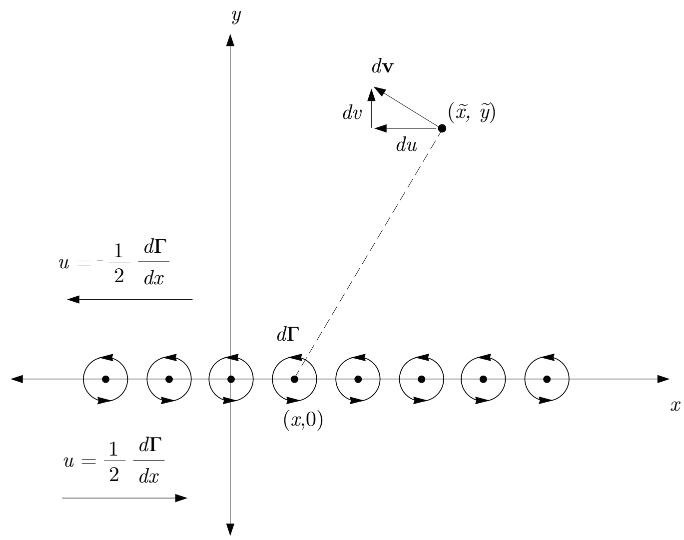

Figure 2.14: Sketch showing schematic of vortex sheet.

Using basic trigonometry, we can deduce that the influence of the single vortex of differential strength on each velocity component is

$$du = \frac{-d\Gamma\tilde{y}}{2\pi\left((\tilde{x}-x)^2+\tilde{y}^2\right)} = \frac{-\frac{d\Gamma}{dx}\tilde{y}}{2\pi\left((\tilde{x}-x)^2+\tilde{y}^2\right)}\,dx, \tag{2.172}$$

$$dv = \frac{d\Gamma(\tilde{x}-x)}{2\pi\left((\tilde{x}-x)^2+\tilde{y}^2\right)} = \frac{\frac{d\Gamma}{dx}(\tilde{x}-x)}{2\pi\left((\tilde{x}-x)^2+\tilde{y}^2\right)}\,dx. \tag{2.173}$$

Here $\frac{d\Gamma}{dx}$ is a measure of the strength of the vortex sheet. Let us account for the effects of *all* of the differential vortices by integrating from $x = -L$ to $x = L$ and then letting $L \to \infty$. We obtain then the total velocity components u and v at each point to be

$$u = \lim_{L\to\infty} -\frac{\frac{d\Gamma}{dx}}{2\pi}\left(\underbrace{\arctan\left(\frac{L-\tilde{x}}{\tilde{y}}\right)}_{\to\pm\frac{\pi}{2}} + \underbrace{\arctan\left(\frac{L+\tilde{x}}{\tilde{y}}\right)}_{\to\pm\frac{\pi}{2}}\right), \tag{2.174}$$

$$= \begin{cases} -\frac{1}{2}\frac{d\Gamma}{dx}, & \text{if} \quad \tilde{y} > 0, \\ \frac{1}{2}\frac{d\Gamma}{dx}, & \text{if} \quad \tilde{y} < 0, \end{cases} \tag{2.175}$$

$$v = \lim_{L\to\infty} \frac{\frac{d\Gamma}{dx}}{4\pi} \ln\frac{(L-\tilde{x})^2+\tilde{y}^2}{(L+\tilde{x})^2+\tilde{y}^2} = 0. \tag{2.176}$$

So the vortex sheet generates no y component of velocity anywhere in the flow field and two uniform x components of velocity of opposite sign above and below the x axis.

2.6.4 Potential of an ideal vortex

Let us calculate the velocity potential function ϕ associated with a single ideal vortex. Consider an ideal vortex centered at the origin, and represent the velocity field here in cylindrical coordinates:

$$v_r = 0, \qquad v_\theta = \frac{\Gamma}{2\pi r}, \qquad v_z = 0. \tag{2.177}$$

Now in cylindrical coordinates the gradient operating on a scalar function gives

$$\begin{aligned}
\nabla\phi &= \mathbf{v}, & (2.178)\\
\frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\phi}{\partial z}\mathbf{e}_z &= 0\mathbf{e}_r + \frac{\Gamma}{2\pi r}\mathbf{e}_\theta + 0\mathbf{e}_z, & (2.179)\\
\frac{\partial\phi}{\partial r} &= 0, & (2.180)\\
\frac{1}{r}\frac{\partial\phi}{\partial\theta} &= \frac{\Gamma}{2\pi r}, \quad \text{so} \quad \phi = \frac{\Gamma}{2\pi}\theta + C(r,z), & (2.181)\\
\frac{\partial\phi}{\partial z} &= 0. & (2.182)
\end{aligned}$$

But since the partials of ϕ with respect to r and z are zero, $C(r,z)$ is at most a constant, which we can set to zero without losing any information regarding the velocity itself

$$\phi = \frac{\Gamma}{2\pi}\theta. \tag{2.183}$$

In Cartesian coordinates, we have

$$\phi = \frac{\Gamma}{2\pi}\arctan\left(\frac{y}{x}\right) \tag{2.184}$$

Lines of constant potential for the ideal vortex centered at the origin are sketched in Figure 2.15.

2.6.5 Interaction of multiple vortices

Here we will consider the interactions of a large number of vortices by using the method of superposition for the velocity potentials.

If we have two vortices with strengths Γ_1 and Γ_2 centered at arbitrary locations (x_1, y_1) and (x_2, y_2) are sketched in Figure 2.16, the potential for each is given by

$$\phi_1 = \frac{\Gamma_1}{2\pi}\arctan\left(\frac{y-y_1}{x-x_1}\right), \tag{2.185}$$

$$\phi_2 = \frac{\Gamma_2}{2\pi}\arctan\left(\frac{y-y_2}{x-x_2}\right). \tag{2.186}$$

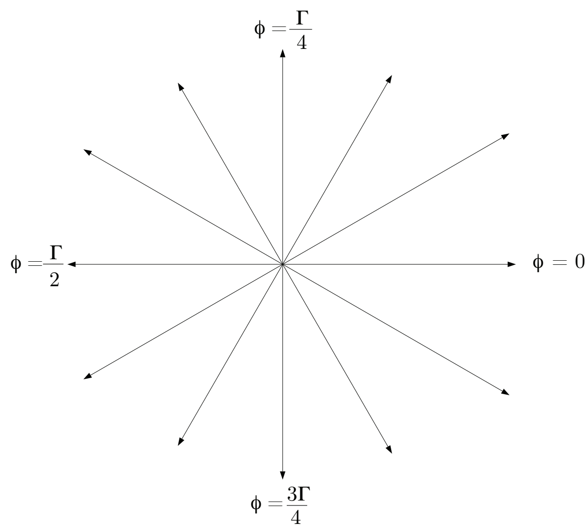

Figure 2.15: Lines of constant potential for ideal irrotational vortex.

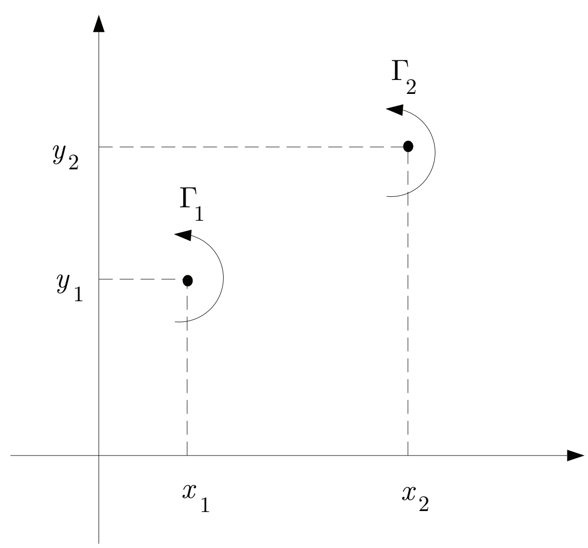

Figure 2.16: Two vortices at arbitrary locations.

Since the equation governing the velocity potential, $\nabla^2\phi = 0$, is linear we can add the two potentials and still satisfy the overall equation so that

$$\phi = \frac{\Gamma_1}{2\pi}\arctan\left(\frac{y-y_1}{x-x_1}\right) + \frac{\Gamma_2}{2\pi}\arctan\left(\frac{y-y_2}{x-x_2}\right), \tag{2.187}$$

is a legitimate solution. Taking the gradient of ϕ,

$$\begin{aligned}\nabla\phi &= \left(-\left(\frac{\Gamma_1}{2\pi}\right)\frac{y-y_1}{(x-x_1)^2+(y-y_1)^2} - \left(\frac{\Gamma_2}{2\pi}\right)\frac{y-y_2}{(x-x_2)^2+(y-y_2)^2}\right)\mathbf{i}\\ &\quad+\left(\left(\frac{\Gamma_1}{2\pi}\right)\frac{x-x_1}{(x-x_1)^2+(y-y_1)^2} + \left(\frac{\Gamma_2}{2\pi}\right)\frac{x-x_2}{(x-x_2)^2+(y-y_2)^2}\right)\mathbf{j},\end{aligned} \tag{2.188}$$

so that

$$u(x,y) = -\left(\frac{\Gamma_1}{2\pi}\right)\frac{y-y_1}{(x-x_1)^2+(y-y_1)^2} - \left(\frac{\Gamma_2}{2\pi}\right)\frac{y-y_2}{(x-x_2)^2+(y-y_2)^2}, \tag{2.189}$$

$$v(x,y) = \left(\frac{\Gamma_1}{2\pi}\right)\frac{x-x_1}{(x-x_1)^2+(y-y_1)^2} + \left(\frac{\Gamma_2}{2\pi}\right)\frac{x-x_2}{(x-x_2)^2+(y-y_2)^2}. \tag{2.190}$$

Extending this to a collection of N vortices located at (x_i, y_i) at a given time, we have the following for the velocity field:

$$u(x,y) = \sum_{i=1}^{N} -\left(\frac{\Gamma_i}{2\pi}\right)\frac{y-y_i}{(x-x_i)^2+(y-y_i)^2}, \tag{2.191}$$

$$v(x,y) = \sum_{i=1}^{N} \left(\frac{\Gamma_i}{2\pi}\right)\frac{x-x_i}{(x-x_i)^2+(y-y_i)^2}. \tag{2.192}$$

Now to convect (that is, to move) the kth vortex, we move it with the velocity induced by the other vortices, since vortices convect with the flow. Recalling that the velocity is the time derivative of the position $u_k = \frac{dx_k}{dt}, v_k = \frac{dy_k}{dt}$, we then get the following $2N$ non-linear ordinary differential equations for the $2N$ unknowns, the x and y positions of each of the N vortices:

$$\frac{dx_k}{dt} = \sum_{i=1, i\neq k}^{N} -\left(\frac{\Gamma_i}{2\pi}\right)\frac{y_k-y_i}{(x_k-x_i)^2+(y_k-y_i)^2}, \quad x_k(0) = x_k^o, \quad k = 1,\ldots,N, \tag{2.193}$$

$$\frac{dy_k}{dt} = \sum_{i=1, i\neq k}^{N} \left(\frac{\Gamma_i}{2\pi}\right)\frac{x_k-x_i}{(x_k-x_i)^2+(y_k-y_i)^2}, \quad y_k(0) = y_k^o, \quad k = 1,\ldots,N. \tag{2.194}$$

This set of equations, except for three or fewer point vortices, must be integrated numerically. These equations form what is commonly termed a Biot-Savart[23] law. These ordinary

[2]Jean-Baptiste Biot, 1774-1862, Paris-born applied mathematician

[3]Felix Savart, 1791-1841, French mathematician who worked on magnetic fields and acoustics.

differential equations are highly non-linear and give rise to chaotic motion of the point vortices in general. It is a very similar calculation to the motion of point masses in a Newtonian gravitational field, except that the essential variation goes as $1/r$ for vortices and $1/r^2$ for Newtonian gravitational fields. Thus the dynamics are different. Nevertheless just as calculations for large numbers of celestial bodies can give rise to solar systems, clusters of planets, and galaxies, similar "galaxies" of vortices can be predicted with the equations for vortex dynamics.

2.6.6 Pressure field

We have thus far examined essentially only the kinematics of vortices. We have actually used dynamics in our incorporation of the Helmholtz equation and Kelvin's theorem, but their simple results really only justify the use of a simple kinematics. Dynamics asks what are the forces which give rise to the motion. Here, we will assume there is no body force and that the fluid is inviscid, in which case it must be pressure forces which give rise to the motion. We have the proper conditions for which Bernoulli's equation can be used to give the pressure field. We consider two cases, a single stationary point vortex, and a group of N moving point vortices.

2.6.6.1 Single stationary vortex

If we take $p = p_\infty$ in the far field and $f_i = \mathbf{g} = 0$, this steady flow gives us

$$\frac{1}{2}\mathbf{v}^T \cdot \mathbf{v} + \frac{p}{\rho} = \frac{1}{2}\mathbf{v}_\infty^T \cdot \mathbf{v}_\infty + \frac{p_\infty}{\rho}, \tag{2.195}$$

$$\frac{1}{2}\left(\frac{\Gamma}{2\pi r}\right)^2 + \frac{p}{\rho} = 0 + \frac{p_\infty}{\rho}, \tag{2.196}$$

$$p(r) = p_\infty - \frac{\rho\Gamma^2}{8\pi^2}\frac{1}{r^2}. \tag{2.197}$$

Note that the pressure goes to negative infinity at the origin. This is obviously unphysical. It can be corrected by including viscous effects, which turn out not to substantially alter our main conclusions.

2.6.6.2 Group of N vortices

For a collection of N vortices, the flow is certainly not steady, and we must in general retain the time dependent velocity potential in Bernoulli's equation yielding

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^T \cdot \nabla\phi + \frac{p}{\rho} = f(t). \tag{2.198}$$

Now we require that as $r \to \infty$ that $p \to p_\infty$. We also know that as $r \to \infty$ that $\phi \to 0$, hence $\nabla\phi \to 0$ as well. Hence as $r \to \infty$, we have $\frac{p_\infty}{\rho} = f(t)$. So our final result is

$$p = p_\infty - \frac{1}{2}\rho(\nabla\phi)^T \cdot \nabla\phi - \frac{\partial\phi}{\partial t}. \tag{2.199}$$

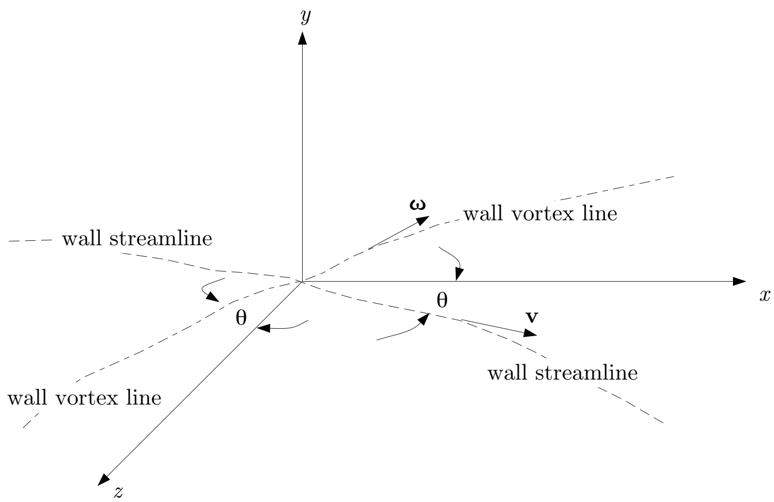

Figure 2.17: Wall streamlines and vortex lines at wall $y = 0$.

So with a knowledge of the velocity field through ϕ, we can determine the pressure field which must have given rise to that velocity field.

2.7 Influence of walls

The Helmholtz equation considers mechanisms that generate vorticity in the interior of a flow. It does not, however, include one of the most important mechanisms, namely the introduction of vorticity due to the no-slip boundary condition at a solid wall. In this section we shall focus on that mechanism.

2.7.1 Streamlines and vortex lines at walls

It seems odd that a streamline can be defined at a wall where the velocity is formally zero, but in the neighborhood of the wall, the fluid velocity is small but non-zero. We can extrapolate the position of streamlines near the wall to the wall to define a wall streamline. We shall also consider a so-called *vortex line*, a line everywhere parallel to the vorticity vector, at the wall.

We consider the geometry sketched in Figure 2.17. Here the $x-z$ plane is locally attached to a wall at $y = 0$, and the y direction is normal to the wall. Wall streamlines and vortex lines are sketched in the figure.

Because the flow satisfies no-slip, we have at the wall

$$u(x, y = 0, z) = 0, \qquad v(x, y = 0, z) = 0, \qquad w(x, y = 0, z) = 0. \tag{2.200}$$

Because of this, partial derivatives of all velocities with respect to either x or z will also be zero at $y = 0$:

$$\left.\frac{\partial u}{\partial x}\right|_{y=0} = \left.\frac{\partial u}{\partial z}\right|_{y=0} = \left.\frac{\partial v}{\partial x}\right|_{y=0} = \left.\frac{\partial v}{\partial z}\right|_{y=0} = \left.\frac{\partial w}{\partial x}\right|_{y=0} = \left.\frac{\partial w}{\partial z}\right|_{y=0} = 0 \tag{2.201}$$

Near the wall, the velocity is near zero, so the Mach number is very small, and the flow is well modeled as incompressible. So here, the mass conservation equation implies that $\nabla^T \cdot \mathbf{v} = 0$, so applying this at the wall, we get

$$\underbrace{\left.\frac{\partial u}{\partial x}\right|_{y=0}}_{=0} + \left.\frac{\partial v}{\partial y}\right|_{y=0} + \underbrace{\left.\frac{\partial w}{\partial z}\right|_{y=0}}_{=0} = 0 \quad \text{so} \tag{2.202}$$

$$\left.\frac{\partial v}{\partial y}\right|_{y=0} = 0. \tag{2.203}$$

Now let us examine the behavior of u, v, and w, as we leave the wall in the y direction. Consider a Taylor series of each:

$$u = \underbrace{u|_{y=0}}_{=0} + \left.\frac{\partial u}{\partial y}\right|_{y=0} y + \frac{1}{2}\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} y^2 + \ldots, \tag{2.204}$$

$$v = \underbrace{v|_{y=0}}_{=0} + \underbrace{\left.\frac{\partial v}{\partial y}\right|_{y=0}}_{=0} y + \frac{1}{2}\left.\frac{\partial^2 v}{\partial y^2}\right|_{y=0} y^2 + \ldots, \tag{2.205}$$

$$w = \underbrace{w|_{y=0}}_{=0} + \left.\frac{\partial w}{\partial y}\right|_{y=0} y + \frac{1}{2}\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0} y^2 + \ldots, \tag{2.206}$$

$$\tag{2.207}$$

So we get

$$u = \left.\frac{\partial u}{\partial y}\right|_{y=0} y + \ldots, \tag{2.208}$$

$$v = \frac{1}{2}\left.\frac{\partial^2 v}{\partial y^2}\right|_{y=0} y^2 + \ldots, \tag{2.209}$$

$$w = \left.\frac{\partial w}{\partial y}\right|_{y=0} y + \ldots. \tag{2.210}$$

Now for streamlines, we must have

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{2.211}$$

For the streamline near the wall, consider just $dx/u = dz/w$, and also tag the streamline as dz_s, so that the slope of the wall streamline, which is the tangent of the angle θ between the wall streamline and the x axis is

$$\tan\theta = \left.\frac{dz_s}{dx}\right|_{y=0} = \lim_{y\to 0}\frac{w}{u} = \frac{\left.\frac{\partial w}{\partial y}\right|_{y=0}}{\left.\frac{\partial u}{\partial y}\right|_{y=0}} \tag{2.212}$$

Now consider the vorticity vector evaluated at the wall:

$$\omega_x|_{y=0} = \left.\frac{\partial w}{\partial y}\right|_{y=0} - \underbrace{\left.\frac{\partial v}{\partial z}\right|_{y=0}}_{=0} = \left.\frac{\partial w}{\partial y}\right|_{y=0}, \tag{2.213}$$

$$\omega_y|_{y=0} = \underbrace{\left.\frac{\partial u}{\partial z}\right|_{y=0}}_{=0} - \underbrace{\left.\frac{\partial w}{\partial x}\right|_{y=0}}_{=0} = 0, \tag{2.214}$$

$$\omega_z|_{y=0} = \underbrace{\left.\frac{\partial v}{\partial x}\right|_{y=0}}_{=0} - \left.\frac{\partial u}{\partial y}\right|_{y=0} = -\left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{2.215}$$

So we see that on the wall at $y = 0$, the vorticity vector has no component in the y direction. Hence, it must be parallel to the wall itself. Further, we can then define the slope of the vortex line, $\frac{dz_v}{dx}$, at the wall in the same fashion as we define a streamline:

$$\left.\frac{dz_v}{dx}\right|_{y=0} = \frac{\omega_z}{\omega_x} = -\frac{\left.\frac{\partial u}{\partial y}\right|_{y=0}}{\left.\frac{\partial w}{\partial y}\right|_{y=0}} = -\frac{1}{\left.\frac{dz_s}{dx}\right|_{y=0}} \tag{2.216}$$

Since the slope of the vortex line is the negative reciprocal of the slope of the streamline, we have that at a no-slip wall, streamlines are orthogonal to vortex lines. We also note that streamlines are orthogonal to vortex lines for flow with variation in the x and y directions only. For general three-dimensional flows away from walls, we do not expect the two lines to be orthogonal.

2.7.2 Generation of vorticity at walls

Now further restrict the coordinate system of the previous subsection so that the x axis is aligned with the wall streamline and the z axis is aligned with the wall vortex line. As before the y axis is normal to the wall. The coordinate system aligned with the wall streamlines and vortex lines is sketched in Figure 2.18, In the figure we take the direction $\mathbf{n}$ to be normal to the wall. Now for this coordinate system, we have

$$\left.\frac{dz_s}{dx}\right|_{y=0} = \left.\frac{w}{u}\right|_{y=0} = \frac{\left.\frac{\partial w}{\partial y}\right|_{y=0}}{\left.\frac{\partial u}{\partial y}\right|_{y=0}} = 0. \tag{2.217}$$

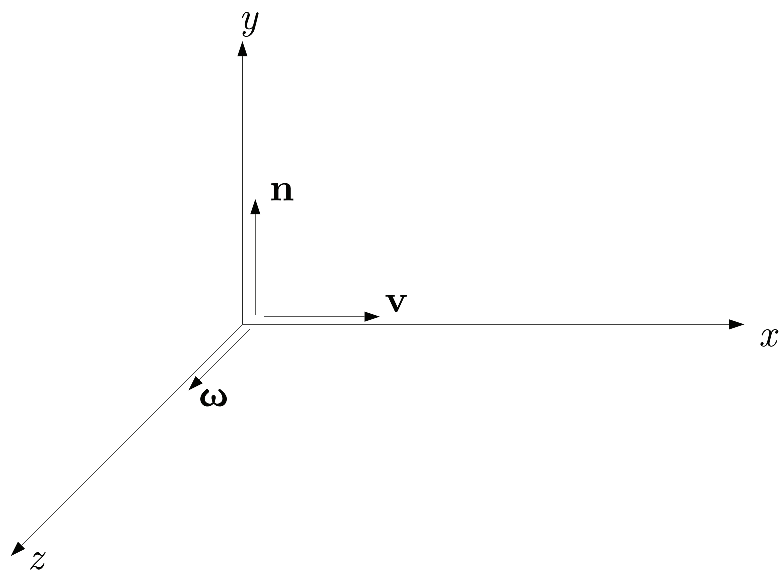

Figure 2.18: Coordinate system aligned with wall streamlines and vortex lines.

Hence, we must have

$$\left.\frac{\partial w}{\partial y}\right|_{y=0} = 0. \tag{2.218}$$

Now consider the viscous traction vector associated with the wall:

$$t_j = n_i\tau_{ij} = n_y\tau_{yj} = \begin{pmatrix} \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \end{pmatrix} = \begin{pmatrix} \mu\left(\left.\frac{\partial u}{\partial y}\right|_{y=0} + \left.\frac{\partial v}{\partial x}\right|_{y=0}\right) \\ \mu\left(\left.\frac{\partial v}{\partial y}\right|_{y=0} + \left.\frac{\partial v}{\partial y}\right|_{y=0}\right) \\ \mu\left(\left.\frac{\partial v}{\partial z}\right|_{y=0} + \left.\frac{\partial w}{\partial y}\right|_{y=0}\right) \end{pmatrix} = \begin{pmatrix} \mu\left.\frac{\partial u}{\partial y}\right|_{y=0} \\ 0 \\ 0 \end{pmatrix} \tag{2.219}$$

So the viscous force is parallel to the surface, hence it is a tangential or shear force; moreover, it points in the same direction as the streamline. Now if we examine the vorticity vector at the surface we find first by our definition of the coordinate systems that $\omega_x = \omega_y = 0$ at $y = 0$ and that

$$\left.\omega_z\right|_{y=0} = \underbrace{\left.\frac{\partial v}{\partial x}\right|_{y=0}}_{=0} - \left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{2.220}$$

For this case, we can say that the viscous force is $t_1 = -\mu\omega_z$. In fact in general, we can say

$$\mathbf{t}_{viscous} = -\mu\mathbf{n}\times\boldsymbol{\omega}, \qquad \text{at a wall.} \tag{2.221}$$

Since the viscous force is orthogonal to both the surface normal and the vorticity vector, it must always at the wall be aligned with the flow direction.

Chapter 3

One-dimensional compressible flow

see Yih, Chapter 6
see Liepmann and Roshko, Chapter 2
see Shapiro, Chapters 4-8

This chapter will focus on one-dimensional flow of a compressible fluid. The following topics will be covered:

- development of generalized one-dimensional flow equations,
- isentropic flow with area change,
- flow with normal shock waves, and
- the method of characteristics for isentropic rarefactions.

We will assume for this chapter:

- $v \equiv 0, w \equiv 0, \partial/\partial y \equiv 0, \partial/\partial z \equiv 0$; one-dimensional flow.

Friction and heat transfer will not be modeled rigorously. Instead, they will be modeled in a fashion which captures the relevant physics and retains analytic tractability. Further, we will ignore the influences of an external body force, $f_i = 0$.

3.1 Generalized one-dimensional equations

Here we will re-derive, in a rather conventional way, the one-dimensional equations of flow with area change. Although for the geometry we use, it will appear that we should be using at least two-dimensional equations, our results will be correct when we interpret them as an average value at a given x location. Our results will be valid as long as the area changes slowly relative to how fast the flow can adjust to area changes.

We could start directly with our equations from an earlier chapter as well. However, the *ad hoc* nature of friction and heat transfer commonly employed makes a re-derivation useful. The flow we wish to consider, flow with area change, heat transfer, and wall friction, is illustrated by the following sketch of a control volume, Figure 3.1.

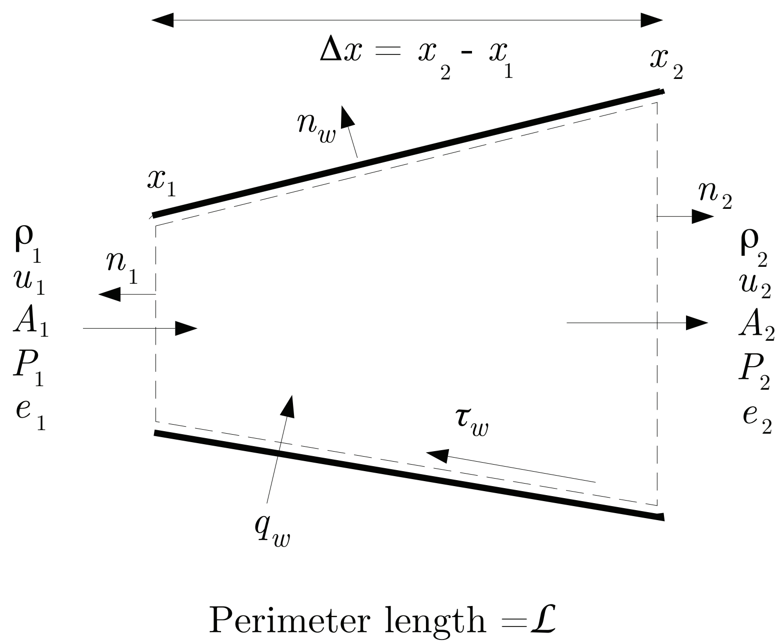

Figure 3.1: Control volume sketch for one-dimensional compressible flow with area change, heat transfer, and wall friction.

For this flow, we will adopt the following conventions

- surface 1 and 2 are open and allow fluxes of mass, momentum, and energy,
- surface w is a closed wall; no mass flux through the wall
- external heat flux q_w (Energy/Area/Time: W/m^2) *through* the wall allowed-q_w known *fixed* parameter,
- diffusive, longitudinal heat transfer ignored, $q_x = 0$,
- wall shear τ_w (Force/Area: N/m^2) allowed–τ_w known, *fixed* parameter,
- diffusive viscous stress not allowed $\tau_{xx} = 0$, and
- cross-sectional area a known *fixed* function: $A(x)$.

3.1.1 Mass

Take the over-bar notation to indicate a volume averaged quantity.

The amount of mass in a control volume after a time increment Δt is equal to the original amount of mass plus that which came in minus that which left:

$$\bar{\rho}\bar{A}\Delta x\Big|_{t+\Delta t} = \bar{\rho}\bar{A}\Delta x\Big|_{t} + \rho_1 A_1 \left(u_1 \Delta t\right) - \rho_2 A_2 \left(u_2 \Delta t\right). \tag{3.1}$$

Rearrange and divide by $\Delta x \Delta t$:

$$\frac{\bar{\rho}\bar{A}\big|_{t+\Delta t} - \bar{\rho}\bar{A}\big|_{t}}{\Delta t} + \frac{\rho_2 A_2 u_2 - \rho_1 A_1 u_1}{\Delta x} = 0. \tag{3.2}$$

Taking the limit as $\Delta t \to 0, \Delta x \to 0$, we get

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0. \tag{3.3}$$

If the flow is steady, then

$$\frac{d}{dx}\left(\rho A u\right) = 0, \tag{3.4}$$

$$A u \frac{d\rho}{dx} + \rho u \frac{dA}{dx} + \rho A \frac{du}{dx} = 0, \tag{3.5}$$

$$\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{A}\frac{dA}{dx} + \frac{1}{u}\frac{du}{dx} = 0. \tag{3.6}$$

Now integrate from x_1 to x_2 to get

$$\int_{x_1}^{x_2} \frac{d}{dx}\left(\rho A u\right)\, dx = \int_{x_1}^{x_2} 0\; dx, \tag{3.7}$$

$$\int_1^2 d\left(\rho A u\right) = 0, \tag{3.8}$$

$$\rho_2 u_2 A_2 - \rho_1 u_1 A_1 = 0, \tag{3.9}$$

$$\rho_2 u_2 A_2 = \rho_1 u_1 A_1 \equiv \dot{m} = C_1. \tag{3.10}$$

3.1.2 Linear momentum

Newton's Second Law says the time rate of change of linear momentum of a body equals the sum of the forces acting on the body. In the x direction this is roughly as follows:

$$\frac{d}{dt}(mu) = F_x. \tag{3.11}$$

In discrete form this would be

$$\frac{mu|_{t+\Delta t} - mu|_t}{\Delta t} = F_x, \tag{3.12}$$

$$mu|_{t+\Delta t} = mu|_t + F_x \Delta t. \tag{3.13}$$

For a control volume containing fluid, we must also account for the momentum which enters and leaves the control volume. The amount of momentum in a control volume after a time increment Δt is equal to the original amount of momentum plus that which came in minus that which left plus that introduced by the forces acting on the control volume.

Note that

- pressure force at surface 1 *pushes* fluid,
- pressure force at surface 2 *restrains* fluid,
- force due to the reaction of the wall to the pressure force *pushes* fluid if area change positive, and
- force due to the reaction of the wall to the shear force *restrains* fluid.

We write the linear momentum principle as

$$\begin{aligned}\left(\bar{\rho}\bar{A}\Delta x\right)\bar{u}\big|_{t+\Delta t} = &\left(\bar{\rho}\bar{A}\Delta x\right)\bar{u}\big|_t \\ &+ \left(\rho_1 A_1 \left(u_1 \Delta t\right)\right) u_1 \\ &- \left(\rho_2 A_2 \left(u_2 \Delta t\right)\right) u_2 \\ &+ \left(p_1 A_1\right)\Delta t - \left(p_2 A_2\right)\Delta t \\ &+ \left(\bar{p}\left(A_2 - A_1\right)\right)\Delta t \\ &- \left(\tau_w \bar{\mathcal{L}} \Delta x\right)\Delta t. \end{aligned} \tag{3.14}$$

Rearrange and divide by $\Delta x \Delta t$ to get

$$\begin{aligned}&\frac{\bar{\rho}\bar{A}\bar{u}\big|_{t+\Delta t} - \bar{\rho}\bar{A}\bar{u}\big|_t}{\Delta t} + \frac{\rho_2 A_2 u_2^2 - \rho_1 A_1 u_1^2}{\Delta x} \\ &= -\frac{p_2 A_2 - p_1 A_1}{\Delta x} + \bar{p}\frac{A_2 - A_1}{\Delta x} - \tau_w \bar{\mathcal{L}}. \end{aligned} \tag{3.15}$$

In the limit $\Delta x \to 0, \Delta t \to 0$ we get

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}\left(\rho A u^2\right) = -\frac{\partial}{\partial x}(pA) + p\frac{\partial A}{\partial x} - \tau_w \mathcal{L}. \tag{3.16}$$

In steady state we find

$$\frac{d}{dx}\left(\rho A u^2\right) = -\frac{d}{dx}(pA) + p\frac{dA}{dx} - \tau_w \mathcal{L}, \tag{3.17}$$

$$\rho A u \frac{du}{dx} + u \frac{d}{dx}(\rho A u) = -p\frac{dA}{dx} - A\frac{dp}{dx} + p\frac{dA}{dx} - \tau_w \mathcal{L}, \tag{3.18}$$

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} - \tau_w \frac{\mathcal{L}}{A}, \tag{3.19}$$

$$\rho u \; du + dp = -\tau_w \frac{\mathcal{L}}{A} \; dx, \tag{3.20}$$

$$du + \frac{1}{\rho u} dp = -\tau_w \frac{\mathcal{L}}{\dot{m}} \; dx, \tag{3.21}$$

$$\rho d\left(\frac{u^2}{2}\right) + dp = -\tau_w \frac{\mathcal{L}}{A} \; dx. \tag{3.22}$$

Wall shear lowers the combination of pressure and dynamic head.

If there is no wall shear, then Eq. (3.22) reduces to

$$dp = -\rho d\left(\frac{u^2}{2}\right). \tag{3.23}$$

An increase in velocity magnitude decreases the pressure.

With no friction $\tau_w \equiv 0$ we can take Eq. (3.19) as have

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = 0. \tag{3.24}$$

Now for flow with no area change $dA = 0$, the steady mass equation, Eq. (3.4), reduces to $d/dx(\rho u) = 0$. The product of u and this mass equation, $u \; d/dx\,(\rho u) = (u)(0) = 0$, can be added to Eq. (3.24) to get

$$\rho u \frac{du}{dx} + u\frac{d}{dx}(\rho u) + \frac{dp}{dx} = 0, \tag{3.25}$$

$$\frac{d}{dx}\left(\rho u^2 + p\right) = 0, \tag{3.26}$$

$$\rho u^2 + p = \rho_o u_o^2 + p_o = C_2. \tag{3.27}$$

3.1.3 Energy

The first law of thermodynamics states that the change of total energy of a body equals the heat transferred to the body minus the work done by the body:

$$E_2 - E_1 = Q - W, \tag{3.28}$$

$$E_2 = E_1 + Q - W. \tag{3.29}$$

So for our control volume this becomes the following when we also account for the energy flux in and out of the control volume in addition to the work and heat transfer:

$$\left(\bar{\rho}\bar{A}\Delta x\right)\left(\bar{e} + \frac{\bar{u}^2}{2}\right)\bigg|_{t+\Delta t} = \left(\bar{\rho}\bar{A}\Delta x\right)\left(\bar{e} + \frac{\bar{u}^2}{2}\right)\bigg|_{t}$$

$$+\rho_1 A_1 (u_1 \Delta t)\left(e_1 + \frac{u_1^2}{2}\right) - \rho_2 A_2 (u_2 \Delta t)\left(e_2 + \frac{u_2^2}{2}\right)$$
$$+q_w \left(\bar{\mathcal{L}}\Delta x\right)\Delta t + (p_1 A_1)(u_1 \Delta t) - (p_2 A_2)(u_2 \Delta t). \quad (3.30)$$

Note:

- the mean pressure times area difference does no work because it is acting on a stationary boundary, and
- the work done by the wall shear force is not included.[1]

Rearrange and divide by $\Delta t \Delta x$:

$$\frac{\bar{\rho}\bar{A}\left(\bar{e} + \frac{\bar{u}^2}{2}\right)\Big|_{t+\Delta t} - \bar{\rho}\bar{A}\left(\bar{e} + \frac{\bar{u}^2}{2}\right)\Big|_{t}}{\Delta t}$$
$$+\frac{\rho_2 A_2 u_2\left(e_2 + \frac{u_2^2}{2} + \frac{p_2}{\rho_2}\right) - \rho_1 A_1 u_1\left(e_1 + \frac{u_1^2}{2} + \frac{p_1}{\rho_1}\right)}{\Delta x} = q_w \bar{\mathcal{L}}. \quad (3.31)$$

In differential form as $\Delta x \to 0, \Delta t \to 0$

$$\frac{\partial}{\partial t}\left(\rho A\left(e + \frac{u^2}{2}\right)\right) + \frac{\partial}{\partial x}\left(\rho A u\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right)\right) = q_w \mathcal{L}. \quad (3.32)$$

In steady state:

$$\frac{d}{dx}\left(\rho A u\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right)\right) = q_w \mathcal{L}, \quad (3.33)$$

$$\rho A u \frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) + \left(e + \frac{u^2}{2} + \frac{p}{\rho}\right)\frac{d}{dx}(\rho A u) = q_w \mathcal{L}, \quad (3.34)$$

$$\rho u \frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) = \frac{q_w \mathcal{L}}{A}, \quad (3.35)$$

$$\rho u\left(\frac{de}{dx} + u\frac{du}{dx} + \frac{1}{\rho}\frac{dp}{dx} - \frac{p}{\rho^2}\frac{d\rho}{dx}\right) = \frac{q_w \mathcal{L}}{A}. \quad (3.36)$$

Now consider the product of velocity and momentum from Eq. (3.19) to get

$$\rho u^2 \frac{du}{dx} + u\frac{dp}{dx} = -\frac{\tau_w \mathcal{L} u}{A}. \quad (3.37)$$

[1]In neglecting work done by the wall shear force, I have taken an approach which is nearly universal, but fundamentally difficult to defend. At this stage of the development of these notes, I am not ready to enter into a grand battle with all established authors and probably confuse the student; consequently, results for flow with friction will be consistent with those of other sources. The argument typically used to justify this is that the real fluid satisfies no-slip at the boundary; thus, the wall shear actually does no work. However, one can easily argue that within the context of the one-dimensional model which has been posed that the shear force behaves as an external force which reduces the fluid's mechanical energy. Moreover, it is possible to show that neglect of this term results in the loss of frame invariance, a serious defect indeed. To model the work of the wall shear, one would include the term $\left(\tau_w\left(\bar{\mathcal{L}}\Delta x\right)\right)(\bar{u}\Delta t)$ in the energy equation.

Subtract this from Eq. (3.36) to get

$$\rho u \frac{de}{dx} - \frac{pu}{\rho}\frac{d\rho}{dx} = \frac{q_w \mathcal{L}}{A} + \frac{\tau_w \mathcal{L} u}{A}, \tag{3.38}$$

$$\frac{de}{dx} - \frac{p}{\rho^2}\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\dot{m}}. \tag{3.39}$$

Since $e = e(p, \rho)$

$$de = \left.\frac{\partial e}{\partial \rho}\right|_p d\rho + \left.\frac{\partial e}{\partial p}\right|_\rho dp, \tag{3.40}$$

$$\frac{de}{dx} = \left.\frac{\partial e}{\partial \rho}\right|_p \frac{d\rho}{dx} + \left.\frac{\partial e}{\partial p}\right|_\rho \frac{dp}{dx}. \tag{3.41}$$

so the steady energy equation becomes

$$\left.\frac{\partial e}{\partial \rho}\right|_p \frac{d\rho}{dx} + \left.\frac{\partial e}{\partial p}\right|_\rho \frac{dp}{dx} - \frac{p}{\rho^2}\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\dot{m}}, \tag{3.42}$$

$$\frac{dp}{dx} + \left(\frac{\left.\frac{\partial e}{\partial \rho}\right|_p - \frac{p}{\rho^2}}{\left.\frac{\partial e}{\partial p}\right|_\rho} \right) \frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\dot{m}\, \left.\frac{\partial e}{\partial p}\right|_\rho}. \tag{3.43}$$

Now let us consider the term in braces in the previous equation. We can put that term in a more common form by considering the Gibbs equation, Eq. (1.541):

$$T\ ds = de - \frac{p}{\rho^2}\ d\rho, \tag{3.44}$$

along with a general caloric equation of state $e = e(p, \rho)$, from which we get

$$de = \left.\frac{\partial e}{\partial p}\right|_\rho dp + \left.\frac{\partial e}{\partial \rho}\right|_p d\rho. \tag{3.45}$$

Substituting into the Gibbs equation, we get

$$T\ ds = \left.\frac{\partial e}{\partial p}\right|_\rho dp + \left.\frac{\partial e}{\partial \rho}\right|_p d\rho - \frac{p}{\rho^2} d\rho. \tag{3.46}$$

Taking s to be constant and dividing by $d\rho$, we get

$$0 = \left.\frac{\partial e}{\partial p}\right|_\rho \left.\frac{\partial p}{\partial \rho}\right|_s + \left.\frac{\partial e}{\partial \rho}\right|_p - \frac{p}{\rho^2}. \tag{3.47}$$

Rearranging, we get

$$\left.\frac{\partial p}{\partial \rho}\right|_s \equiv c^2 = - \left(\frac{\left.\frac{\partial e}{\partial \rho}\right|_p - \frac{p}{\rho^2}}{\left.\frac{\partial e}{\partial p}\right|_\rho} \right), \tag{3.48}$$

so

$$\frac{dp}{dx} - c^2 \frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\dot{m}\,\left.\frac{\partial e}{\partial p}\right|_\rho}, \tag{3.49}$$

$$\frac{dp}{dx} - c^2 \frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\rho u A\,\left.\frac{\partial e}{\partial p}\right|_\rho}. \tag{3.50}$$

In the above c is the isentropic sound speed, a thermodynamic property of the material. We shall see in a later section why it is appropriate to interpret this property as the propagation speed of small disturbances. At this point, it should simply be thought of as a state property.

Consider now the special case of flow with no heat transfer $q_w \equiv 0$. We still allow area change and wall friction allowed (see earlier footnote):

$$\rho u \frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) = 0, \tag{3.51}$$

$$e + \frac{u^2}{2} + \frac{p}{\rho} = e_o + \frac{u_o^2}{2} + \frac{p_o}{\rho_o} = C_3, \tag{3.52}$$

$$h + \frac{u^2}{2} = h_o + \frac{u_o^2}{2} = C_3. \tag{3.53}$$

3.1.4 Summary of equations

We can summarize the one-dimensional compressible flow equations in various forms here. In the equations below, we assume $A(x)$, τ_w, q_w, and $\mathcal{L}$ are all known.

3.1.4.1 Unsteady conservative form

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0, \tag{3.54}$$

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}\left(\rho A u^2 + pA\right) = p\frac{\partial A}{\partial x} - \tau_w \mathcal{L}, \tag{3.55}$$

$$\frac{\partial}{\partial t}\left(\rho A\left(e + \frac{u^2}{2}\right)\right) + \frac{\partial}{\partial x}\left(\rho A u\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right)\right) = q_w \mathcal{L}, \tag{3.56}$$

$$e = e(\rho, p), \tag{3.57}$$

$$p = p(\rho, T). \tag{3.58}$$

3.1.4.2 Unsteady non-conservative form

$$\frac{d\rho}{dt} = -\frac{\rho}{A}\frac{\partial}{\partial x}(Au), \tag{3.59}$$

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \frac{\tau_w \mathcal{L}}{A}, \tag{3.60}$$
$$\rho \frac{de}{dt} = -p\frac{\partial u}{\partial x} + \frac{q_w \mathcal{L} - \tau_w \mathcal{L} u}{A}, \tag{3.61}$$
$$e = e(\rho, p), \tag{3.62}$$
$$p = p(\rho, T). \tag{3.63}$$

3.1.4.3 Steady conservative form

$$\frac{d}{dx}(\rho A u) = 0, \tag{3.64}$$
$$\frac{d}{dx}\left(\rho A u^2 + pA\right) = p\frac{dA}{dx} - \tau_w \mathcal{L}, \tag{3.65}$$
$$\frac{d}{dx}\left(\rho A u\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right)\right) = q_w \mathcal{L}, \tag{3.66}$$
$$e = e(\rho, p), \tag{3.67}$$
$$p = p(\rho, T). \tag{3.68}$$

3.1.4.4 Steady non-conservative form

$$u\frac{d\rho}{dx} = -\frac{\rho}{A}\frac{d}{dx}(Au), \tag{3.69}$$
$$\rho u \frac{du}{dx} = -\frac{dp}{dx} + \frac{\tau_w \mathcal{L}}{A}, \tag{3.70}$$
$$\rho u \frac{de}{dx} = -p\frac{du}{dx} + \frac{q_w \mathcal{L} - \tau_w \mathcal{L} u}{A}, \tag{3.71}$$
$$e = e(\rho, p), \tag{3.72}$$
$$p = p(\rho, T). \tag{3.73}$$

In whatever form we consider, we have five equations in five unknown dependent variables: ρ, u, p, e, and T. We can always use the thermal and caloric state equations to eliminate e and T to give rise to three equations in three unknowns.

Example 3.1

Flow of air with heat addition

Given: Air initially at $p_1 = 100$ kPa, $T_1 = 300$ K, $u_1 = 10$ m/s flows in a duct of length 100 m. The duct has a constant circular cross sectional area of $A = 0.02$ m^2 and is isobarically heated with a constant heat flux q_w along the entire surface of the duct. At the end of the duct the flow has $p_2 = 100$ kPa, $T_2 = 500$ K

Find: the mass flow rate $\dot{m}$, the wall heat flux q_w and the entropy change $s_2 - s_1$; check for satisfaction of the second law.

Assume: Calorically perfect ideal gas, $R = 0.287$ kJ/(kg K), $c_p = 1.0035$ kJ/(kg K)

Analysis:

$$A = \pi r^2, \tag{3.74}$$

$$r = \sqrt{\frac{A}{\pi}} \tag{3.75}$$

$$\mathcal{L} = 2\pi r = 2\sqrt{\pi A} = 2\sqrt{\pi\left(0.02 \text{ m}^2\right)} = 0.501 \text{ m}. \tag{3.76}$$

Now get the mass flux.

$$p_1 = \rho_1 R T_1, \tag{3.77}$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{100 \text{ kPa}}{\left(0.287 \frac{\text{kJ}}{\text{kg K}}\right)(300 \text{ K})}, \tag{3.78}$$

$$= 1.161 \frac{\text{kg}}{\text{m}^3} \tag{3.79}$$

So

$$\dot{m} = \rho_1 u_1 A_1 = \left(1.161 \frac{\text{kg}}{\text{m}^3}\right)\left(10 \frac{\text{m}}{\text{s}}\right)\left(0.02 \text{ m}^2\right) = 0.2322 \frac{\text{kg}}{\text{s}}. \tag{3.80}$$

Now get the flow variables at state 2:

$$\rho_2 = \frac{p_2}{RT_2}, = \frac{100 \text{ kPa}}{\left(0.287 \frac{\text{kJ}}{\text{kg K}}\right)(500 \text{ K})}, \tag{3.81}$$

$$= 0.6969 \frac{\text{kg}}{\text{m}^3}, \tag{3.82}$$

$$\rho_2 u_2 A_2 = \rho_1 u_1 A_1, \tag{3.83}$$

$$u_2 = \frac{\rho_1 u_1 A_1}{\rho_2 A_2} = \frac{\rho_1 u_1}{\rho_2}, \tag{3.84}$$

$$= \frac{\left(1.161 \frac{\text{kg}}{\text{m}^3}\right)\left(10 \frac{\text{m}}{\text{s}}\right)}{0.6969 \frac{\text{kg}}{\text{m}^3}} = 16.67 \frac{\text{m}}{\text{s}}. \tag{3.85}$$

Now consider the energy equation:

$$\rho u \frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) = \frac{q_w \mathcal{L}}{A}, \tag{3.86}$$

$$\frac{d}{dx}\left(h + \frac{u^2}{2}\right) = \frac{q_w \mathcal{L}}{\dot{m}}, \tag{3.87}$$

$$\int_0^L \frac{d}{dx}\left(h + \frac{u^2}{2}\right) dx = \int_0^L \frac{q_w \mathcal{L}}{\dot{m}} dx, \tag{3.88}$$

$$h_2 + \frac{u_2^2}{2} - h_1 - \frac{u_1^2}{2} = \frac{q_w L \mathcal{L}}{\dot{m}}, \tag{3.89}$$

$$c_p\left(T_2 - T_1\right) + \frac{u_2^2}{2} - \frac{u_1^2}{2} = \frac{q_w L \mathcal{L}}{\dot{m}}. \tag{3.90}$$

Solving for q_w, we get

$$
\begin{aligned}
q_w &= \left(\frac{\dot{m}}{L\mathcal{L}}\right)\left(c_p\left(T_2-T_1\right)+\frac{u_2^2}{2}-\frac{u_1^2}{2}\right), &(3.91)\\
&= \left(\frac{0.2322\ \frac{\text{kg}}{\text{s}}}{(100\ \text{m})\,(0.501\ \text{m})}\right)\left(\left(1003.5\ \frac{\text{J}}{\text{kg K}}\right)(500\ \text{K}-300\ \text{K})+\frac{\left(16.67\ \frac{\text{m}}{\text{s}}\right)^2}{2}-\frac{\left(10\ \frac{\text{m}}{\text{s}}\right)^2}{2}\right), &(3.92)\\
&= 0.004635\ \frac{\text{kg}}{\text{m}^2\ \text{s}}\left(200700\ \frac{\text{J}}{\text{kg}}-88.9\ \frac{\text{m}^2}{\text{s}^2}\right), &(3.93)\\
&= 0.004635\ \frac{\text{kg}}{\text{m}^2\ \text{s}}\left(200700\ \frac{\text{J}}{\text{kg}}-88.9\ \frac{\text{J}}{\text{kg}}\right), &(3.94)\\
&= 930\ \frac{\text{W}}{\text{m}^2}. &(3.95)
\end{aligned}
$$

The heat flux is positive, which indicates a transfer of thermal energy *into* the air.

Now find the entropy change.

$$
\begin{aligned}
s_2-s_1 &= c_p\ln\left(\frac{T_2}{T_1}\right)-R\ln\left(\frac{p_2}{p_1}\right), &(3.96)\\
s_2-s_1 &= \left(1003.5\ \frac{\text{J}}{\text{kg K}}\right)\ln\left(\frac{500\ \text{K}}{300\ \text{K}}\right)-\left(287\ \frac{\text{J}}{\text{kg K}}\right)\ln\left(\frac{100\ \text{kPa}}{100\ \text{kPa}}\right), &(3.97)\\
s_2-s_1 &= 512.6-0=512.6\ \frac{\text{J}}{\text{kg K}}. &(3.98)
\end{aligned}
$$

Is the second law satisfied? Assume the heat transfer takes place from a reservoir held at 500 K. The reservoir would have to be *at least* at 500 K in order to bring the fluid to its final state of 500 K. It could be greater than 500 K and still satisfy the second law.

$$
\begin{aligned}
S_2-S_1 &\geq \frac{Q_{12}}{T}, &(3.99)\\
\dot{S}_2-\dot{S}_1 &\geq \frac{\dot{Q}_{12}}{T}, &(3.100)\\
\dot{m}\left(s_2-s_1\right) &\geq \frac{\dot{Q}_{12}}{T}, &(3.101)\\
&\geq \frac{q_w A_{tot}}{T}, &(3.102)\\
&\geq \frac{q_w L\mathcal{L}}{T}, &(3.103)\\
s_2-s_1 &\geq \frac{q_w L\mathcal{L}}{\dot{m}T}, &(3.104)\\
512.6\ \frac{\text{J}}{\text{kg K}} &\geq \frac{\left(930\ \frac{\text{J}}{\text{s m}^2}\right)(100\ \text{m})\,(0.501\ \text{m})}{\left(0.2322\ \frac{\text{kg}}{\text{s}}\right)(500\ \text{K})}, &(3.105)\\
512.6\ \frac{\text{J}}{\text{kg K}} &\geq 401.3\ \frac{\text{J}}{\text{kg K}}. &(3.106)
\end{aligned}
$$

3.1.5 Influence coefficients

Now, let us uncouple the steady one-dimensional equations. First let us summarize again, in a slightly different manner than before:

$$u\frac{d\rho}{dx} + \rho\frac{du}{dx} = -\frac{\rho u}{A}\frac{dA}{dx}, \tag{3.107}$$

$$\rho u\frac{du}{dx} + \frac{dp}{dx} = -\frac{\tau_w \mathcal{L}}{A}, \tag{3.108}$$

$$\frac{dp}{dx} - c^2\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\rho u A \left.\frac{\partial e}{\partial p}\right|_\rho}. \tag{3.109}$$

In matrix form this is

$$\begin{pmatrix} u & \rho & 0 \\ 0 & \rho u & 1 \\ -c^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{d\rho}{dx} \\ \frac{du}{dx} \\ \frac{dp}{dx} \end{pmatrix} = \begin{pmatrix} -\frac{\rho u}{A}\frac{dA}{dx} \\ -\frac{\tau_w \mathcal{L}}{A} \\ \frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A \left.\frac{\partial e}{\partial p}\right|_\rho} \end{pmatrix}. \tag{3.110}$$

Use Cramer's Rule to solve for the derivatives. First calculate the determinant of the coefficient matrix:

$$u\left((\rho u)(1) - (1)(0)\right) - \rho\left((0)(1) - (-c^2)(1)\right) = \rho\left(u^2 - c^2\right). \tag{3.111}$$

Implementing Cramer's Rule:

$$\frac{d\rho}{dx} = \frac{\rho u\left(-\frac{\rho u}{A}\frac{dA}{dx}\right) - \rho\left(-\frac{\tau_w\mathcal{L}}{A}\right) + \rho\left(\frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A \left.\frac{\partial e}{\partial p}\right|_\rho}\right)}{\rho\left(u^2 - c^2\right)}, \tag{3.112}$$

$$\frac{du}{dx} = \frac{-c^2\left(-\frac{\rho u}{A}\frac{dA}{dx}\right) + u\left(-\frac{\tau_w\mathcal{L}}{A}\right) - u\left(\frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A \left.\frac{\partial e}{\partial p}\right|_\rho}\right)}{\rho\left(u^2 - c^2\right)}, \tag{3.113}$$

$$\frac{dp}{dx} = \frac{\rho u c^2\left(-\frac{\rho u}{A}\frac{dA}{dx}\right) - \rho c^2\left(-\frac{\tau_w\mathcal{L}}{A}\right) + \rho u^2\left(\frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A \left.\frac{\partial e}{\partial p}\right|_\rho}\right)}{\rho\left(u^2 - c^2\right)}. \tag{3.114}$$

Simplify to find

$$\frac{d\rho}{dx} = \frac{1}{A}\frac{-\rho u^2\frac{dA}{dx} + \tau_w\mathcal{L} + \frac{(q_w+\tau_w u)\mathcal{L}}{\rho u \left.\frac{\partial e}{\partial p}\right|_\rho}}{(u^2 - c^2)}, \tag{3.115}$$

$$\frac{du}{dx} = \frac{1}{A}\frac{c^2\rho u\frac{dA}{dx} - u\tau_w\mathcal{L} - \frac{(q_w+\tau_w u)\mathcal{L}}{\rho \left.\frac{\partial e}{\partial p}\right|_\rho}}{\rho\left(u^2 - c^2\right)}. \tag{3.116}$$

$$\frac{dp}{dx} = \frac{1}{A}\frac{-c^2\rho u^2\frac{dA}{dx} + c^2\tau_w\mathcal{L} + \frac{(q_w+\tau_w u)\mathcal{L}u}{\rho \left.\frac{\partial e}{\partial p}\right|_\rho}}{(u^2 - c^2)}. \tag{3.117}$$

Note, we have

- a system of coupled non-linear ordinary differential equations,
- in standard form for dynamic system analysis: $d\mathbf{u}/dx = \mathbf{f}(\mathbf{u})$,
- valid for *general* equations of state, and
- *singular* when velocity sonic $u = c$.

3.2 Flow with area change

This section will consider flow with area change with an emphasis on isentropic flow. Some problems will involve non-isentropic flow but a detailed discussion of such flows will be delayed.

3.2.1 Isentropic Mach number relations

Take the special case of

- $\tau_w = 0$,
- $q_w = 0$,
- calorically perfect ideal gas (CPIG).

Then

$$\frac{d}{dx}(\rho u A) = 0, \tag{3.118}$$

$$\frac{d}{dx}\left(\rho u^2 + p\right) = 0, \tag{3.119}$$

$$\frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{p}{\rho}\right) = 0. \tag{3.120}$$

Integrate the energy equation using Eq. (1.531) $h = e + p/\rho$ to get

$$h + \frac{u^2}{2} = h_o + \frac{u_o^2}{2}. \tag{3.121}$$

If we define the “o” condition to be a condition of rest, then $u_o \equiv 0$. This is a *stagnation* condition. So

$$h + \frac{u^2}{2} = h_o, \tag{3.122}$$

$$(h - h_o) + \frac{u^2}{2} = 0. \tag{3.123}$$

Since we have a CPIG,

$$c_p \left(T - T_o\right) + \frac{u^2}{2} = 0, \tag{3.124}$$

$$T - T_o + \frac{u^2}{2c_p} = 0, \tag{3.125}$$

$$1 - \frac{T_o}{T} + \frac{u^2}{2c_p T} = 0. \tag{3.126}$$

Now note that

$$c_p = c_p \frac{c_p - c_v}{c_p - c_v} = \frac{c_p}{c_v} \frac{c_p - c_v}{\frac{c_p}{c_v} - 1} = \frac{\gamma R}{\gamma - 1}, \tag{3.127}$$

so

$$1 - \frac{T_o}{T} + \frac{\gamma - 1}{2} \frac{u^2}{\gamma R T} = 0, \tag{3.128}$$

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} \frac{u^2}{\gamma R T}. \tag{3.129}$$

Recall the sound speed and Mach number for a CPIG:

$$c^2 = \gamma R T \qquad \text{if} \qquad p = \rho R T, \qquad e = c_v T + e_o, \tag{3.130}$$

$$M^2 \equiv \left(\frac{u}{c}\right)^2. \tag{3.131}$$

Thus,

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2, \tag{3.132}$$

$$\frac{T}{T_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1}. \tag{3.133}$$

Now if the flow is isentropic, we have

$$\frac{T}{T_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma - 1} = \left(\frac{p}{p_o}\right)^{\frac{\gamma - 1}{\gamma}}. \tag{3.134}$$

Thus,

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{\gamma - 1}}, \tag{3.135}$$

$$\frac{p}{p_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma}{\gamma - 1}}. \tag{3.136}$$

For air $\gamma = 7/5$, so

$$\frac{T}{T_o} = \left(1 + \frac{1}{5}M^2\right)^{-1}, \tag{3.137}$$

$$\frac{\rho}{\rho_o} = \left(1 + \frac{1}{5}M^2\right)^{-\frac{5}{2}}, \tag{3.138}$$

$$\frac{p}{p_o} = \left(1 + \frac{1}{5}M^2\right)^{-\frac{7}{2}}. \tag{3.139}$$

Figures 3.2, 3.3 3.4 show the variation of T, ρ and p with M^2 for isentropic flow.

Other thermodynamic properties can be determined from these, e.g. the sound speed:

$$\frac{c}{c_o} = \sqrt{\frac{\gamma RT}{\gamma RT_o}} = \sqrt{\frac{T}{T_o}} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{-1/2}. \tag{3.140}$$

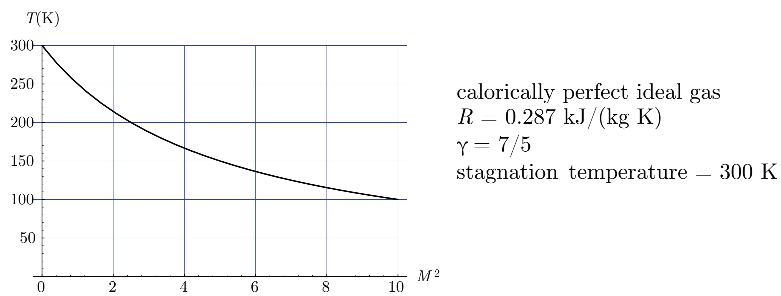

Figure 3.2: Static temperature versus Mach number squared.

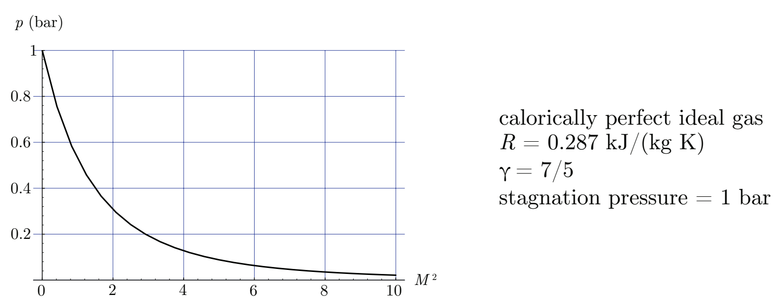

Figure 3.3: Static pressure versus Mach number squared.

Example 3.2

Airplane problem

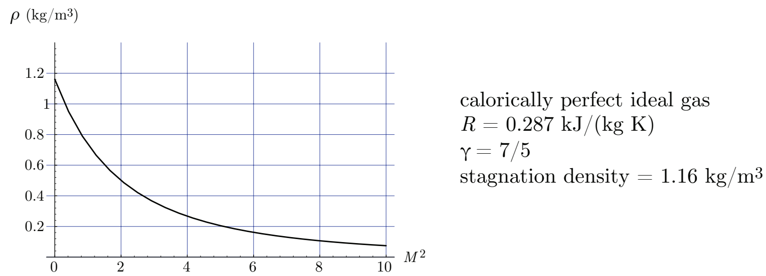

Figure 3.4: Static density versus Mach number squared.

Given: An airplane is flying into still air at $u = 200$ m/s. The ambient air is at 288 K and 101.3 kPa.
Find: Temperature, pressure, and density at nose of airplane
Assume: Steady isentropic flow of C.P.I.G.

Analysis: In the steady wave frame, the ambient conditions are *static* while the nose conditions are *stagnation*.

$$M = \frac{u}{c} = \frac{u}{\sqrt{\gamma RT}} = \frac{200\ \frac{\mathrm{m}}{\mathrm{s}}}{\sqrt{\frac{7}{5}\left(287\ \frac{\mathrm{J}}{\mathrm{kg\ K}}\right) 288\ \mathrm{K}}} = 0.588 \tag{3.141}$$

so

$$T_o = T\left(1 + \frac{1}{5}M^2\right) = (288\ \mathrm{K})\left(1 + \frac{1}{5}0.588^2\right) = 307.9\ \mathrm{K} \tag{3.142}$$

$$\rho_o = \rho\left(1 + \frac{1}{5}M^2\right)^{\frac{5}{2}} = \frac{101.3\ \mathrm{kPa}}{\left(0.287\ \frac{\mathrm{kJ}}{\mathrm{kg\ K}}\right)(288\ \mathrm{K})\left(1 + \frac{1}{5}0.588^2\right)^{\frac{5}{2}}} = 1.45\ \frac{kg}{m^3} \tag{3.143}$$

$$p_o = p\left(1 + \frac{1}{5}M^2\right)^{\frac{7}{2}} = (101.3\ \mathrm{kPa})\left(1 + \frac{1}{5}0.588^2\right)^{\frac{7}{2}} = 128\ \mathrm{kPa} \tag{3.144}$$

Note the temperature, pressure, and density all rise in the isentropic process. In this wave frame, the kinetic energy of the flow is being converted isentropically to thermal energy.

3.2.2 Sonic properties

Let "*" denote a property at the sonic state $M^2 \equiv 1$

$$\frac{T_*}{T_o} = \left(1 + \frac{\gamma - 1}{2}1^2\right)^{-1} = \frac{2}{\gamma + 1}, \tag{3.145}$$

$$\frac{\rho_*}{\rho_o} = \left(1 + \frac{\gamma - 1}{2}1^2\right)^{-\frac{1}{\gamma-1}} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma-1}}, \tag{3.146}$$

$$\frac{p_*}{p_o} = \left(1+\frac{\gamma-1}{2}1^2\right)^{-\frac{\gamma}{\gamma-1}} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}, \tag{3.147}$$

$$\frac{c_*}{c_o} = \left(1+\frac{\gamma-1}{2}1^2\right)^{-1/2} = \sqrt{\frac{2}{\gamma+1}}, \tag{3.148}$$

$$u_* = c_* = \sqrt{\gamma R T_*} = \sqrt{\frac{2\gamma}{\gamma+1}RT_o}. \tag{3.149}$$

If the fluid is air, we have $\gamma = 7/5$ and

$$\frac{T_*}{T_o} = 0.8333, \tag{3.150}$$

$$\frac{\rho_*}{\rho_o} = 0.6339, \tag{3.151}$$

$$\frac{p_*}{p_o} = 0.5283, \tag{3.152}$$

$$\frac{c_*}{c_o} = 0.9123. \tag{3.153}$$

3.2.3 Effect of area change

To understand the effect of area change, the influence of the mass equation must be considered. So far we have really only looked at energy. In the isentropic limit the mass, momentum, and energy equation for a C.P.I.G. reduce to

$$\frac{d\rho}{\rho}+\frac{du}{u}+\frac{dA}{A} = 0, \tag{3.154}$$

$$\rho u\ du + dp = 0, \tag{3.155}$$

$$\frac{dp}{p} = \gamma\frac{d\rho}{\rho}. \tag{3.156}$$

Substitute energy, then mass into momentum:

$$\rho u\ du + \gamma\frac{p}{\rho}d\rho = 0, \tag{3.157}$$

$$\rho u\ du + \gamma\frac{p}{\rho}\left(-\frac{\rho}{u}\ du - \frac{\rho}{A}\ dA\right) = 0, \tag{3.158}$$

$$du + \gamma\frac{p}{\rho}\left(-\frac{1}{u^2}\ du - \frac{1}{uA}\ dA\right) = 0, \tag{3.159}$$

$$du\left(1-\frac{\gamma p/\rho}{u^2}\right) = \gamma\frac{p}{\rho}\frac{dA}{uA}, \tag{3.160}$$

$$\frac{du}{u}\left(1-\frac{\gamma p/\rho}{u^2}\right) = \frac{\gamma p/\rho}{u^2}\frac{dA}{A}, \tag{3.161}$$

$$\frac{du}{u}\left(1-\frac{1}{M^2}\right) = \frac{1}{M^2}\frac{dA}{A}, \tag{3.162}$$

$$\frac{du}{u}\left(M^2-1\right) = \frac{dA}{A}, \tag{3.163}$$
$$\frac{du}{u} = \frac{1}{M^2-1}\frac{dA}{A}. \tag{3.164}$$

Figure 3.5 gives show the performance of a fluid in a variable area duct. We note

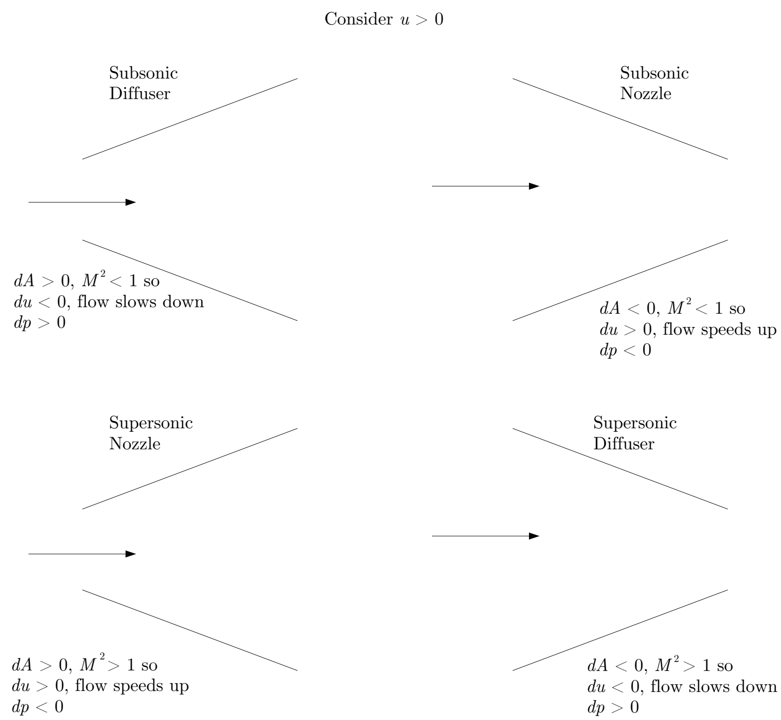

Figure 3.5: Behavior of fluid in sub- and supersonic nozzles and diffusers.

- there is a singularity when $M^2 = 1$,
- if $M^2 = 1$, we need $dA = 0$,
- area minimum necessary to transition from subsonic to supersonic flow,
- it can be shown an area maximum is not relevant.

Consider A at a sonic state. From the mass equation:

$$\rho u A = \rho_* u_* A_*, \tag{3.165}$$
$$\rho u A = \rho_* c_* A_*, \tag{3.166}$$
$$\frac{A}{A_*} = \frac{\rho_*}{\rho} c_* \frac{1}{u} = \frac{\rho_*}{\rho}\sqrt{\gamma R T_*}\frac{1}{u} = \frac{\rho_*}{\rho}\frac{\sqrt{\gamma R T_*}}{\sqrt{\gamma R T}}\frac{\sqrt{\gamma R T}}{u}, \tag{3.167}$$
$$\frac{A}{A_*} = \frac{\rho_*}{\rho}\sqrt{\frac{T_*}{T}}\frac{1}{M} = \frac{\rho_*}{\rho_o}\frac{\rho_o}{\rho}\sqrt{\frac{T_* T_o}{T_o T}}\frac{1}{M}. \tag{3.168}$$

Substitute from earlier-developed relations and get

$$\frac{A}{A_*} = \frac{1}{M}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}. \quad (3.169)$$

Figure 3.6 shows the performance of a fluid in a variable area duct.

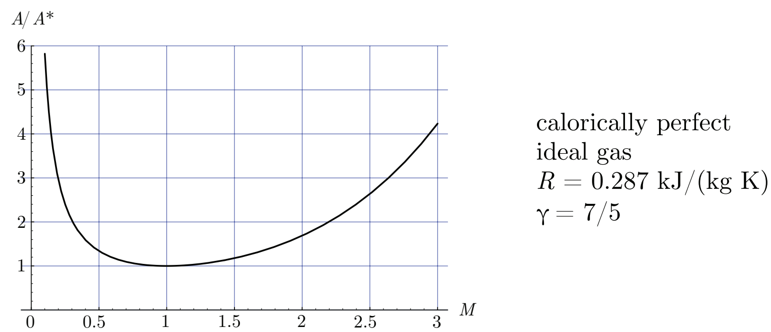

Figure 3.6: Area versus Mach number for a calorically perfect ideal gas.

Note that

- A/A_* has a minimum value of 1 at $M = 1$,
- For each $A/A* > 1$, there exist *two* values of M, and
- $A/A_* \to \infty$ as $M \to 0$ or $M \to \infty$.

3.2.4 Choking

Consider mass flow rate variation with pressure difference. We have then

- small pressure difference gives small velocity and small mass flow,
- as pressure difference grows, velocity and mass flow rate grow,
- velocity is limited to sonic at a particular duct location,
- this provides fundamental restriction on mass flow rate,
- it can be proven rigorously that sonic condition gives maximum mass flow rate.

$$\dot{m}_{max} = \rho_* u_* A_*, \quad (3.170)$$

$$\textit{if}\text{ ideal gas:} \quad = \; = \rho_o\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left(\sqrt{\frac{2\gamma}{\gamma+1}RT_o}\right)A_*, \quad (3.171)$$

$$= \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\frac{2}{\gamma+1}\right)^{1/2} \sqrt{\gamma R T_o} A_*, \quad (3.172)$$

$$= \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o} A_*. \quad (3.173)$$

A flow which has a maximum mass flow rate is known as **choked** flow. Flows will choke at area minima in a duct.

Example 3.3

Isentropic area change problem with choking [2]

Given: Air with stagnation conditions $p_o = 200$ kPa $T_o = 500$ K flows through a throat to an exit Mach number of 2.5. The desired mass flow is 3.0 kg/s,

Find: a) throat area, b) exit pressure, c) exit temperature, d) exit velocity, and e) exit area.

Assume: C.P.I.G., isentropic flow, $\gamma = 7/5$

Analysis:

$$\rho_o = \frac{p_o}{R T_o} = \frac{200 \text{ kPa}}{\left(0.287 \frac{\text{kJ}}{\text{kg K}}\right)(500 \text{ K})} = 1.394 \frac{\text{kg}}{\text{m}^3}. \quad (3.174)$$

Since it necessarily flows through a sonic throat:

$$\dot{m}_{max} = \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o} A_*, \quad (3.175)$$

$$A_* = \frac{\dot{m}_{max}}{\rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o}}, \quad (3.176)$$

$$A_* = \frac{3 \frac{\text{kg}}{\text{s}}}{\left(1.394 \frac{\text{kg}}{\text{m}^3}\right)(0.5787)\sqrt{1.4\left(287 \frac{\text{J}}{\text{kg K}}\right)(500 \text{ K})}} = 0.008297 \text{ m}^2. \quad (3.177)$$

Since we know M_e use isentropic relations to find other exit conditions.

$$p_e = p_o \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{-\frac{\gamma}{\gamma-1}} = (200 \text{ kPa})\left(1 + \frac{1}{5} 2.5^2\right)^{-3.5} = 11.71 \text{ kPa}, \quad (3.178)$$

$$T_e = T_o \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{-1} = (500 \text{ K})\left(1 + \frac{1}{5} 2.5^2\right)^{-1} = 222.2 \text{ K}. \quad (3.179)$$

Note

$$\rho_e = \frac{p_e}{R T_e} = \frac{11.71 \text{ kPa}}{\left(0.287 \frac{\text{kJ}}{\text{kg K}}\right)(222.2 \text{ K})} = 0.1834 \frac{\text{kg}}{\text{m}^3}. \quad (3.180)$$

[2] adopted from White, *Fluid Mechanics* McGraw-Hill: New York, 1986, p. 529, Ex. 9.5

Now the exit velocity is simply

$$u_e = M_e c_e = M_e\sqrt{\gamma R T_e} = 2.5\sqrt{1.4\left(287\ \frac{\text{J}}{\text{kg K}}\right)(222.2\ \text{K})} = 747.0\ \frac{\text{m}}{\text{s}}. \tag{3.181}$$

Now determine the exit area.

$$A = \frac{A_*}{M_e}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}, \tag{3.182}$$

$$= \frac{0.008297\ \text{m}^2}{2.5}\left(\frac{5}{6}\left(1+\frac{1}{5}2.5^2\right)\right)^3 = 0.0219\ \text{m}^2. \tag{3.183}$$

3.3 Normal shock waves

This section will develop relations for normal shock waves in fluids with general equations of state. It will be specialized to calorically perfect ideal gases to illustrate the general features of the waves.

Assume for this section we have

- one-dimensional flow,
- steady flow,
- no area change,
- viscous effects and wall friction do not have time to influence flow, and
- heat conduction and wall heat transfer do not have time to influence flow.

We will consider the problem in the context of the piston problem as sketched in Figure 3.7.

The physical problem is as follows:

- Drive a piston with known velocity v_p into a fluid at rest ($v_1 = 0$) with known properties, p_1, ρ_1 in the x^* laboratory frame,
- Determine the disturbance speed D,
- Determine the disturbance properties v_2, p_2, ρ_2,
- in this frame of reference we have an *unsteady* problem.

Transformed Problem:

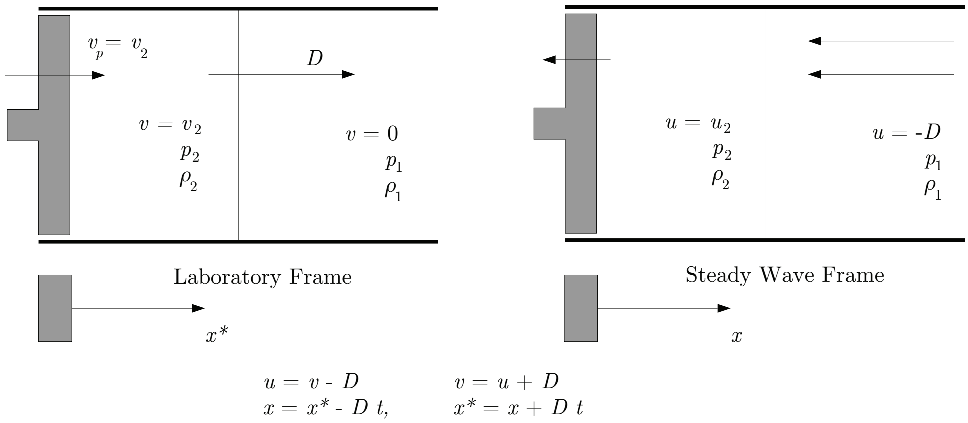

Figure 3.7: Normal shock sketch.

- use a Galilean transformation $x = x^* - Dt$, $u = v - D$ to transform to the frame in which the wave is at rest, therefore rending the problem *steady* in this frame,
- solve as though D is known to get downstream "2" conditions: $u_2(D), p_2(D), ...$,
- invert to solve for D as function of u_2, the transformed piston velocity: $D(u_2)$,
- back transform to get all variables as function of v_2, the laboratory piston velocity: $D(v_2), p_2(v_2), \rho_2(v_2), ...$

3.3.1 Rankine-Hugoniot equations

Under these assumptions the conservation principles in conservative form and equation of state are in the steady frame as follows:

$$\frac{d}{dx}(\rho u) = 0, \tag{3.184}$$

$$\frac{d}{dx}\left(\rho u^2 + p\right) = 0, \tag{3.185}$$

$$\frac{d}{dx}\left(\rho u\left(h + \frac{u^2}{2}\right)\right) = 0, \tag{3.186}$$

$$h = h(p, \rho). \tag{3.187}$$

Upstream conditions are $\rho = \rho_1$, $p = p_1$, $u = -D$. With knowledge of the equation of state, we get $h = h_1$. In what is a natural, but in fact *naive*, step we can integrate the equations from upstream to state "2" to give the correct *Rankine-Hugoniot jump equations.*[3][4]

$$\rho_2 u_2 = -\rho_1 D, \quad (3.188)$$

$$\rho_2 u_2^2 + p_2 = \rho_1 D^2 + p_1, \quad (3.189)$$

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{D^2}{2}, \quad (3.190)$$

$$h_2 = h(p_2, \rho_2). \quad (3.191)$$

This analysis is straightforward and yields the correct result. In actuality, however, the analysis should be more nuanced. We are going to solve these algebraic equations to arrive at *discontinuous* shock jumps. Thus, we should be concerned about the validity of of differential equations in the vicinity of a discontinuity.

As described by LeVeque,[5] the proper way to arrive at the shock jump equations is to use a more primitive form of the conservation laws, expressed in terms of integrals of conserved quantities balanced by fluxes of those quantities. If $\mathbf{q}$ is a set of conserved variables, and $\mathbf{f}(\mathbf{q})$ is the flux of $\mathbf{q}$ (e.g. for mass conservation, ρ is a conserved variable and ρu is the flux), then the primitive form of the conservation law can be written as

$$\frac{d}{dt}\int_{x_1}^{x_2} \mathbf{q}(x,t)\ dx = \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)). \quad (3.192)$$

Here we have considered flow into and out of a one-dimensional box for $x \in [x_1, x_2]$. For our Euler equations we have

$$\mathbf{q} = \left(\begin{array}{c} \rho \\ \rho u \\ \rho\left(e + \frac{1}{2}u^2\right) \end{array}\right), \qquad \mathbf{f}(\mathbf{q}) = \left(\begin{array}{c} \rho u \\ \rho u^2 + p \\ \rho u\left(e + \frac{1}{2}u^2 + \frac{p}{\rho}\right) \end{array}\right). \quad (3.193)$$

If we assume there is a discontinuity in the region $x \in [x_1, x_2]$ propagating at speed D, we can break up the integral into the form

$$\frac{d}{dt}\int_{x_1}^{x_1+Dt^-} \mathbf{q}(x,t)\ dx + \frac{d}{dt}\int_{x_1+Dt^+}^{x_2} \mathbf{q}(x,t)\ dx = \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)). \quad (3.194)$$

Here $x_1 + Dt^-$ lies just before the discontinuity and $x_1 + Dt^+$ lies just past the discontinuity. Using Leibniz's rule, we get

$$\mathbf{q}(x_1 + Dt^-, t)D + 0 + \int_{x_1}^{x_1+Dt^-} \frac{\partial \mathbf{q}}{\partial t}\ dx + 0 - \mathbf{q}(x_1 + Dt^+, t)D + \int_{x_1+Dt^+}^{x_2} \frac{\partial \mathbf{q}}{\partial t}\ dx \quad (3.195)$$
$$= \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)).$$

[3]William John Macquorn Rankine, 1820-1872, Scottish engineer and mechanician, pioneer of thermodynamics and steam engine theory, taught at University of Glasgow, studied fatigue in railway engine axles.

[4]Pierre Henri Hugoniot, 1851-1887, French engineer.

[5]LeVeque, R. J., 1992, *Numerical Methods for Conservation Laws*, Birkhäuser, Basel.

Now if we assume that on either side of the discontinuity the volume of integration is sufficiently small so that the time and space variation of $\mathbf{q}$ is negligibly small, we get

$$\begin{aligned}\mathbf{q}(x_1)D - \mathbf{q}(x_2)D &= \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)), && (3.196)\\ D\left(\mathbf{q}(x_1) - \mathbf{q}(x_2)\right) &= \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)). && (3.197)\end{aligned}$$

Defining next the notation for a jump as

$$[\![\mathbf{q}(x)]\!] \equiv \mathbf{q}(x_2) - \mathbf{q}(x_1), \tag{3.198}$$

the jump conditions are rewritten as

$$D\,[\![\mathbf{q}(x)]\!] = [\![\mathbf{f}(\mathbf{q}(x))]\!]\,. \tag{3.199}$$

If $D = 0$, as is the case when we transform to the frame where the wave is at rest, we simply recover

$$\begin{aligned}0 &= \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)), && (3.200)\\ \mathbf{f}(\mathbf{q}(x_1)) &= \mathbf{f}(\mathbf{q}(x_2)), && (3.201)\\ [\![\mathbf{f}(\mathbf{q}(x))]\!] &= 0. && (3.202)\end{aligned}$$

That is the fluxes on either side of the discontinuity are equal. This is precisely what we obtained by our naive analysis. We also get a more general result for $D \neq 0$, which is the well-known

$$D = \frac{\mathbf{f}(\mathbf{q}(x_2)) - \mathbf{f}(\mathbf{q}(x_1))}{\mathbf{q}(x_2) - \mathbf{q}(x_1)} = \frac{[\![\mathbf{f}(\mathbf{q}(x))]\!]}{[\![\mathbf{q}(x)]\!]}. \tag{3.203}$$

The general Rankine-Hugoniot equation then for the one-dimensional Euler equations across a non-stationary jump is given by

$$D\begin{pmatrix}\rho_2 - \rho_1\\ \rho_2 u_2 - \rho_1 u_1\\ \rho_2\left(e_2 + \frac{1}{2}u_2^2\right) - \rho_1\left(e_1 + \frac{1}{2}u_1^2\right)\end{pmatrix} = \begin{pmatrix}\rho_2 u_2 - \rho_1 u_1\\ \rho_2 u_2^2 + p_2 - \rho_1 u_1^2 - p_1\\ \rho_2 u_2\left(e_2 + \frac{1}{2}u_2^2 + \frac{p_2}{\rho_2}\right) - \rho_1 u_1\left(e_1 + \frac{1}{2}u_1^2 + \frac{p_1}{\rho_1}\right)\end{pmatrix}. \tag{3.204}$$

3.3.2 Rayleigh line

If we operate on the momentum equation as follows

$$\begin{aligned}p_2 &= p_1 + \rho_1 D^2 - \rho_2 u_2^2, && (3.205)\\ p_2 &= p_1 + \frac{\rho_1^2 D^2}{\rho_1} - \frac{\rho_2^2 u_2^2}{\rho_2}. && (3.206)\end{aligned}$$

Since mass gives us $\rho_2^2 u_2^2 = \rho_1^2 D^2$ we get an equation for the *Rayleigh Line*,[6] a line in $(p, 1/\rho)$ space:

$$p_2 = p_1 + \rho_1^2 D^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right). \tag{3.207}$$

Note that the Rayleigh line

- passes through ambient state,
- has *negative* slope,
- has a slope with magnitude proportional to square of the wave speed, and
- is independent of state and energy equations.

3.3.3 Hugoniot curve

Let us now work on the energy equation, using both mass and momentum to eliminate velocity. First eliminate u_2 via the mass equation:

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{D^2}{2}, \tag{3.208}$$

$$h_2 + \frac{1}{2} \left(\frac{\rho_1 D}{\rho_2} \right)^2 = h_1 + \frac{D^2}{2}, \tag{3.209}$$

$$h_2 - h_1 + \frac{D^2}{2} \left(\left(\frac{\rho_1}{\rho_2} \right)^2 - 1 \right) = 0, \tag{3.210}$$

$$h_2 - h_1 + \frac{D^2}{2} \left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2} \right) = 0, \tag{3.211}$$

$$h_2 - h_1 + \frac{D^2}{2} \left(\frac{(\rho_1 - \rho_2)(\rho_1 + \rho_2)}{\rho_2^2} \right) = 0. \tag{3.212}$$

Now use the Rayleigh line, Eq. (3.207), to eliminate D^2:

$$D^2 = (p_2 - p_1) \left(\frac{1}{\rho_1^2} \right) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)^{-1}, \tag{3.213}$$

$$D^2 = (p_2 - p_1) \left(\frac{1}{\rho_1^2} \right) \left(\frac{\rho_2 - \rho_1}{\rho_1 \rho_2} \right)^{-1}, \tag{3.214}$$

$$D^2 = (p_2 - p_1) \left(\frac{1}{\rho_1^2} \right) \left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1} \right). \tag{3.215}$$

[6] John William Strutt (Lord Rayleigh), 1842-1919, aristocratic-born English mathematician and physicist, studied at Cambridge, influenced by Stokes, toured the United States rather than the traditional continent of Europe, described correctly why the sky is blue, appointed Cavendish professor experimental physics at Cambridge, won the Nobel prize for the discovery of Argon, described traveling waves and solitons.

So the energy equation becomes

$$h_2 - h_1 + \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{\rho_1\rho_2}{\rho_2 - \rho_1}\right)\left(\frac{(\rho_1 - \rho_2)(\rho_1 + \rho_2)}{\rho_2^2}\right) = 0, \quad (3.216)$$

$$h_2 - h_1 - \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_1}\right)\left(\frac{\rho_1 + \rho_2}{\rho_2}\right) = 0, \quad (3.217)$$

$$h_2 - h_1 - \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) = 0. \quad (3.218)$$

Regrouping to see what induces enthalpy changes, we get

$$h_2 - h_1 = (p_2 - p_1)\left(\frac{1}{2}\right)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right). \quad (3.219)$$

This equation is the *Hugoniot* equation. It

- holds that enthalpy change equals the product of the pressure difference and mean volume,
- is independent of wave speed D and velocity u_2, and
- is independent of the equation of state.

3.3.4 Solution procedure for general equations of state

The shocked state can be determined by the following procedure:

- specify the equation of state $h(p, \rho)$,
- substitute the equation of state into the Hugoniot, Eq. (3.219), to get a second relation between p_2 and ρ_2,
- use the Rayleigh line, Eq. (3.207), to eliminate p_2 in the Hugoniot so that the Hugoniot is a single equation in ρ_2,
- solve for ρ_2 as functions of "1" and D,
- back substitute to solve for p_2, u_2, h_2, T_2 as functions of "1" and D,
- invert to find D as function of "1" state and u_2,
- back transform to laboratory frame to get D as function of "1" state and piston velocity $v_2 = v_p$.

3.3.5 Calorically perfect ideal gas solutions

Let us follow this procedure for the special case of a calorically perfect ideal gas.

$$h = c_p(T - T_o) + h_o, \tag{3.220}$$
$$p = \rho R T. \tag{3.221}$$

Thus,

$$h = c_p\left(\frac{p}{R\rho} - \frac{p_o}{R\rho_o}\right) + h_o, \tag{3.222}$$
$$h = \frac{c_p}{R}\left(\frac{p}{\rho} - \frac{p_o}{\rho_o}\right) + h_o, \tag{3.223}$$
$$h = \frac{c_p}{c_p - c_v}\left(\frac{p}{\rho} - \frac{p_o}{\rho_o}\right) + h_o, \tag{3.224}$$
$$h = \frac{\gamma}{\gamma - 1}\left(\frac{p}{\rho} - \frac{p_o}{\rho_o}\right) + h_o. \tag{3.225}$$

Evaluate at states 1 and 2 and substitute into the Hugoniot equation, Eq. (3.219):

$$\left(\frac{\gamma}{\gamma-1}\left(\frac{p_2}{\rho_2} - \frac{p_o}{\rho_o}\right) + h_o\right) - \left(\frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1} - \frac{p_o}{\rho_o}\right) + h_o\right)$$
$$= (p_2 - p_1)\left(\frac{1}{2}\right)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right),$$
$$\frac{\gamma}{\gamma-1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) - (p_2 - p_1)\left(\frac{1}{2}\right)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) = 0,$$
$$p_2\left(\frac{\gamma}{\gamma-1}\frac{1}{\rho_2} - \frac{1}{2\rho_2} - \frac{1}{2\rho_1}\right) - p_1\left(\frac{\gamma}{\gamma-1}\frac{1}{\rho_1} - \frac{1}{2\rho_2} - \frac{1}{2\rho_1}\right) = 0,$$
$$p_2\left(\frac{\gamma+1}{2(\gamma-1)}\frac{1}{\rho_2} - \frac{1}{2\rho_1}\right) - p_1\left(\frac{\gamma+1}{2(\gamma-1)}\frac{1}{\rho_1} - \frac{1}{2\rho_2}\right) = 0,$$
$$p_2\left(\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) - p_1\left(\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = 0,$$
$$p_2 = p_1\frac{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1} - \frac{1}{\rho_2}}{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2} - \frac{1}{\rho_1}}.$$

- a *hyperbola* in $(p, 1/\rho)$ space,
- $1/\rho_2 \to (\gamma - 1)/(\gamma + 1)(1/\rho_1)$ causes $p_2 \to \infty$, note $= \gamma = 1.4, \rho_2 \to 6$ for infinite pressure, and
- as $1/\rho_2 \to \infty, p_2 \to -p_1(\gamma - 1)/(\gamma + 1)$, note negative pressure, not physical here.

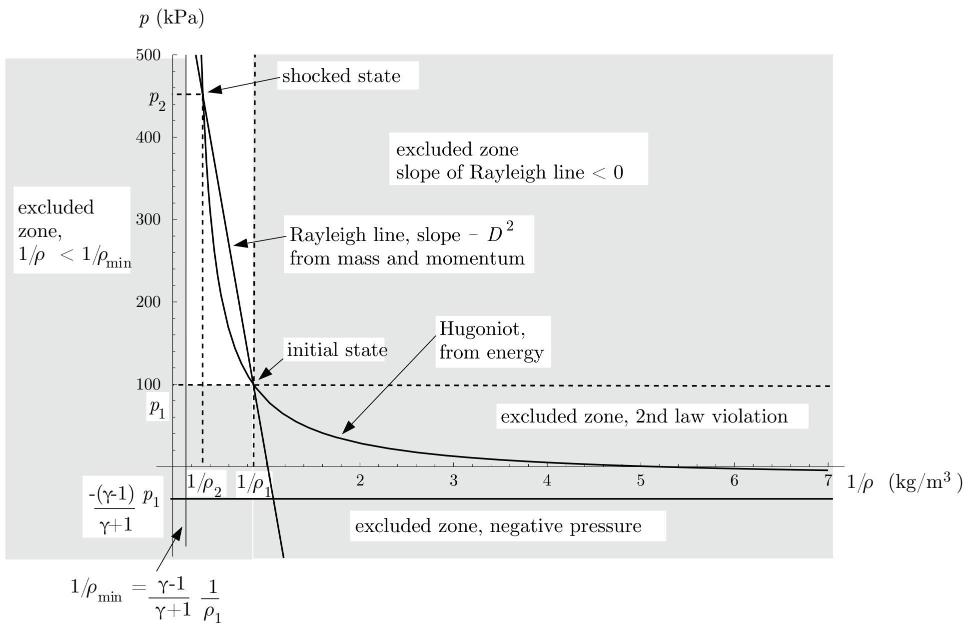

Figure 3.8: Rayleigh line and Hugoniot curve for a typical shocked gas.

The Rayleigh line and Hugoniot curve are sketched in Figure 3.8.
Note:

- intersections of the two curves are solutions to the equations,
- the ambient state "1" is one solution,
- the other solution "2" is known as the shock solution,
- the shock solution has higher pressure and higher density,
- higher wave speed implies higher pressure and higher density,
- a minimum wave speed exists, it
  - occurs when the Rayleigh line is tangent to the Hugoniot curve,
  - occurs for infinitesimally small pressure changes,
  - corresponds to a sonic wave speed, and
  - has disturbances which are *acoustic*.
- if pressure increases, it can be shown that entropy increases, and
- if pressure decreases (for wave speeds which are less than sonic), entropy decreases; this is non-physical.

Substitute the Rayleigh line into the Hugoniot equation to get a single equation for ρ_2:

$$p_1 + \rho_1^2 D^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = p_1 \frac{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1} - \frac{1}{\rho_2}}{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2} - \frac{1}{\rho_1}}. \tag{3.226}$$

This equation is quadratic in $\frac{1}{\rho_2}$ and factorizable. Use computer algebra to solve and get two solutions, one ambient $\frac{1}{\rho_2} = \frac{1}{\rho_1}$ and one shocked solution:

$$\frac{1}{\rho_2} = \frac{1}{\rho_1}\frac{\gamma-1}{\gamma+1}\left(1 + \frac{2\gamma}{(\gamma-1)\,D^2}\frac{p_1}{\rho_1}\right). \tag{3.227}$$

The shocked density ρ_2 is plotted against wave speed D for CPIG air in Figure 3.9.

Note

- density solution allows allows all wave speeds $0 < D < \infty$,
- plot range, however, is $c_1 < D < \infty$,
- Rayleigh line and Hugoniot show $D \geq c_1$,
- solution for $D = D(v_p)$, *to be shown*, rigorously shows $D \geq c_1$,
- *strong shock limit*: $D^2 \to \infty, \rho_2 \to (\gamma+1)/(\gamma-1)$,
- *acoustic limit*: $D^2 \to \gamma p_1/\rho_1, \rho_2 \to \rho_1$, and
- *non-physical limit*: $D^2 \to 0, \rho_2 \to 0$.

Back substitute into Rayleigh line and mass conservation to solve for the shocked pressure and the fluid velocity in the shocked wave frame:

$$p_2 = \frac{2}{\gamma+1}\rho_1 D^2 - \frac{\gamma-1}{\gamma+1}p_1, \tag{3.228}$$

$$u_2 = -D\frac{\gamma-1}{\gamma+1}\left(1 + \frac{2\gamma}{(\gamma-1)\,D^2}\frac{p_1}{\rho_1}\right). \tag{3.229}$$

The shocked pressure p_2 is plotted against wave speed D for CPIG air in Figure 3.10 including both the exact solution and the solution in the strong shock limit. Note for these parameters, the results are indistinguishable. The shocked wave frame fluid particle velocity u_2 is plotted against wave speed D for CPIG air in Figure 3.11. The shocked wave frame fluid particle Mach number, $M_2^2 = \rho_2 u_2^2/(\gamma p_2)$, is plotted against wave speed D for CPIG air in Figure 3.12.

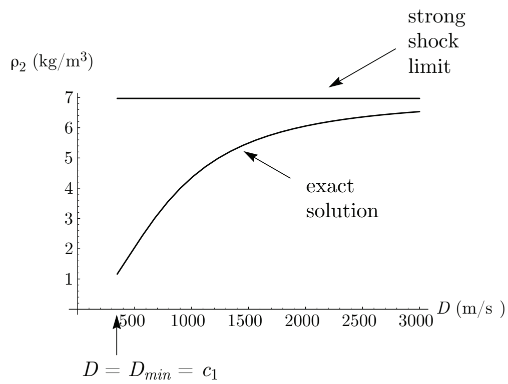

Figure 3.9: Shock density versus shock wave speed for calorically perfect ideal air.

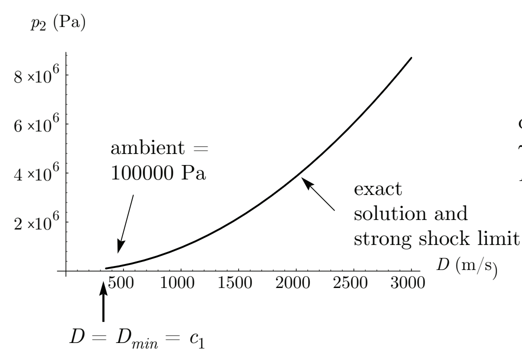

Figure 3.10: Shock pressure versus shock wave speed for calorically perfect ideal air.

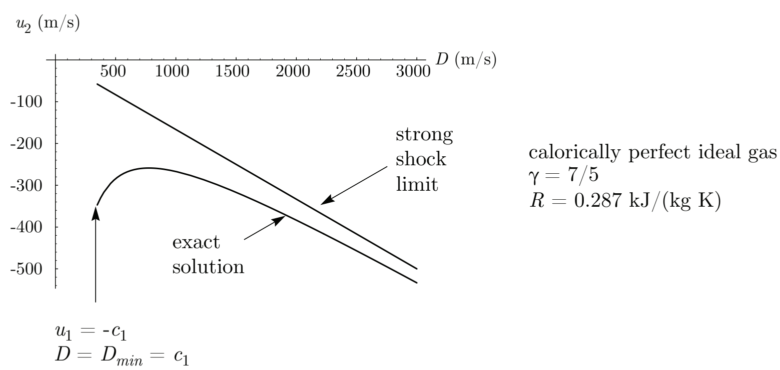

Figure 3.11: Shock wave frame fluid particle velocity versus shock wave speed for calorically perfect ideal air.

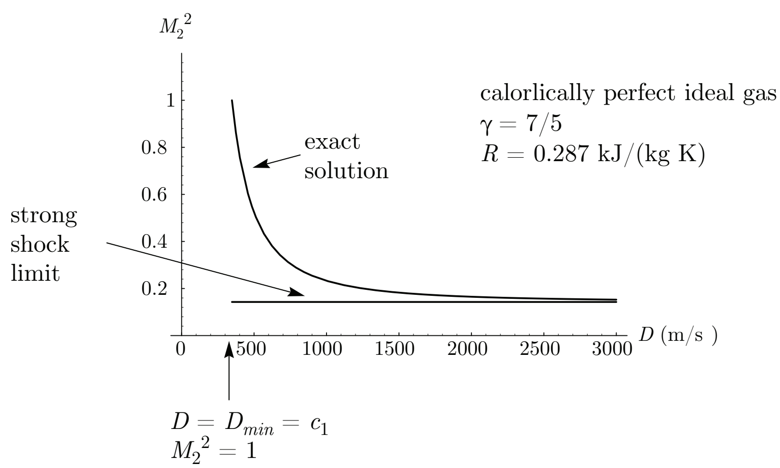

Figure 3.12: Mach number squared of shocked fluid particle versus shock wave speed for calorically perfect ideal air.

Note in the steady frame that the Mach number of the

- undisturbed flow is (and must be) > 1: *supersonic*, and
- shocked flow is (and must be) < 1: *subsonic*.

Transform back to the laboratory frame $u = v - D$:

$$v_2 - D = -D\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)D^2}\frac{p_1}{\rho_1}\right), \tag{3.230}$$

$$v_2 = D - D\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)D^2}\frac{p_1}{\rho_1}\right). \tag{3.231}$$

Manipulate the above equation and solve the resulting quadratic equation for D and get

$$D = \frac{\gamma+1}{4}v_2 \pm \sqrt{\frac{\gamma p_1}{\rho_1} + v_2^2\left(\frac{\gamma+1}{4}\right)^2}. \tag{3.232}$$

Now if $v_2 > 0$, we expect $D > 0$ so take positive root, also set the velocity equal to the piston velocity $v_2 = v_p$.

$$D = \frac{\gamma+1}{4}v_p + \sqrt{\frac{\gamma p_1}{\rho_1} + v_p^2\left(\frac{\gamma+1}{4}\right)^2}. \tag{3.233}$$

Note:

- *acoustic limit*: as $v_p \to 0$, $D \to c_1$; the shock speed approaches the sound speed, and
- *strong shock limit*: as $v_p \to \infty, D \to v_p(\gamma+1)/2$.

The shock speed D is plotted against piston velocity v_p for CPIG air in Figure 3.13. Both the exact solution and strong shock limit are shown.

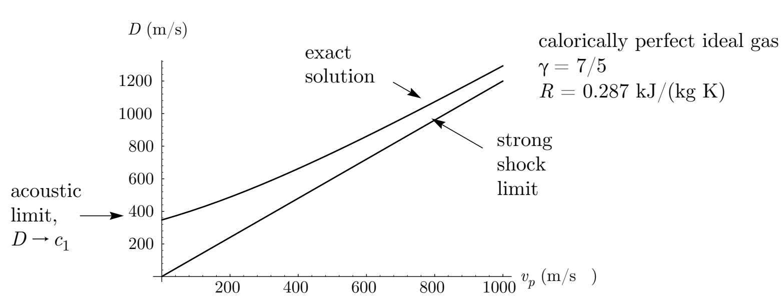

Figure 3.13: Shock speed versus piston velocity for calorically perfect ideal air.

If we define the Mach number of the shock as

$$M_s \equiv \frac{D}{c_1}, \tag{3.234}$$

we get

$$M_s = \frac{\gamma+1}{4}\frac{v_p}{\sqrt{\gamma R T_1}} + \sqrt{1 + \frac{v_p^2}{\gamma R T_1}\left(\frac{\gamma+1}{4}\right)^2}. \tag{3.235}$$

The shock Mach number M_s is plotted against piston velocity v_p for CPIG air in Figure 3.14. Both the exact solution and strong shock limit are shown.

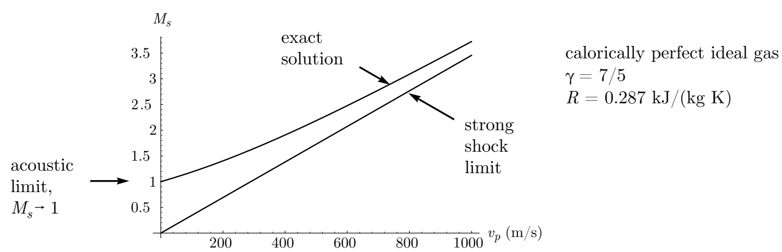

Figure 3.14: Shock Mach number versus piston velocity for calorically perfect ideal air.

3.3.6 Acoustic limit

Consider that state 2 is a small perturbation of state 1 so that

$$\rho_2 = \rho_1 + \Delta\rho, \tag{3.236}$$
$$u_2 = u_1 + \Delta u_1, \tag{3.237}$$
$$p_2 = p_1 + \Delta p. \tag{3.238}$$

Substituting into the normal shock equations, we get

$$(\rho_1 + \Delta\rho)(u_1 + \Delta u) = \rho_1 u_1, \tag{3.239}$$
$$(\rho_1 + \Delta\rho)(u_1 + \Delta u)^2 + (p_1 + \Delta p) = \rho_1 {u_1}^2 + p_1, \tag{3.240}$$
$$\frac{\gamma}{\gamma-1}\frac{p_1 + \Delta p}{\rho_1 + \Delta\rho} + \frac{1}{2}(u_1 + \Delta u)^2 = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}{u_1}^2. \tag{3.241}$$

Expanding, we get

$$\rho_1 u_1 + u_1(\Delta\rho) + \rho_1(\Delta u) + (\Delta\rho)(\Delta u) = \rho_1 u_1$$
$$\left(\rho_1 {u_1}^2 + 2\rho_1 u_1(\Delta u) + {u_1}^2(\Delta\rho) + \rho_1(\Delta u)^2 + 2u_1(\Delta u)(\Delta\rho) + (\Delta\rho)(\Delta u)^2\right)$$

$$+\left(p_1+\Delta p\right) = \rho_1 {u_1}^2 + p_1$$

$$\frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1}+\frac{1}{\rho_1}\Delta p - \frac{p_1}{\rho_1^2}\Delta\rho + ...\right) + \frac{1}{2}\left({u_1}^2 + 2u_1\left(\Delta u\right) + \left(\Delta u\right)^2\right) = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}{u_1}^2$$

Subtracting the base state and eliminating products of small quantities yields

$$u_1\left(\Delta\rho\right) + \rho_1\left(\Delta u\right) = 0, \tag{3.242}$$

$$2\rho_1 u_1\left(\Delta u\right) + {u_1}^2\left(\Delta\rho\right) + \Delta p = 0, \tag{3.243}$$

$$\frac{\gamma}{\gamma-1}\left(\frac{1}{\rho_1}\Delta p - \frac{p_1}{\rho_1^2}\Delta\rho\right) + u_1\left(\Delta u\right) = 0. \tag{3.244}$$

In matrix form this is

$$\begin{pmatrix} u_1 & \rho_1 & 0 \\ {u_1}^2 & 2\rho_1 u_1 & 1 \\ -\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1^2} & u_1 & \frac{\gamma}{\gamma-1}\frac{1}{\rho_1} \end{pmatrix} \begin{pmatrix} \Delta\rho \\ \Delta u \\ \Delta p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.245}$$

As the right hand side is zero, the determinant must be zero and there must be a linear dependency of the solution. First check the determinant:

$$u_1\left(\frac{2\gamma}{\gamma-1}\; u_1 - u_1\right) - \rho_1\left(\frac{\gamma}{\gamma-1}\;\frac{{u_1}^2}{\rho_1} + \frac{\gamma}{\gamma-1}\;\frac{p_1}{\rho_1^2}\right) = 0, \tag{3.246}$$

$$\frac{{u_1}^2}{\gamma-1}\;\left(2\gamma - \left(\gamma-1\right)\right) - \frac{1}{\gamma-1}\left(\gamma\; {u_1}^2 + \gamma\;\frac{p_1}{\rho_1}\right) = 0, \tag{3.247}$$

$${u_1}^2\left(\gamma+1\right) - \left(\gamma\; {u_1}^2 + \gamma\;\frac{p_1}{\rho_1}\right) = 0, \tag{3.248}$$

$${u_1}^2 = \gamma\frac{p_1}{\rho_1} = c_1^2. \tag{3.249}$$

So the velocity is necessarily sonic for a small disturbance.

Take Δu to be known and solve a resulting 2×2 system:

$$\begin{pmatrix} u_1 & 0 \\ -\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1^2} & \frac{\gamma}{\gamma-1}\frac{1}{\rho_1} \end{pmatrix} \begin{pmatrix} \Delta\rho \\ \Delta p \end{pmatrix} = \begin{pmatrix} -\rho_1\Delta u \\ -u_1\Delta u \end{pmatrix}. \tag{3.250}$$

Solving yields

$$\Delta\rho = -\frac{\rho_1\Delta u}{\sqrt{\gamma\frac{p_1}{\rho_1}}} = -\rho_1\frac{\Delta u}{c_1} \tag{3.251}$$

$$\Delta p = -\rho_1\sqrt{\gamma\frac{p_1}{\rho_1}}\;\Delta u = -\rho_1 c_1\Delta u. \tag{3.252}$$

3.4 Flow with area change and normal shocks

This section will consider flow from a reservoir with the fluid at stagnation conditions to a constant pressure environment. The pressure of the environment is commonly known as the *back pressure*: p_b.

Generic problem: Given $A(x)$, stagnation conditions and p_b, find the pressure, temperature, density at all points in the duct and the mass flow rate.

3.4.1 Converging nozzle

A converging nozzle operating at several different values of p_b is sketched in Figure 3.15. The

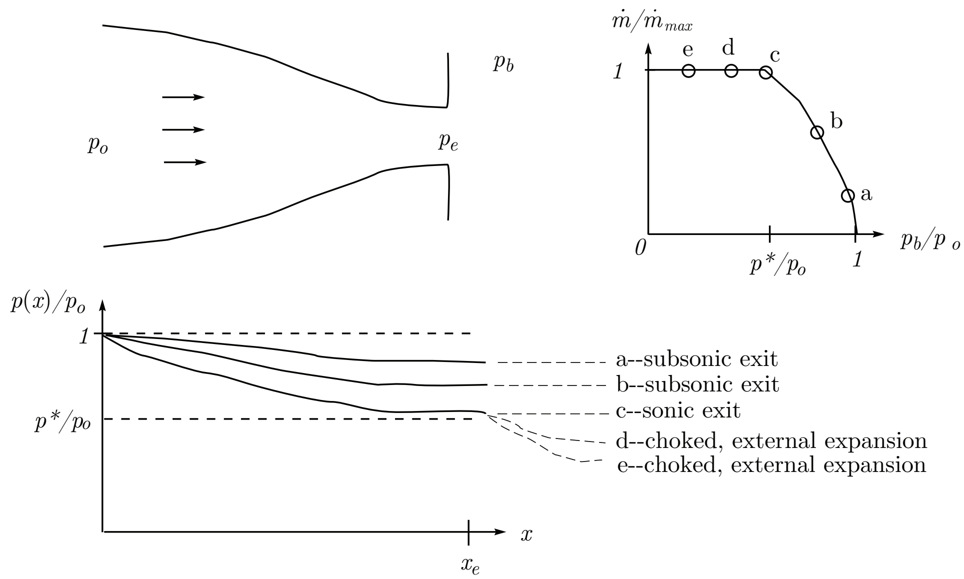

Figure 3.15: Converging nozzle sketch.

flow through the duct can be solved using the following procedure:

- check if $p_b \geq p_*$,
- if so, set $p_e = p_b$,
- determine M_e from isentropic flow relations,
- determine A_* from A/A_* relation,
- at any point in the flow where A is known, compute A/A_* and then invert A/A_* relation to find local M.

Note:

- These flows are subsonic throughout and correspond to points a and b in Figure 3.15.
- If $p_b = p_*$ then the flow is sonic at the exit and just choked. This corresponds to point c in Figure 3.15.
- If $p_b < p_*$, then the flow chokes, is sonic at the exit, and continues to expand outside of the nozzle. This corresponds to points d and e in Figure 3.15.

3.4.2 Converging-diverging nozzle

A converging-diverging nozzle operating at several different values of p_b is sketched in Figure 3.16.

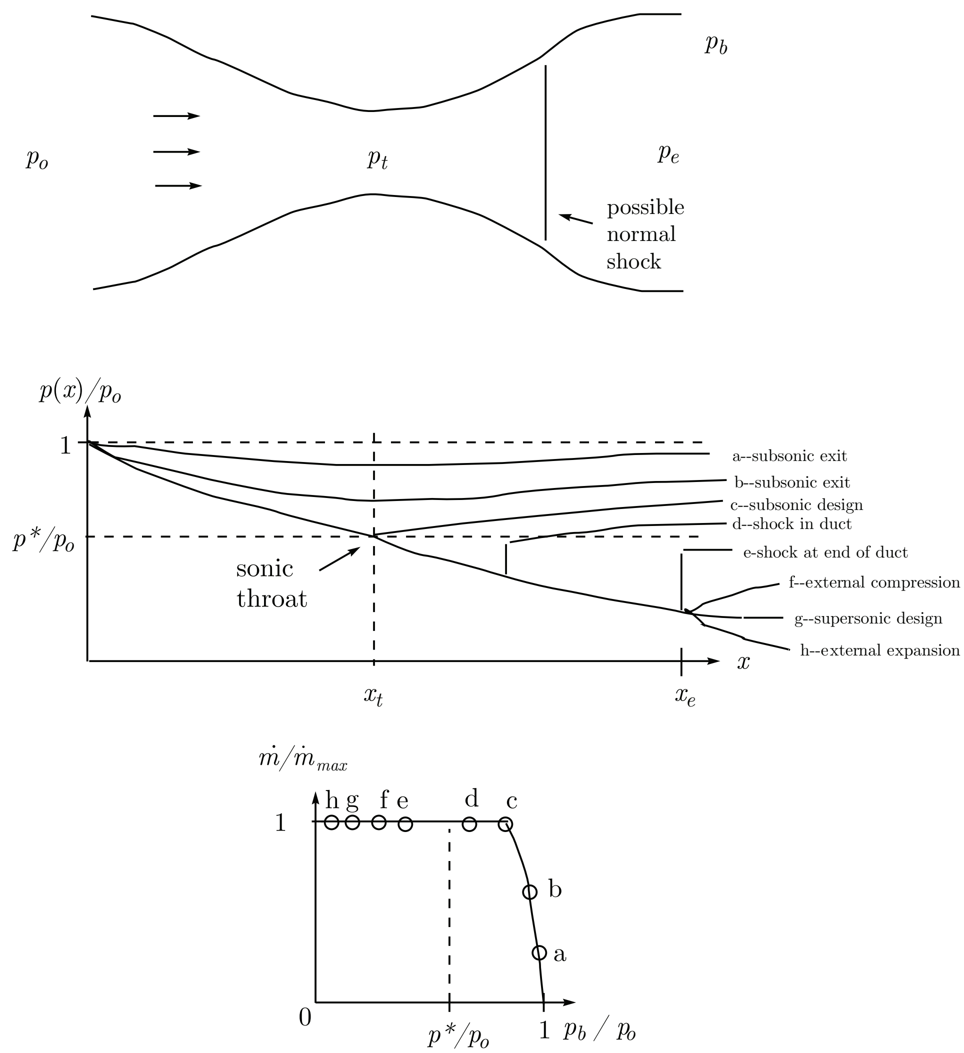

Figure 3.16: Converging-diverging nozzle sketch.

The flow through the duct can be solved using the a very similar following procedure

- set $A_t = A_*$,
- with this assumption, calculate A_e/A_*,
- determine M_{esub}, M_{esup}, both supersonic and subsonic, from A/A_* relation,
- determine p_{esub}, p_{esup}, from M_{esub}, M_{esup}; these are the supersonic and subsonic design pressures,
- if $p_b > p_{esub}$, the flow is subsonic throughout and the throat is not sonic. Use same procedure as for converging duct: Determine M_e by setting $p_e = p_b$ and using isentropic relations,
- if $p_{esub} > p_b > p_{esup}$, the procedure is complicated.
  - estimate the pressure with a normal shock at the end of the duct, p_{esh}.
  - If $p_b \geq p_{esh}$, there is a normal shock inside the duct,
  - If $p_b < p_{esh}$, the duct flow is shockless, and there may be compression outside the duct.
- if $p_{esup} = p_b$, the flow is at supersonic design conditions and the flow is shockless, and
- if $p_b < p_{esup}$, the flow in the duct is isentropic and there is expansion outside the duct.

3.5 Rarefactions and the method of characteristics

Here we discuss how to model expansion waves in a one-dimensional unsteady, inviscid, non-heat conducting fluid. This analysis is a good deal more rigorous than much of traditional one-dimensional gas dynamics, and draws upon some of the more difficult mathematical methods we will encounter.

In assuming no diffusive transport, we have eliminated all mechanisms for entropy generation; consequently, we will be able to model the process as isentropic. We note that even without diffusion, shocks can generate entropy. However, the expansion waves are inherently continuous, and do remain isentropic. We will consider a general equation of state, and later specialize to a calorically perfect ideal gas. The problem is inherently non-linear and is modeled by partial differential equations of the type which is known as *hyperbolic*. Such problems, in contrast to say Laplace's equation, which requires boundary conditions, require initial data only, and no boundary data.

3.5.1 Inviscid one-dimensional equations

The equations to be considered are shown here in non-conservative form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0, && (3.253)\\ \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} &= 0, && (3.254)\\ \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} &= 0, && (3.255)\\ p &= p(\rho, s). && (3.256) \end{aligned}$$

Here we have written the energy equation in terms of entropy. The development of this was shown in Chapter 1. We have also utilized the general result from thermodynamics that any intensive property can be written as a function of two other independent thermodynamic properties. Here we have chosen to write pressure as a function of density and entropy. Thus we have four equations for the four unknowns, ρ, u, p, s.

Now we note that

$$\begin{aligned} dp &= \left.\frac{\partial p}{\partial \rho}\right|_s d\rho + \left.\frac{\partial p}{\partial s}\right|_\rho ds, \qquad \text{so,} && (3.257)\\ \left.\frac{\partial p}{\partial x}\right|_t &= \left.\frac{\partial p}{\partial \rho}\right|_s \left.\frac{\partial \rho}{\partial x}\right|_t + \left.\frac{\partial p}{\partial s}\right|_\rho \left.\frac{\partial s}{\partial x}\right|_t . && (3.258) \end{aligned}$$

Now, let us define thermodynamic properties c^2 and ζ as follows

$$c^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_s, \qquad \zeta \equiv \left.\frac{\partial p}{\partial s}\right|_\rho . \qquad (3.259)$$

We will see that ζ will be unimportant, and will be able to ascribe to c^2 the physical significance of the speed of propagation of small disturbances, the so-called sound speed, which we have already encountered in acoustics. If we know the equation of state, then we can think of c^2 and ζ as known thermodynamic functions of ρ and s. Our definitions give us

$$\frac{\partial p}{\partial x} = c^2\frac{\partial \rho}{\partial x} + \zeta\frac{\partial s}{\partial x}. \qquad (3.260)$$

Substituting into our governing equations, we see that pressure can be eliminated to give three equations in three unknowns:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0, && (3.261)\\ \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + c^2\frac{\partial \rho}{\partial x} + \zeta\frac{\partial s}{\partial x} &= 0, && (3.262)\\ \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} &= 0. && (3.263) \end{aligned}$$

Now if $\partial s/\partial t + u(\partial s/\partial x) = 0$, we can say that if $s = s(x,t)$,

$$ds = \frac{\partial s}{\partial t}dt + \frac{\partial s}{\partial x}dx, \tag{3.264}$$

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \frac{dx}{dt}\frac{\partial s}{\partial x}. \tag{3.265}$$

Thus, on curves where $dx/dt = u$, we have from substituting Eq. (3.265) into the energy equation (3.263)

$$\frac{ds}{dt} = 0. \tag{3.266}$$

Thus we have converted the partial differential equation into an ordinary differential equation. This can be integrated to give us

$$s = C, \qquad \text{on a particle pathline, } \frac{dx}{dt} = u. \tag{3.267}$$

This scenario is sketched on the so-called $x - t$ diagram of Figure 3.17.

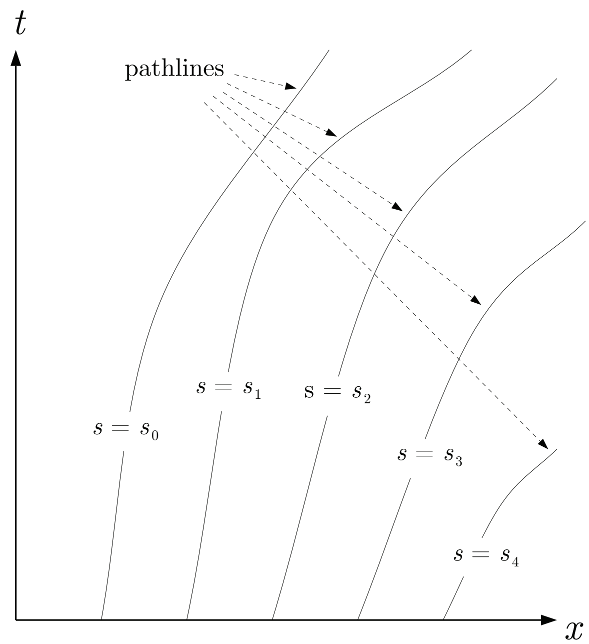

Figure 3.17: $x - t$ diagram showing maintenance of entropy s along particle pathlines $dx/dt = u$ for isentropic flow.

This result is satisfying, but not complete, as we do not in general know where the pathlines are. Let us try to apply this technique to the system in general. Consider our equations in matrix form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \begin{pmatrix} u & \rho & 0 \\ c^2 & \rho u & \zeta \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.268}$$

These equations are of the form

$$A_{ij}\frac{\partial u_j}{\partial t} + B_{ij}\frac{\partial u_j}{\partial x} = C_i. \tag{3.269}$$

As described by Whitham,[7] there is a general technique to analyze such equations. First pre-multiply both sides of the equation by a yet to be determined vector of variables ℓ_i:

$$\ell_i A_{ij}\frac{\partial u_j}{\partial t} + \ell_i B_{ij}\frac{\partial u_j}{\partial x} = \ell_i C_i. \tag{3.270}$$

Now, this method will work if we can choose ℓ_i to render the above product to be of the form similar to $\partial/\partial t + u(\partial/\partial x)$. Let us take

$$\ell_i A_{ij}\frac{\partial u_j}{\partial t} + \ell_i B_{ij}\frac{\partial u_j}{\partial x} = m_j\left(\frac{\partial u_j}{\partial t} + \lambda\frac{\partial u_j}{\partial x}\right), \tag{3.271}$$

$$= m_j\frac{du_j}{dt} \qquad \text{on} \qquad \frac{dx}{dt} = \lambda. \tag{3.272}$$

So comparing terms, we see that

$$\ell_i A_{ij} = m_j, \qquad \ell_i B_{ij} = \lambda m_j, \tag{3.273}$$

$$\lambda \ell_i A_{ij} = \lambda m_j, \tag{3.274}$$

so, we get by eliminating m_j that

$$\ell_i\left(\lambda A_{ij} - B_{ij}\right) = 0. \tag{3.275}$$

This is a left eigenvalue problem. We set the determinant of $\lambda A_{ij} - B_{ij}$ to zero for a non-trivial solution and find

$$\begin{vmatrix} \lambda - u & -\rho & 0 \\ -c^2 & \rho(\lambda - u) & -\zeta \\ 0 & 0 & \lambda - u \end{vmatrix} = 0. \tag{3.276}$$

Evaluating, we get

$$(\lambda - u)\left(\rho(\lambda - u)^2\right) + \rho(\lambda - u)(-c^2) = 0, \tag{3.277}$$

$$\rho(\lambda - u)\left((\lambda - u)^2 - c^2\right) = 0. \tag{3.278}$$

Solving we get

$$\lambda = u, \qquad \lambda = u \pm c. \tag{3.279}$$

[7]Gerald Beresford Whitham, 1927-, applied mathematician and developer of theory for non-linear wave propagation.

Now the left eigenvectors ℓ_i give us the actual equations. First for $\lambda = u$, we get

$$\left(\ell_1 \quad \ell_2 \quad \ell_3 \right) \begin{pmatrix} u-u & -\rho & 0 \\ -c^2 & \rho(u-u) & -\zeta \\ 0 & 0 & u-u \end{pmatrix} = \left(0 \quad 0 \quad 0 \right), \tag{3.280}$$

$$\left(\ell_1 \quad \ell_2 \quad \ell_3 \right) \begin{pmatrix} 0 & -\rho & 0 \\ -c^2 & 0 & -\zeta \\ 0 & 0 & 0 \end{pmatrix} = \left(0 \quad 0 \quad 0 \right). \tag{3.281}$$

Two of the equations require that $\ell_1 = 0$ and $\ell_2 = 0$. There is no restriction on ℓ_3. We will select a normalized solution so that

$$\ell_i = (0, 0, 1). \tag{3.282}$$

Thus $\ell_i A_{ij}(\partial u_j/\partial t) + \ell_i B_{ij}(\partial u_j/\partial x) = \ell_i C_i$ gives

$$\begin{aligned}
\left(0 \quad 0 \quad 1 \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \left(0 \quad 0 \quad 1 \right) \begin{pmatrix} u & \rho & 0 \\ c^2 & \rho u & \zeta \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} &= \left(0 \quad 0 \quad 1 \right) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
\left(0 \quad 0 \quad 1 \right) \begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \left(0 \quad 0 \quad u \right) \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} &= 0, \\
\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} &= 0.
\end{aligned} \tag{3.283}$$

So as before with $s = s(x,t)$, we have $ds = (\partial s/\partial t)dt + (\partial s/\partial x)dx$, and $ds/dt = \partial s/\partial t + (dx/dt)(\partial s/\partial x)$. Now *if* we require dx/dt to be a particle pathline, $dx/dt = u$, then our energy equation gives us

$$\frac{ds}{dt} = 0, \qquad \text{on} \qquad \frac{dx}{dt} = u. \tag{3.284}$$

The special case in which the pathlines are straight in $x-t$ space, corresponding to a uniform velocity field of $u(x,t) = u_o$, is sketched in the $x-t$ diagram of Figure 3.18.

Now let us look at the remaining eigenvalues, $\lambda = u \pm c$.

$$\left(\ell_1 \quad \ell_2 \quad \ell_3 \right) \begin{pmatrix} u \pm c - u & -\rho & 0 \\ -c^2 & \rho(u \pm c - u) & -\zeta \\ 0 & 0 & u \pm c - u \end{pmatrix} = \left(0 \quad 0 \quad 0 \right), \tag{3.285}$$

$$\left(\ell_1 \quad \ell_2 \quad \ell_3 \right) \begin{pmatrix} \pm c & -\rho & 0 \\ -c^2 & \pm \rho c & -\zeta \\ 0 & 0 & \pm c \end{pmatrix} = \left(0 \quad 0 \quad 0 \right). \tag{3.286}$$

As one of the components of the left eigenvector should be arbitrary, we will take $\ell_1 = 1$; we arrive at the following equations then

$$\pm c - c^2 \ell_2 = 0, \Longrightarrow \ell_2 = \pm \frac{1}{c}, \tag{3.287}$$

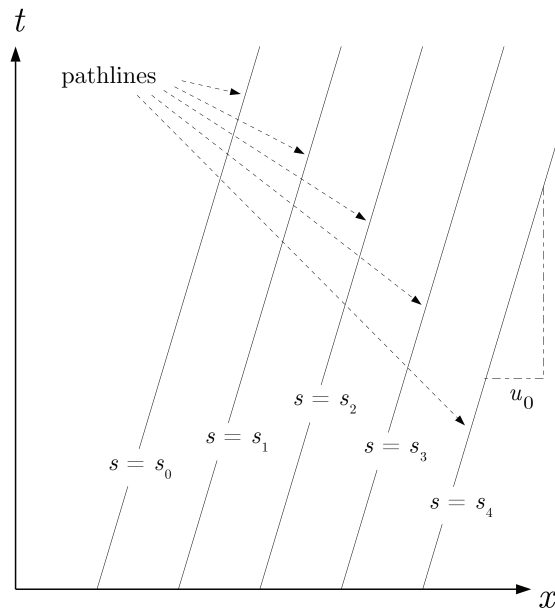

Figure 3.18: $x - t$ diagram showing maintenance of entropy s along particle pathlines $dx/dt = u_o$ for isentropic flow.

$$-\rho \pm \rho c \ell_2 = 0, \Longrightarrow \ell_2 = \pm\frac{1}{c}, \tag{3.288}$$

$$-\zeta \ell_2 \pm c \ell_3 = 0, \Longrightarrow \ell_3 = \frac{\zeta}{c^2}. \tag{3.289}$$

Thus $\ell_i A_{ij}(\partial u_j/\partial t) + \ell_i B_{ij}(\partial u_j/\partial x) = \ell_i C_i$ gives

$$\begin{aligned}
\left(1 \quad \pm\tfrac{1}{c} \quad \tfrac{\zeta}{c^2} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \left(1 \quad \pm\tfrac{1}{c} \quad \tfrac{\zeta}{c^2} \right) \begin{pmatrix} u & \rho & 0 \\ c^2 & \rho u & \zeta \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} &= \left(1 \quad \pm\tfrac{1}{c} \quad \tfrac{\zeta}{c^2} \right) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
\left(1 \quad \pm\tfrac{\rho}{c} \quad \tfrac{\zeta}{c^2} \right) \begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial u}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} + \left(u \pm c \quad \rho \pm \rho\tfrac{u}{c} \quad \pm\tfrac{\zeta}{c} + \tfrac{\zeta u}{c^2} \right) \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial x} \end{pmatrix} &= 0,
\end{aligned}$$

$$\frac{\partial \rho}{\partial t} + (u \pm c)\frac{\partial \rho}{\partial x} \pm \frac{\rho}{c}\frac{\partial u}{\partial t} + \rho\left(1 \pm \frac{u}{c}\right)\frac{\partial u}{\partial x} + \frac{\zeta}{c^2}\frac{\partial s}{\partial t} + \left(\frac{\zeta u}{c^2} \pm \frac{\zeta}{c}\right)\frac{\partial s}{\partial x} = 0, \tag{3.290}$$

$$\left(\frac{\partial \rho}{\partial t} + (u \pm c)\frac{\partial \rho}{\partial x}\right) \pm \frac{\rho}{c}\left(\frac{\partial u}{\partial t} + (u \pm c)\frac{\partial u}{\partial x}\right) + \frac{\zeta}{c^2}\left(\frac{\partial s}{\partial t} + (u \pm c)\frac{\partial s}{\partial x}\right) = 0, \tag{3.291}$$

$$c^2\left(\frac{\partial \rho}{\partial t} + (u \pm c)\frac{\partial \rho}{\partial x}\right) \pm \rho c\left(\frac{\partial u}{\partial t} + (u \pm c)\frac{\partial u}{\partial x}\right) + \zeta\left(\frac{\partial s}{\partial t} + (u \pm c)\frac{\partial s}{\partial x}\right) = 0. \tag{3.292}$$

Now on lines where $dx/dt = u \pm c$, we get a transformation of the partial differential equations to ordinary differential equations:

$$c^2 \frac{d\rho}{dt} \pm \rho c \frac{du}{dt} + \zeta \frac{ds}{dt} = 0, \qquad \text{on} \qquad \frac{dx}{dt} = u \pm c. \tag{3.293}$$

A sketch of the *characteristics*, the lines on which the differential equations are obtained, are sketched in the $x - t$ diagram of Figure 3.19.

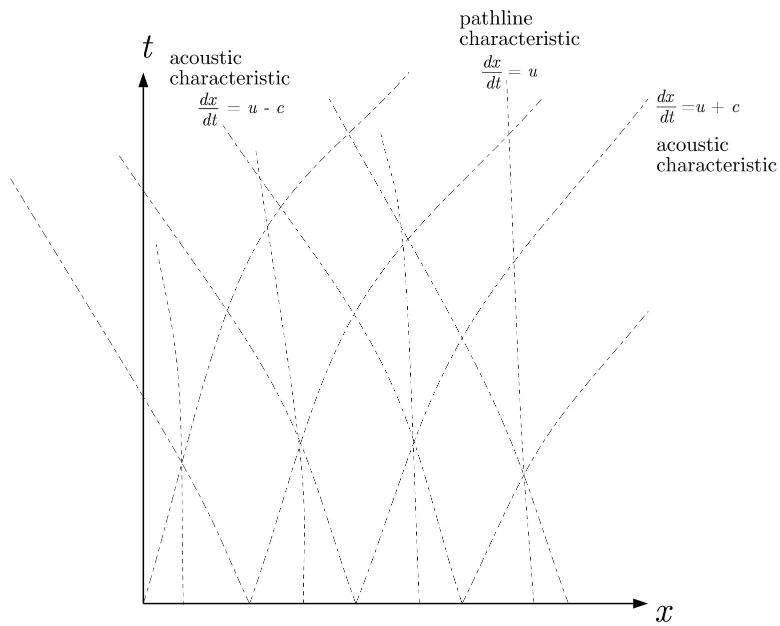

Figure 3.19: $x - t$ diagram showing characteristics for pathlines $dx/dt = u$ and acoustic waves $dx/dt = u \pm c$.

3.5.2 Homeoentropic flow of an ideal gas

The equations developed so far are valid for a general equation of state. Here let us now consider the flow of a calorically perfect ideal gas, so $p = \rho RT$ and $e = c_v T + \hat{e}$. Further let us take the flow to be *homeoentropic*, that is to say, not only does the entropy remain constant on pathlines, which is isentropic, but it has the same value on each streamline. That is the entropy field is a constant. Consequently, we have the standard relations for a calorically perfect ideal gas:

$$c^2 = \gamma \frac{p}{\rho}, \tag{3.294}$$

$$\frac{p}{\rho^\gamma} = A, \tag{3.295}$$

where A is a constant. Because of homeoentropy, we no longer need consider the energy equation, and the linear combination of mass and linear momentum equations reduces to

$$c^2 \frac{d\rho}{dt} \pm \rho c \frac{du}{dt} = 0, \qquad \text{on} \qquad \frac{dx}{dt} = u \pm c. \tag{3.296}$$

Rearranging, we get

$$\frac{du}{dt} = \mp \frac{d\rho}{dt} \frac{c}{\rho}, \qquad \text{on} \qquad \frac{dx}{dt} = u \pm c. \tag{3.297}$$

Now $c^2 = \gamma p/\rho = \gamma A \rho^{\gamma-1}$, and $c = \sqrt{\gamma A} \rho^{\frac{\gamma-1}{2}}$, so

$$\frac{du}{dt} = \mp \sqrt{\gamma A} \rho^{\frac{\gamma-1}{2}} \rho^{-1} \frac{d\rho}{dt} = \mp \sqrt{\gamma A} \frac{2}{\gamma - 1} \frac{d}{dt} \left(\rho^{\frac{\gamma-1}{2}} \right). \tag{3.298}$$

Regrouping, we find

$$\frac{d}{dt} \left(u \pm \sqrt{\gamma A} \frac{2}{\gamma - 1} \rho^{\frac{\gamma-1}{2}} \right) = 0, \tag{3.299}$$

$$\frac{d}{dt} \left(u \pm \frac{2}{\gamma - 1} c \right) = 0. \tag{3.300}$$

Following notation used by Courant[8] and Friedrichs,[9] we then integrate each of these equations, which are homogeneous, along characteristics to obtain algebraic relations

$$u + \frac{2}{\gamma - 1} c = 2r, \quad \text{on} \quad \frac{dx}{dt} = u + c, \quad C^+ \text{ characteristic}, \tag{3.301}$$

$$u - \frac{2}{\gamma - 1} c = -2s, \quad \text{on} \quad \frac{dx}{dt} = u - c, \quad C^- \text{ characteristic}. \tag{3.302}$$

A sketch of the characteristics is given in the $x - t$ diagram of Figure 3.20. Now r and s can take on different values, depending on which characteristic we are on. On a given characteristic, they remain constant. Let us define additional parameters α and β to identify which characteristic we are on. So we have

$$u + \frac{2}{\gamma - 1} c = 2r(\beta), \quad \text{on} \quad \frac{dx}{dt} = u + c, \quad C^+ \text{ characteristic}, \tag{3.303}$$

$$u - \frac{2}{\gamma - 1} c = -2s(\alpha), \quad \text{on} \quad \frac{dx}{dt} = u - c, \quad C^- \text{ characteristic}. \tag{3.304}$$

[8]Richard Courant, 1888-1972, Prussian-born German mathematician, received Ph.D. under David Hilbert at Göttingen, compiled Hilbert's course notes into classic two-volume text of applied mathematics, drafted into German army in World War I, where half of his unit was killed in action, developed telegraph system which used the earth as a conductor for use in the trenches of the Western front, expelled from Göttingen by the Nazis in 1933, fled Germany, and founded the Courant Institute of Mathematical Sciences at New York University, author of classic mathematical text on supersonic fluid mechanics.

[9]Kurt Otto Friedrichs, 1901-1982, German-born mathematician who emigrated to the United States in 1937, student of Richard Courant's at Göttingen, taught at Aachen, Braunschweig, and New York University, worked on partial differential equations of mathematical physics and fluid mechanics.

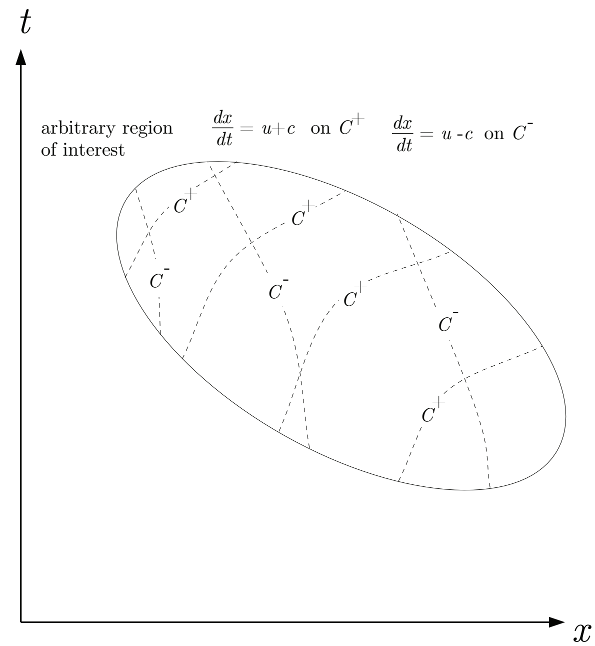

Figure 3.20: $x - t$ diagram showing C^+ and C^- characteristics $dx/dt = u \pm c$.

These quantities are known as *Riemann invariants*.[10]

3.5.3 Simple waves

Simple waves are defined to exist when either $r(\beta)$ or $s(\alpha)$ are constant everywhere in $x - t$ space and not just on characteristics. For example say $s(\alpha) = s_o$. Then the Riemann invariant

$$u - \frac{2}{\gamma - 1} c = -2 s_o, \tag{3.305}$$

is actually invariant over all of $x - t$ space. Now the other Riemann invariant,

$$u + \frac{2}{\gamma - 1} c = 2 r(\beta), \tag{3.306}$$

takes on many values depending on β. However, it is easily shown that for the simple wave that the characteristics have a constant slope in the $x - t$ plane as sketched in the $x - t$ diagram of Figure 3.21.

Now consider a rarefaction with a *prescribed* piston motion $u = u_p(t)$. A sketch is given in the $x - t$ diagram of Figure 3.22.

[10]Georg Friedrich Bernhard Riemann, 1826-1866, German mathematician and geometer whose work in non-Euclidean geometry was critical to Einstein's theory of general relativity, produced the first major study of shock waves.

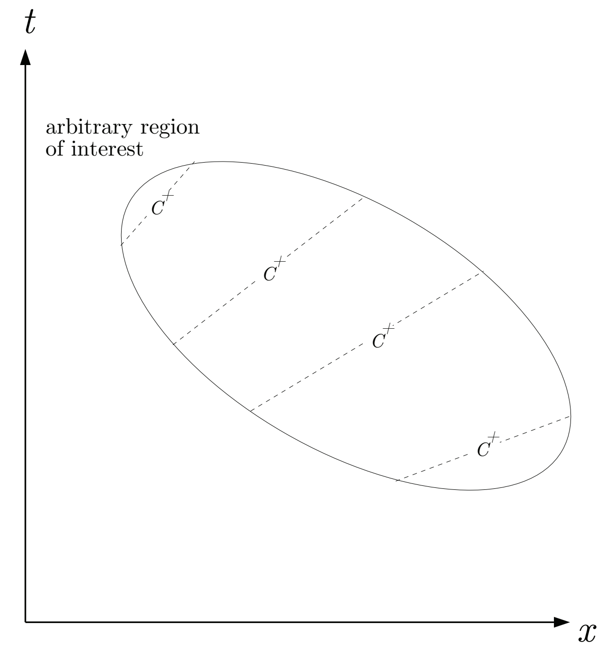

Figure 3.21: $x - t$ diagram showing C^+ for a simple wave.

For this configuration, the Riemann invariant $u - 2c/(\gamma - 1) = -2s_o$ is valid everywhere. Now when $t = 0$, we have $u = 0$, $c = c_o$, so

$$u - \frac{2}{\gamma - 1}c = -\frac{2}{\gamma - 1}c_o. \tag{3.307}$$

Consider now a special characteristic $\hat{C}^+$ at $t = \hat{t}$. At this time the piston moves with velocity $\hat{u}_p$, and the fluid velocity at the piston face is

$$u_{face}(\hat{t}) = \hat{u}_p. \tag{3.308}$$

We get $c_{face}(\hat{t})$ from Eq. (3.307):

$$\underbrace{u_{face}}_{\hat{u}_p} - \frac{2}{\gamma - 1}c_{face} = -\frac{2}{\gamma - 1}c_o, \tag{3.309}$$

$$c_{face}(t = \hat{t}) = c_o + \frac{\gamma - 1}{2}\hat{u}_p. \tag{3.310}$$

Also from Eq. (3.307), we have

$$c = c_o + \frac{\gamma - 1}{2}u, \tag{3.311}$$

which is valid everywhere.

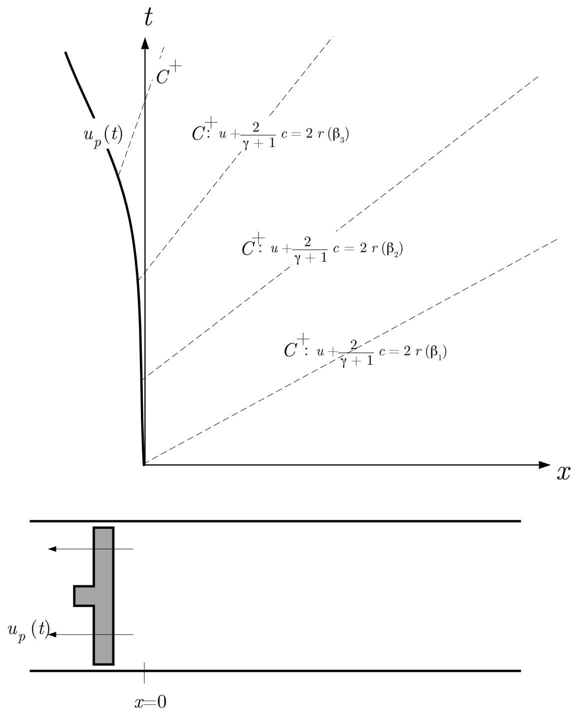

Figure 3.22: $x - t$ diagram showing C^+ characteristics for isentropic rarefaction problem, along with piston cylinder arrangement.

Now on $\hat{C}^+$, we have

$$u + \frac{2}{\gamma - 1} c = \left(u_{face} + \frac{2}{\gamma - 1} c_{face} \right)_{t=\hat{t}}, \tag{3.312}$$

$$u + \frac{2}{\gamma - 1} \left(c_o + \frac{\gamma - 1}{2} u \right) = \hat{u}_p + \frac{2}{\gamma - 1} \left(c_o + \frac{\gamma - 1}{2} \hat{u}_p \right), \tag{3.313}$$

$$2u + \frac{2}{\gamma - 1} c_o = 2\hat{u}_p + \frac{2}{\gamma - 1} c_o, \tag{3.314}$$

$$u = \hat{u}_p \quad \text{on} \quad \hat{C}^+. \tag{3.315}$$

So on $\hat{C}^+$, we have

$$c = c_o + \frac{\gamma - 1}{2} \hat{u}_p. \tag{3.316}$$

So for $\hat{C}^+$, we get

$$\frac{dx}{dt} = u + c = \hat{u}_p + c_o + \frac{\gamma - 1}{2} \hat{u}_p = \frac{\gamma + 1}{2} \hat{u}_p + c_o. \tag{3.317}$$

for a particular characteristic, this slope is a constant, as was earlier suggested.

Now for *prescribed* motion, $\hat{u}_p$ decreases with time and becomes more negative; hence the slope of our $\hat{C}^+$ characteristic decreases, and they *diverge* in $x - t$ space. The slope of the leading characteristic is c_o, the ambient sound speed. The characteristic we consider, $\hat{C}^+$, along with a few other is sketched in the $x-t$ diagram of Figure 3.23. We can use our

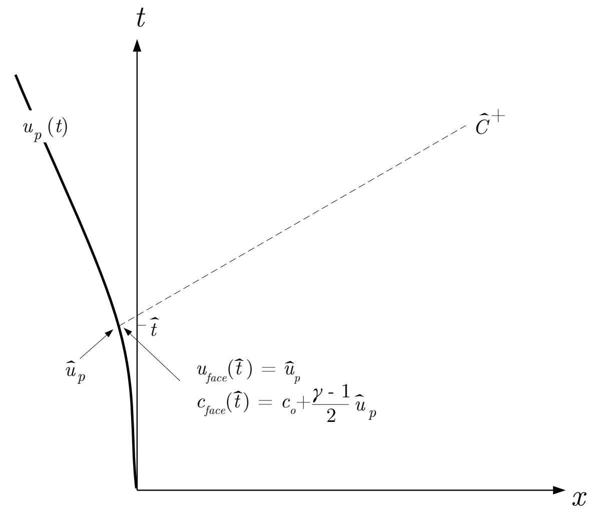

Figure 3.23: $x-t$ diagram showing $\hat{C}^+$ for our rarefaction problem.

Riemann invariant along with isentropic relations to obtain other flow variables. From Eq. (3.307), we get

$$\frac{c}{c_o} = 1 + \frac{\gamma - 1}{2} \frac{u}{c_o}. \tag{3.318}$$

Since the flow is homeoentropic, we have $c/c_o = (\rho/\rho_o)^{\frac{\gamma-1}{2}}$ and $p/p_o = (\rho/\rho_o)^{\gamma}$, so

$$\frac{p}{p_o} = \left(1 + \frac{\gamma - 1}{2}\frac{u}{c_o}\right)^{\frac{2\gamma}{\gamma-1}}, \tag{3.319}$$

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma - 1}{2}\frac{u}{c_o}\right)^{\frac{2}{\gamma-1}}. \tag{3.320}$$

3.5.4 Centered rarefaction

If the piston is *suddenly* accelerated to a constant velocity, then a family of characteristics clusters at the origin on the $x-t$ diagram and fans out in a *centered rarefaction*. This can also be studied using the similarity transformation $\xi = x/t$ which reduces the partial differential equations to ordinary differential equations. Relevant sketches comparing centered to non-centered rarefactions are shown in in the $x-t$ diagram of Figure 3.24.

Example 3.4

Analyze a centered rarefaction fan propagating into calorically perfect ideal air for a piston suddenly accelerated from rest to $u_p = -100$ m/s. Take the ambient air to be at $p_o = 10^5$ Pa, $T_o = 300$ K.

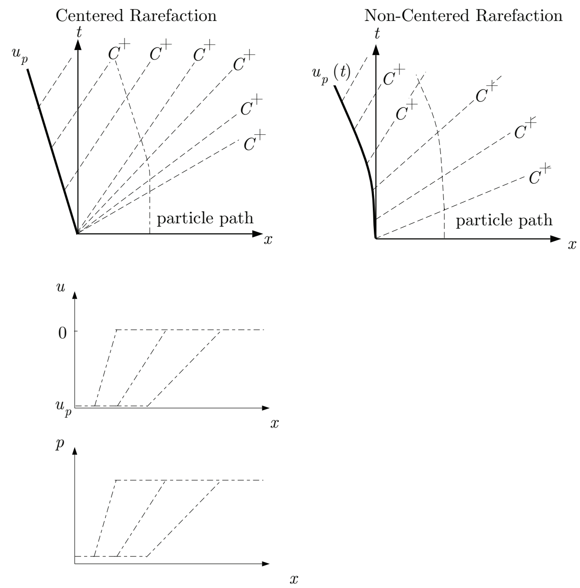

Figure 3.24: $x-t$ diagram centered and non-centered rarefactions, along with pressure and velocity profiles for centered fans.

The ideal gas law gives $\rho_o = p_o/RT_o = 10^5$ Pa/((287 kJ/kg/K)(300 K)) = 1.16 kg/m^3. Now

$$c_o = \sqrt{\gamma R T_o} = \sqrt{\frac{7}{5}\left(287\ \frac{\text{J}}{\text{kg K}}\right)(300\ \text{K})} = 347\ \frac{\text{m}}{\text{s}}. \tag{3.321}$$

On the final characteristic of the fan, C_f^+: $u = u_p = -100\ \frac{\text{m}}{\text{s}}$. So

$$c = c_o + \frac{\gamma-1}{2}u_p = \left(347\ \frac{\text{m}}{\text{s}}\right) + \frac{7/5-1}{2}\left(-100\ \frac{\text{m}}{\text{s}}\right) = 327\ \frac{\text{m}}{\text{s}}. \tag{3.322}$$

Now the final pressure is

$$\frac{p_f}{p_o} = \left(1 + \frac{\gamma-1}{2}\frac{u_f}{c_o}\right)^{\frac{2\gamma}{\gamma-1}} = \left(1 + \frac{7/5-1}{2}\frac{\left(-100\ \frac{\text{m}}{\text{s}}\right)}{347\ \frac{\text{m}}{\text{s}}}\right)^{\frac{2(7/5)}{7/5-1}} = 0.660 \tag{3.323}$$

Hence $p_f = 6.6 \times 10^4$ Pa. Since the flow is homeoentropic, we get

$$\rho_f = \rho_o\left(\frac{p_f}{p_o}\right)^{\frac{1}{\gamma}} = \left(1.16\ \frac{\text{kg}}{\text{m}^3}\right)(0.660)^{5/7} = 0.863\ \frac{\text{kg}}{\text{m}^3}. \tag{3.324}$$

And the final temperature is

$$T_f = \frac{p_f}{\rho_f R} = \frac{66.0 \times 10^3\ \text{Pa}}{\left(0.863\ \frac{\text{kg}}{\text{m}^3}\right)\left(287\ \frac{\text{J}}{\text{kg K}}\right)} = 266.4\ \text{K}. \tag{3.325}$$

From linear acoustic theory, Sec. 3.3.6, we deduce that

$$\Delta\rho \sim -\rho_o\frac{\Delta u}{c_o}, \qquad \Delta p \sim -\rho_o c_o \Delta u, \qquad \Delta T \sim -(\gamma-1)T_o\frac{\Delta u}{c_o}. \tag{3.326}$$

We compare the results of this problem with the estimates of linear acoustic theory. and see

$$\Delta\rho_{exact} = -0.298 \frac{\text{kg}}{\text{m}^3}, \qquad \Delta\rho_{linear} = -0.335 \frac{\text{kg}}{\text{m}^3}, \tag{3.327}$$

$$\Delta p_{exact} = -34.0 \times 10^3 \text{ Pa}, \qquad \Delta p_{linear} = -40.3 \times 10^3 \text{ Pa}, \tag{3.328}$$

$$\Delta T_{exact} = -33.6 \text{ K}, \qquad \Delta T_{linear} = -34.6 \text{ K}. \tag{3.329}$$

3.5.5 Simple compression

We sketch a simple compression in the $x - t$ diagram of Figure 3.25.

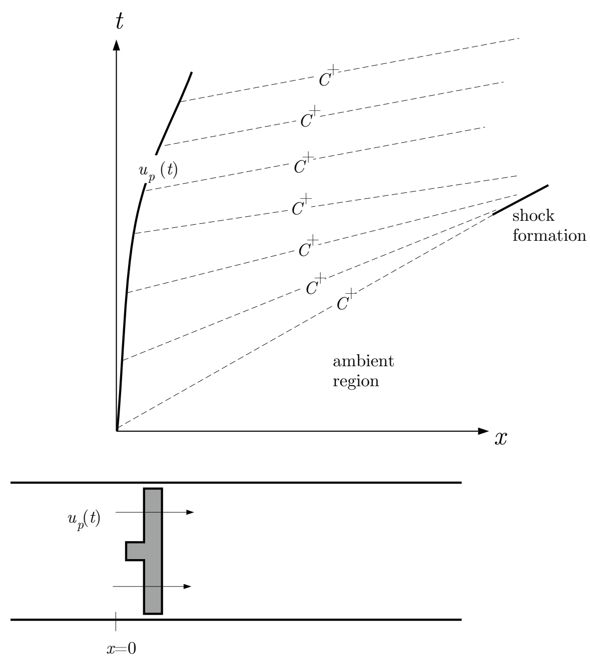

Figure 3.25: $x - t$ diagram for simple compression.

3.5.6 Two interacting expansions

We sketch two interacting expansion waves in the $x - t$ diagram of Figure 3.26.

3.5.7 Wall interactions

We sketch an expansion wall interaction in the $x - t$ diagram of Figure 3.27.

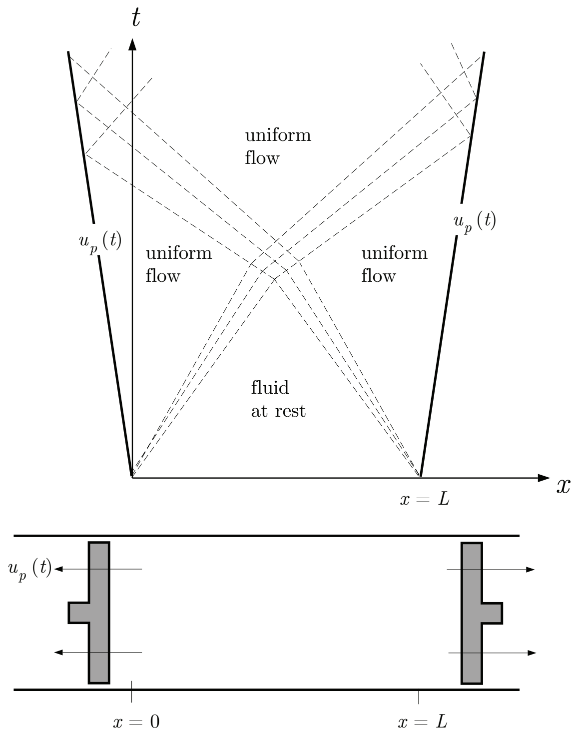

Figure 3.26: $x - t$ diagram for two interacting expansion waves.

3.5.8 Shock tube

We sketch the behavior of a shock tube in the diagrams of Figure 3.28.

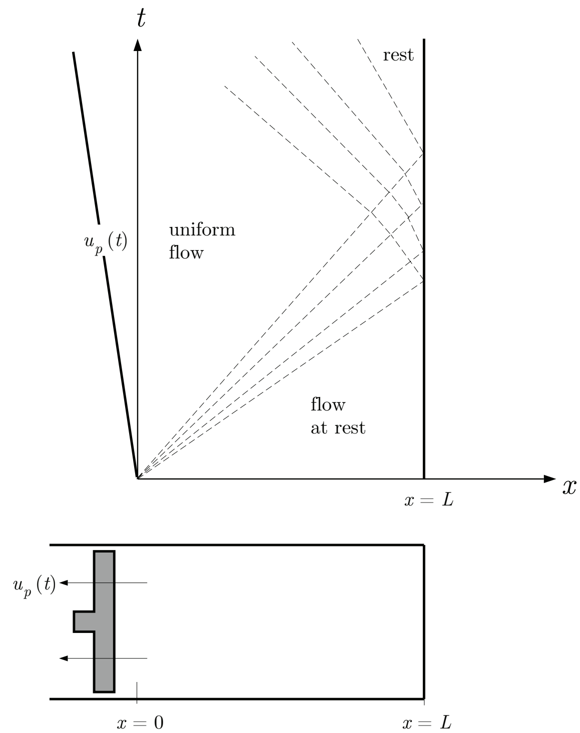

Figure 3.27: $x - t$ diagram for expansion wall interaction.

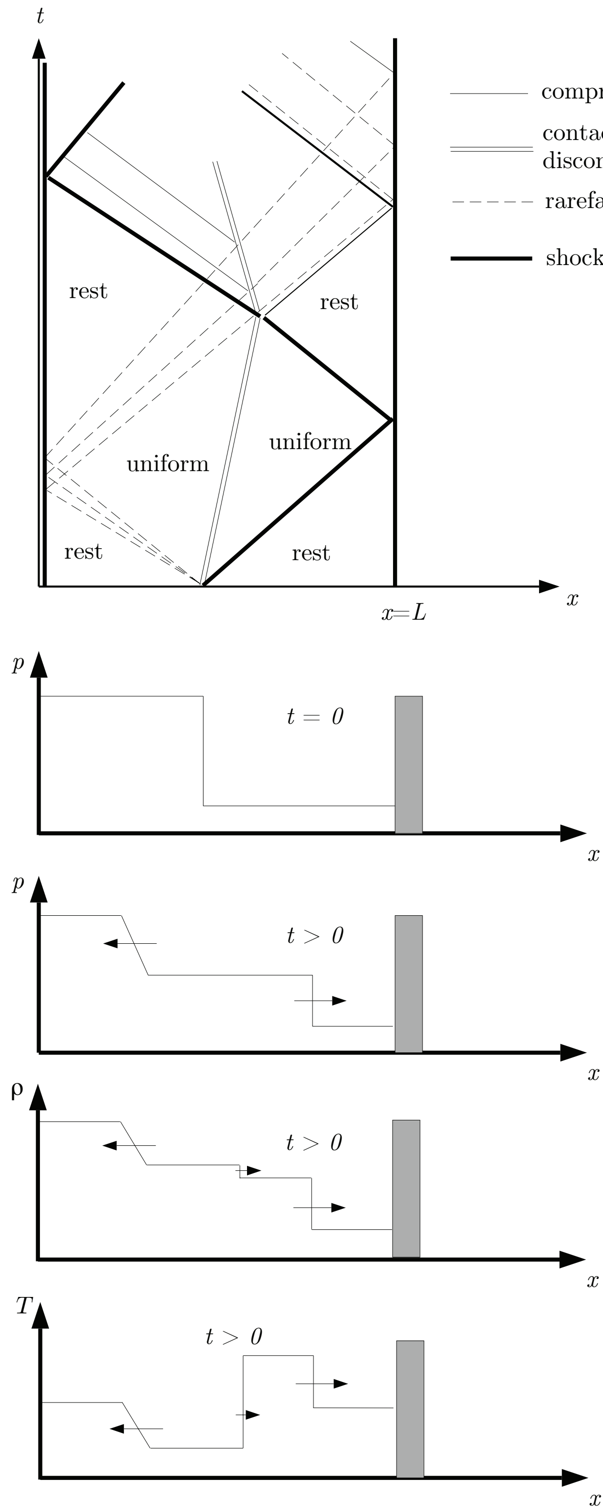

Figure 3.28: $x - t$ and p, ρ, T versus x behavior for a shock tube.

3.5.9 Final note on method of characteristics

We have described here a common and traditional approach to the method of characteristics (MOC). Using common notation, we have written what began as partial differential equations (PDEs) in the form of ordinary differential equations (ODEs), and it is often said that the method of characteristics is a way to *transform* PDEs into ODEs. However, the equations which result are certainly not in a standard form for ODEs; they are burdened with unusual side conditions.

It is in fact more sound to state that the MOC transforms the PDEs in (x, t) space to another set of PDEs in a new space (r, s) in which the integration is much easier. Consider for example a model equation which is hyperbolic, the inviscid Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{3.330}$$

Now consider a general transformation $(x, t) \to (r, s)$. Applying the chain rule, we get

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t}, \tag{3.331}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}. \tag{3.332}$$

In transformed space, the inviscid Burgers' equation becomes

$$\frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} + u\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}\right) = 0. \tag{3.333}$$

Now we also have

$$dx = \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial s}\,ds, \tag{3.334}$$

$$dt = \frac{\partial t}{\partial r}\,dr + \frac{\partial t}{\partial s}\,ds. \tag{3.335}$$

With the Jacobian[11]

$$J = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r}, \tag{3.336}$$

we invert to find

$$dr = \frac{1}{J}\left(\frac{\partial t}{\partial s}\,dx - \frac{\partial x}{\partial s}\,dt\right), \tag{3.337}$$

$$ds = \frac{1}{J}\left(-\frac{\partial t}{\partial r}\,dx + \frac{\partial x}{\partial r}\,dt\right). \tag{3.338}$$

[11]Carl Gustav Jacob Jacobi, 1804-1851, Prussian born, prolific German mathematician. The Jacobian determinant was extensively studied by Jacobi, but first identified by Cauchy.

So it is easy to see that we get the following for the partial derivatives

$$\frac{\partial r}{\partial t} = -\frac{1}{J}\frac{\partial x}{\partial s}, \qquad \frac{\partial r}{\partial x} = \frac{1}{J}\frac{\partial t}{\partial s}, \tag{3.339}$$

$$\frac{\partial s}{\partial t} = \frac{1}{J}\frac{\partial x}{\partial r}, \qquad \frac{\partial s}{\partial x} = -\frac{1}{J}\frac{\partial t}{\partial r}. \tag{3.340}$$

Substituting into the inviscid Burgers' equation, we get

$$\frac{1}{J}\left(-\frac{\partial u}{\partial r}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial s}\frac{\partial x}{\partial r} + u\left(\frac{\partial u}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial t}{\partial r}\right)\right) = 0, \tag{3.341}$$

$$-\frac{\partial u}{\partial r}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial s}\frac{\partial x}{\partial r} + u\frac{\partial u}{\partial r}\frac{\partial t}{\partial s} - u\frac{\partial u}{\partial s}\frac{\partial t}{\partial r} = 0. \tag{3.342}$$

Up to this point we have a perfectly general transformation and a perfectly general inviscid Burgers' equation, now cast in the transformed space. Let us now demand of our transformation that

$$\frac{\partial x}{\partial s} = u\frac{\partial t}{\partial s}, \qquad t(r,s) = s. \tag{3.343}$$

The first of these says that on any line on which r is a constant that for a given change in s, the ratio of the change in x to that in t will be equal to u. This is a generalization of our more standard statement that on characteristics, $dx/dt = u$. The second is a convenience, and we actually need not be as restrictive. With this specification, our inviscid Burgers' equation becomes

$$-\frac{\partial u}{\partial r}\underbrace{\frac{\partial x}{\partial s}}_{=u\frac{\partial t}{\partial s}=u} + \frac{\partial u}{\partial s}\frac{\partial x}{\partial r} + u\frac{\partial u}{\partial r}\underbrace{\frac{\partial t}{\partial s}}_{=1} - u\frac{\partial u}{\partial s}\underbrace{\frac{\partial t}{\partial r}}_{=0} = 0, \tag{3.344}$$

$$-u\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\frac{\partial x}{\partial r} + u\frac{\partial u}{\partial r} = 0, \tag{3.345}$$

$$\frac{\partial u}{\partial s}\frac{\partial x}{\partial r} = 0. \tag{3.346}$$

Now, let us require that $\partial x/\partial r \neq 0$; hence in this special transformed space, we have that

$$\frac{\partial u}{\partial s} = 0. \tag{3.347}$$

This has solution

$$u = f(r), \tag{3.348}$$

where $f(r)$ is an arbitrary function. We now substitute this into $\partial x/\partial s = u(\partial t/\partial s)$ to get

$$\frac{\partial x}{\partial s} = f(r)\frac{\partial t}{\partial s}, \tag{3.349}$$

which can be integrated to get

$$x = f(r)t + g(r). \tag{3.350}$$

Now substituting $t = s$ and setting $g(r) = r$ arbitrarily so that our transformation maps x into r when $t = s = 0$, we get

$$x = f(r)s + r. \tag{3.351}$$

In summary we can write a solution parametrically in terms of our transformed space as

$$u(r, s) = f(r), \tag{3.352}$$
$$x(r, s) = f(r)s + r, \tag{3.353}$$
$$t(r, s) = s. \tag{3.354}$$

So given an initial distribution of u, we can select a domain in (r, s) and parametrically determine u as a function of x and t. While this formulation maps every (r, s) into (u, x, t), we cannot be assured that in physical space that the same (x, t) may not map into non-unique values of u! This multi-valuedness actually indicates that a shock has formed, and correct insertion of a shock will eliminate the difficulty.

Example 3.5

If we have the inviscid Burgers' equation $\partial u/\partial t + u(\partial u/\partial x) = 0$ with $u(x, 0) = \sin(\pi x)$, find u, and plot $u(x)$ for $t = 0, 1, 2$.

When $t = s = 0$, we have $x = r$, so $f(r) = \sin(\pi r)$, and our solution is

$$u(r, s) = \sin(\pi r), \tag{3.355}$$
$$x(r, s) = s \sin(\pi r) + r, \tag{3.356}$$
$$t(r, s) = s. \tag{3.357}$$

We can use this solution to form parametric plots and effectively form $u(x)$ for various values of t. These are shown in Figure 3.29. It is clear that as time advances the left side of the wave is flattening and the right side is steepening. The left side is undergoing what is equivalent to a rarefaction, and the right side is undergoing what is equivalent to a compression. At $t = 3$, the wave has steepened enough so that u is a multivalued function of x. In a physical problem, this would indicate that a shock has formed.

This procedure can be extended to the Euler equations, though it is somewhat more complicated. For isentropic Euler equations, Courant and Friedrichs give some special solutions for rarefactions.

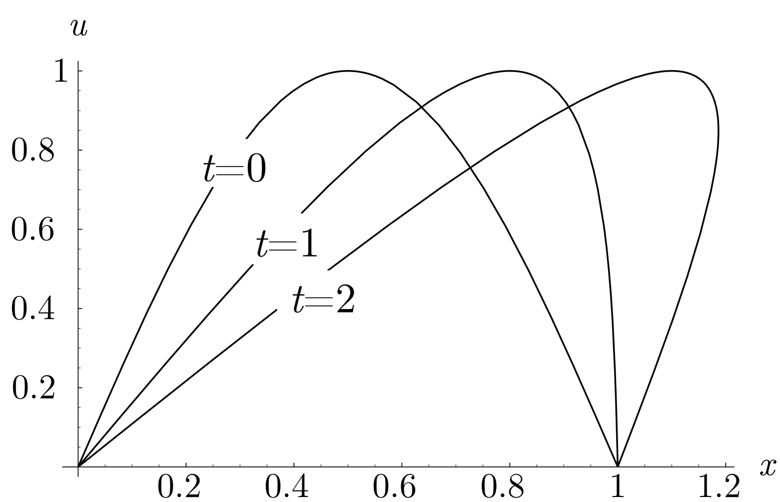

Figure 3.29: $u(x)$ for $t = 0, 1, 2$ for inviscid Burgers' equation problem.

Chapter 4

Potential flow

see Panton, Chapter 18
see Yih, Chapter 4

This chapter will consider potential flow. A good deal of highly developed and beautiful mathematical theory was generated for potential flows in the nineteenth century. Additionally, these solutions can be applied in highly disparate fields, as the equations governing potential flow of a fluid are identical in form to those governing some forms of energy and mass diffusion, as well as electro-magnetics.

Despite its beauty, in some ways it is impractical for many engineering applications, though not all. As the theory necessarily ignores all vorticity generating mechanisms, it must ignore viscous effects. Consequently, the theory is incapable of predicting drag forces on solid bodies. Consequently, those who needed to know the drag, resorted in the nineteenth century to far more empirically based methods.

In the early twentieth century, Prandtl took steps to reconcile the practical viscous world of engineering with the more mathematical world of potential flow with his viscous boundary layer theory. He showed that indeed potential flow solutions could be of great value away from no-slip walls, and provided a recipe to fix the solutions in the neighborhood of the wall. In so doing, he opened up a new field of applied mathematics known as matched asymptotic analysis.

So why study potential flows? The following arguments offer some justification.

- portions of real flow fields are well described by this theory, and those that are not can often be remedied by application of a viscous boundary layer theory,

- study of potential flow solutions can give great insight into fluid behavior and aid in the honing of a more precise intuition,

- fundamental solutions are useful as test cases for verification of numerical methods, and

- there is pedantic and historical value in knowing potential flow.

4.1 Stream functions and velocity potentials

We first consider stream functions and velocity potentials. We have seen velocity potentials before in study of ideal vortices. In this chapter, we will adopt the same assumption of irrotationality, and further require that the flow be two-dimensional. Recall if a flow velocity is confined to the $x-y$ plane, then the vorticity vector is confined to the z direction and takes the form

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}. \tag{4.1}$$

Now if the flow is two-dimensional and irrotational, we have

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{4.2}$$

Moreover, because of irrotationality, we can express the velocity vector $\mathbf{v}$ as the gradient of a potential ϕ, the *velocity potential*:

$$\mathbf{v} = \nabla \phi. \tag{4.3}$$

Note that with this definition, fluid flows from regions of low velocity potential to regions of high velocity potential. Thus,

$$u = \frac{\partial \phi}{\partial x}, \tag{4.4}$$

$$v = \frac{\partial \phi}{\partial y}. \tag{4.5}$$

We see by substitution into the equation for vorticity, that this is true identically:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0. \tag{4.6}$$

Now for two-dimensional incompressible flows, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.7}$$

Substituting for u and v in favor of ϕ, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0, \tag{4.8}$$

$$\nabla^2 \phi = 0. \tag{4.9}$$

Now if the flow is incompressible, we can also define the *stream function* ψ as follows:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{4.10}$$

Direct substitution into the mass conservation equation shows that this yields an identity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0. \tag{4.11}$$

Now, in an equation which will be critically important soon, we can set our definitions of u and v in terms of ϕ and ψ equal to each other, as they must be:

$$\underbrace{\frac{\partial \phi}{\partial x}}_{u} = \underbrace{\frac{\partial \psi}{\partial y}}_{u}, \tag{4.12}$$

$$\underbrace{\frac{\partial \phi}{\partial y}}_{v} = \underbrace{-\frac{\partial \psi}{\partial x}}_{v}. \tag{4.13}$$

Now if we differentiate the first equation with respect to y, and the second with respect to x we see

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial y^2}, \tag{4.14}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial^2 \psi}{\partial x^2}, \tag{4.15}$$

$$\text{now subtract the second from the first to get} \tag{4.16}$$

$$0 = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}, \tag{4.17}$$

$$\nabla^2 \psi = 0. \tag{4.18}$$

Let us know examine lines of constant ϕ (equipotential lines) and lines of constant ψ (which we will see are streamlines). So take $\phi = C_1$, $\psi = C_2$. For ϕ we get

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y}\, dy = 0, \tag{4.19}$$

$$d\phi = u\, dx + v\, dy = 0, \tag{4.20}$$

$$\left.\frac{dy}{dx}\right|_{\phi = C_1} = -\frac{u}{v}. \tag{4.21}$$

Now for ψ we similarly get

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy = 0, \tag{4.22}$$

$$d\psi = -v\, dx + u\, dy = 0, \tag{4.23}$$

$$\left.\frac{dy}{dx}\right|_{\psi = C_2} = \frac{v}{u} \tag{4.24}$$

We note two features

- $\left.\frac{dy}{dx}\right|_{\phi=C_1} = -\frac{1}{\left.\frac{dy}{dx}\right|_{\psi=C_2}}$; hence, lines of constant ϕ are orthogonal to lines of constant ψ, and

- on $\psi = C_2$, we see that $dx/u = dy/v$; hence, lines of $\psi = C_2$ must be streamlines.

As an aside, we note that the definition of the stream function $u = \partial\psi/\partial y, v = -\partial\psi/\partial x$, can be rewritten as

$$\frac{\partial\psi}{\partial y} = \frac{dx}{dt}, \qquad \frac{\partial\psi}{\partial x} = -\frac{dy}{dt}. \tag{4.25}$$

This is a common form from classical dynamics in which we can interpret ψ as the *Hamiltonian*[1] of the system. We shall not pursue this path, but note that a significant literature exists for Hamiltonian systems.

Now the study of ϕ and ψ is essentially kinematics. The only incursion of dynamics is that we must have irrotational flow. Recalling the Helmholtz equation, Eq. (2.132), we realize that we can only have potential flow when the vorticity generating mechanisms (three-dimensional effects, non-conservative body forces, baroclinic effects, and viscous effects) are suppressed. In that case, the dynamics, that is the driving force for the fluid motion, can be understood in the context of the unsteady Bernoulli equation, Eq. (1.966, taken for incompressible flow and negligible body force, in which limit, Eq. (1.957) reduces to $\Upsilon = p/\rho$:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^T \cdot \nabla\phi + \frac{p}{\rho} = f(t). \tag{4.26}$$

Note that we do not have to require steady flow to have a potential flow field. It is also easy to correct for the presence of a conservative body force.

Now solutions to the two key equations of potential flow $\nabla^2\phi = 0, \nabla^2\psi = 0$, are most efficiently studied using methods involving complex variables. We will delay discussing solutions until we have reviewed the necessary mathematics.

4.2 Mathematics of complex variables

Here we briefly introduce relevant elements of complex variable theory. Recall that the imaginary number i is defined such that

$$i^2 = -1, \qquad i = \sqrt{-1}. \tag{4.27}$$

[1]William Rowan Hamilton, 1805-1865, Anglo-Irish mathematician.

4.2.1 Euler's formula

We can get a very useful formula *Euler's formula*, by considering the following Taylor[2] expansions of common functions about $t = 0$:

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 \ldots, \tag{4.28}$$

$$\sin t = 0 + t + 0\frac{1}{2!}t^2 - \frac{1}{3!}t^3 + 0\frac{1}{4!}t^4 + \frac{1}{5!}t^5 \ldots, \tag{4.29}$$

$$\cos t = 1 + 0t - \frac{1}{2!}t^2 + 0\frac{1}{3!}t^3 + \frac{1}{4!}t^4 + 0\frac{1}{5!}t^5 \ldots \tag{4.30}$$

With these expansions now consider the following combinations: $(\cos t + i \sin t)_{t=\theta}$ and $e^t|_{t=i\theta}$:

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 + \ldots, \tag{4.31}$$

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \ldots, \tag{4.32}$$

$$= 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 + \ldots \tag{4.33}$$

As the two series are identical, we have Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{4.34}$$

4.2.2 Polar and Cartesian representations

Now if we take x and y to be real numbers and define the complex number z to be

$$z = x + iy, \tag{4.35}$$

we can multiply and divide by $\sqrt{x^2 + y^2}$ to obtain

$$z = \sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}}\right). \tag{4.36}$$

Noting the similarities between this and the transformation between Cartesian and polar coordinates suggests we adopt

$$r = \sqrt{x^2 + y^2}, \qquad \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}. \tag{4.37}$$

[2]Brook Taylor, 1685-1731, English mathematician and artist, Cambridge educated, published on capillary action, magnetism, and thermometers, adjudicated the dispute between Newton and Leibniz over priority in developing calculus, contributed to the method of finite differences, invented integration by parts, name ascribed to Taylor series of which variants were earlier discovered by Gregory, Newton, Leibniz, Johann Bernoulli, and de Moivre.

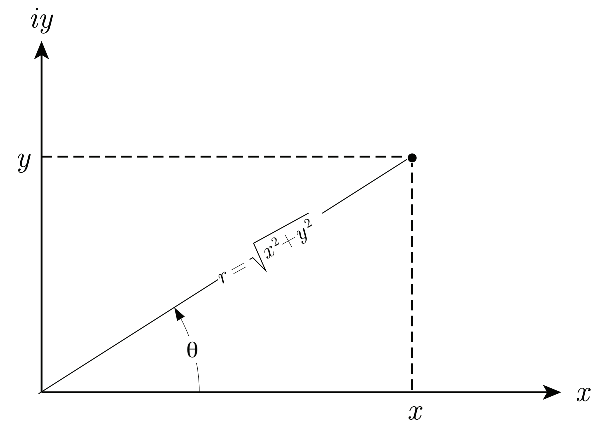

Figure 4.1: Polar and Cartesian representation of a complex number z.

Thus we have

$$z = r\left(\cos\theta + i\sin\theta\right), \tag{4.38}$$
$$z = re^{i\theta}. \tag{4.39}$$

The polar and Cartesian representation of a complex number z is shown in Figure 4.1. Now we can define the *complex conjugate* $\overline{z}$ as

$$\overline{z} = x - iy, \tag{4.40}$$
$$\overline{z} = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} - i\frac{y}{\sqrt{x^2+y^2}}\right), \tag{4.41}$$
$$\overline{z} = r\left(\cos\theta - i\sin\theta\right), \tag{4.42}$$
$$\overline{z} = r\left(\cos(-\theta) + i\sin(-\theta)\right), \tag{4.43}$$
$$\overline{z} = re^{-i\theta}. \tag{4.44}$$

Note now that

$$z\overline{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2, \tag{4.45}$$
$$= re^{i\theta}re^{-i\theta} = r^2 = |z|^2. \tag{4.46}$$

We also have

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}, \tag{4.47}$$
$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}. \tag{4.48}$$

4.2.3 Cauchy-Riemann equations

Now it is possible to define complex functions of complex variables $W(z)$. For example take a complex function to be defined as

$$\begin{aligned} W(z) &= z^2 + z, &(4.49)\\ &= (x+iy)^2 + (x+iy), &(4.50)\\ &= x^2 + 2xyi - y^2 + x + iy, &(4.51)\\ &= \left(x^2 + x - y^2\right) + i\,(2xy + y)\,. &(4.52) \end{aligned}$$

In general, we can say

$$W(z) = \phi(x,y) + i\psi(x,y). \tag{4.53}$$

Here ϕ and ψ are *real* functions of *real* variables.

Now $W(z)$ is defined as *analytic* at z_o if dW/dz exists at z_o and is independent of the direction in which it was calculated. That is, using the definition of the derivative

$$\left.\frac{dW}{dz}\right|_{z_o} = \frac{W(z_o + \Delta z) - W(z_o)}{\Delta z}. \tag{4.54}$$

Now there are many paths that we can choose to evaluate the derivative. Let us consider two distinct paths, $y = C_1$ and $x = C_2$. We will get a result which can be shown to be valid for arbitrary paths. For $y = C_1$, we have $\Delta z = \Delta x$, so

$$\left.\frac{dW}{dz}\right|_{z_o} = \frac{W(x_o + iy_o + \Delta x) - W(x_o + iy_o)}{\Delta x} = \left.\frac{\partial W}{\partial x}\right|_y. \tag{4.55}$$

For $x = C_2$, we have $\Delta z = i\Delta y$, so

$$\left.\frac{dW}{dz}\right|_{z_o} = \frac{W(x_o + iy_o + i\Delta y) - W(x_o + iy_o)}{i\Delta y} = \frac{1}{i}\left.\frac{\partial W}{\partial y}\right|_x = -i\left.\frac{\partial W}{\partial y}\right|_x. \tag{4.56}$$

Now for an analytic function, we need

$$\left.\frac{\partial W}{\partial x}\right|_y = -i\left.\frac{\partial W}{\partial y}\right|_x, \tag{4.57}$$

or, expanding, we need

$$\begin{aligned} \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} &= -i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right), &(4.58)\\ &= \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}. &(4.59) \end{aligned}$$

For equality, and thus path independence of the derivative, we require

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{4.60}$$

These are the well known *Cauchy-Riemann*[3] equations for analytic functions of complex variables. *They are identical to our kinematic equations for incompressible irrotational fluid mechanics.* Consequently, *any analytic complex function is guaranteed to be a physical solution.* There are essentially an infinite number of functions to choose from.

Thus we define the *complex velocity potential* as

$$W(z) = \phi(x, y) + i\psi(x, y), \tag{4.61}$$

and taking a derivative of the analytic potential, we have

$$\frac{dW}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv. \tag{4.62}$$

We can equivalently say

$$\frac{dW}{dz} = -i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right) = \left(\frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}\right) = u - iv. \tag{4.63}$$

Now most common functions are easily shown to be analytic. For example for the function $W(z) = z^2 + z$, which can be expressed as $W(z) = (x^2 + x - y^2) + i(2xy + y)$, we have

$$\phi(x, y) = x^2 + x - y^2, \qquad \psi(x, y) = 2xy + y, \tag{4.64}$$

$$\frac{\partial \phi}{\partial x} = 2x + 1, \qquad \frac{\partial \psi}{\partial x} = 2y, \tag{4.65}$$

$$\frac{\partial \phi}{\partial y} = -2y, \qquad \frac{\partial \psi}{\partial y} = 2x + 1. \tag{4.66}$$

Note that the Cauchy-Riemann equations are satisfied since $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = -\partial\psi/\partial x$. So the derivative is independent of direction, and we can say

$$\frac{dW}{dz} = \left.\frac{\partial W}{\partial x}\right|_y = (2x + 1) + i(2y) = 2(x + iy) + 1 = 2z + 1. \tag{4.67}$$

We could get this result by ordinary rules of derivatives for real functions.

For example of a non-analytic function consider $W(z) = \overline{z}$. Thus

$$W(z) = x - iy. \tag{4.68}$$

So $\phi = x$ and $\psi = -y$, $\partial\phi/\partial x = 1$, $\partial\phi/\partial y = 0$, and $\partial\psi/\partial x = 0$, $\partial\psi/\partial y = -1$. Since $\partial\phi/\partial x \neq \partial\psi/\partial y$, the Cauchy-Riemann equations are not satisfied, and the derivative depends on direction.

[3]Augustin-Louis Cauchy, 1789-1857, French mathematician and military engineer, worked in complex analysis, optics, and theory of elasticity.

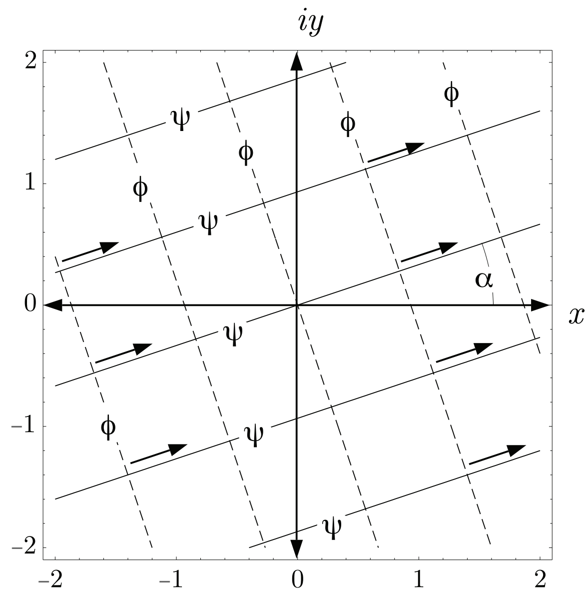

Figure 4.2: Streamlines for uniform flow.

4.3 Elementary complex potentials

Let us examine some simple analytic functions and see the fluid mechanics to which they correspond.

4.3.1 Uniform flow

Take

$$W(z) = Az, \qquad \text{with} \qquad A \in \mathbb{C}^1. \tag{4.69}$$

Then

$$\frac{dW}{dz} = A = u - iv. \tag{4.70}$$

Since A is complex, we can say

$$A = U_o e^{-i\alpha} = U_o \cos\alpha - iU_o \sin\alpha. \tag{4.71}$$

Thus we get

$$u = U_o \cos\alpha, \qquad v = U_o \sin\alpha. \tag{4.72}$$

This represents a spatially uniform flow with streamlines inclined at angle α to the x axis. The flow is sketched in Figure 4.2.

4.3.2 Sources and sinks

Take

$$W(z) = A\ln z, \qquad \text{with} \qquad A \in \mathbb{R}^1. \tag{4.73}$$

With $z = re^{i\theta}$ we have $\ln z = \ln r + i\theta$. So

$$W(z) = A\ln r + iA\theta. \tag{4.74}$$

Consequently, we have for the velocity potential and stream function

$$\phi = A\ln r, \qquad \psi = A\theta. \tag{4.75}$$

Now $\mathbf{v} = \nabla\phi$, so

$$v_r = \frac{\partial\phi}{\partial r} = \frac{A}{r}, \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0. \tag{4.76}$$

So the velocity is all radial, and becomes infinite at $r = 0$. We can show that the volume flow rate is bounded, and is in fact a constant. The volume flow rate Q through a surface is

$$Q = \int_A \mathbf{v}^T \cdot \mathbf{n}\ dA = \int_0^{2\pi} v_r r\ d\theta = \int_0^{2\pi} \frac{A}{r} r\ d\theta = 2\pi A. \tag{4.77}$$

The volume flow rate is a constant. If $A > 0$, we have a source. If $A < 0$, we have a sink. The potential for a source/sink is often written as

$$W(z) = \frac{Q}{2\pi}\ln z. \tag{4.78}$$

For a source located at a point z_o which is not at the origin, we can say

$$W(z) = \frac{Q}{2\pi}\ln(z - z_o). \tag{4.79}$$

The flow is sketched in Figure 4.3.

4.3.3 Point vortices

For an ideal point vortex, identical to what we studied in an earlier chapter, we have

$$W(z) = iB\ln z, \qquad \text{with} \qquad B \in \mathbb{R}^1. \tag{4.80}$$

So

$$W(z) = iB\left(\ln r + i\theta\right) = -B\theta + iB\ln r. \tag{4.81}$$

Consequently,

$$\phi = -B\theta, \qquad \psi = B\ln r. \tag{4.82}$$

We get the velocity field from

$$v_r = \frac{\partial\phi}{\partial r} = 0, \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{B}{r}. \tag{4.83}$$

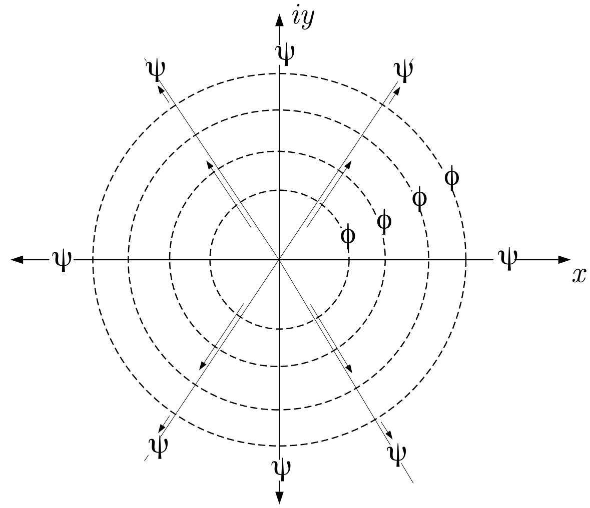

Figure 4.3: Velocity vectors and equipotential lines for source flow.

So we see that the streamlines are circles about the origin, and there is no radial component of velocity. Consider the circulation of this flow

$$\Gamma = \oint_C \mathbf{v}^T \cdot d\mathbf{r} = \int_0^{2\pi} -\frac{B}{r} r \; d\theta = -2\pi B. \tag{4.84}$$

So we often write the complex potential in terms of the ideal vortex strength Γ:

$$W(z) = -\frac{i\Gamma}{2\pi} \ln z. \tag{4.85}$$

For an ideal vortex not at $z = z_o$, we say

$$W(z) = -\frac{i\Gamma}{2\pi} \ln(z - z_o). \tag{4.86}$$

The point vortex flow is sketched in Figure 4.4.

4.3.4 Superposition of sources

Since the equation for velocity potential is linear, we can use the method of superposition to create new solutions as summations of elementary solutions. Say we want to model the effect of a wall on a source as sketched in Figure 4.5. At the wall we want $u(0, y) = 0$. That is

$$\Re\left\{\frac{dW}{dz}\right\} = \Re\left\{u - iv\right\} = 0, \qquad \text{on} \qquad z = iy. \tag{4.87}$$

Here $\Re$ denotes the real part of a complex function. Now let us place a source at $z = a$ and superpose a source at $z = -a$, where a is a real number. So we have for the complex

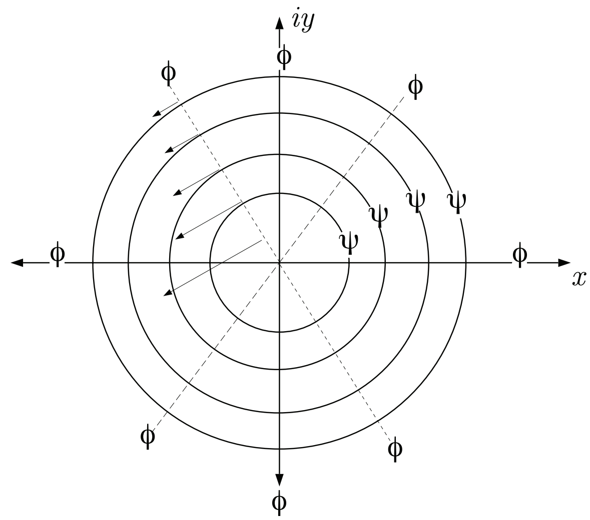

Figure 4.4: Streamlines, equipotential, and velocity vectors lines for a point vortex.

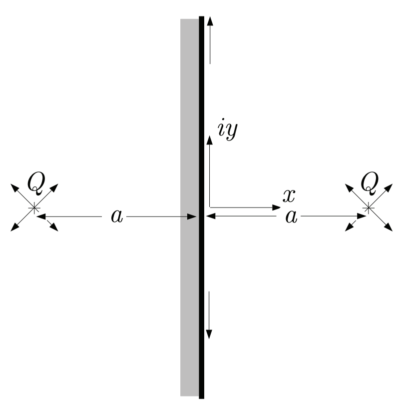

Figure 4.5: Sketch for source-wall interaction.

potential

$$W(z) = \underbrace{\frac{Q}{2\pi}\ln(z-a)}_{\text{original}} + \underbrace{\frac{Q}{2\pi}\ln(z+a)}_{\text{image}}, \tag{4.88}$$

$$= \frac{Q}{2\pi}\left(\ln(z-a)+\ln(z+a)\right), \tag{4.89}$$

$$= \frac{Q}{2\pi}\left(\ln(z-a)(z+a)\right), \tag{4.90}$$

$$= \frac{Q}{2\pi}\ln(z^2-a^2), \tag{4.91}$$

$$\frac{dW}{dz} = \frac{Q}{2\pi}\frac{2z}{z^2-a^2}. \tag{4.92}$$

Now on $z = iy$, which is the location of the wall, we have

$$\frac{dW}{dz} = \frac{Q}{2\pi}\left(\frac{2iy}{-y^2-a^2}\right). \tag{4.93}$$

The term is purely imaginary; hence, the real part is zero, and we have $u = 0$ on the wall, as desired.

On the wall we do have a non-zero y component of velocity. Hence the wall is not a no-slip wall. On the wall we have then

$$v = \frac{Q}{\pi}\frac{y}{y^2+a^2}. \tag{4.94}$$

We find the location on the wall of the maximum v velocity by setting the derivative with respect to y to be zero,

$$\frac{\partial v}{\partial y} = \frac{Q}{\pi}\frac{(y^2+a^2)-y(2y)}{(y^2+a^2)^2} = 0. \tag{4.95}$$

Solving, we find a critical point at $y = \pm a$, which can be shown to be a maximum.

So on the wall we have

$$\frac{1}{2}(u^2+v^2) = \frac{1}{2}\frac{Q^2}{\pi^2}\frac{y^2}{(y^2+a^2)^2}. \tag{4.96}$$

We can use Bernoulli's equation to find the pressure field, assuming steady flow and that $p \to p_o$ as $r \to \infty$. So Bernoulli's equation in this limit

$$\frac{1}{2}(\nabla\phi)^T\cdot\nabla\phi + \frac{p}{\rho} = \frac{p_o}{\rho}, \tag{4.97}$$

reduces to

$$p = p_o - \frac{1}{2}\rho\frac{Q^2}{\pi^2}\frac{y^2}{\left(y^2+a^2\right)^2}. \tag{4.98}$$

Note that the pressure is p_o at $y = 0$ and is p_o as $y \to \infty$. By integrating the pressure over the wall surface, one would find that the source exerted a net force on the wall.

4.3.5 Flow in corners

Flow in or around a corner can be modeled by the complex potential

$$\begin{aligned} W(z) &= Az^n, \qquad \text{with } A \in \mathbb{R}^1, && (4.99)\\ &= A\left(re^{i\theta}\right)^n, && (4.100)\\ &= Ar^n e^{in\theta}, && (4.101)\\ &= Ar^n(\cos(n\theta) + i\sin(n\theta)). && (4.102) \end{aligned}$$

So we have

$$\phi = Ar^n \cos n\theta, \qquad \psi = Ar^n \sin n\theta. \qquad (4.103)$$

Now recall that lines on which ψ is constant are streamlines. Examining the stream function, we obviously have streamlines when $\psi = 0$ which occurs whenever $\theta = 0$ or $\theta = \pi/n$.

For example if $n = 2$, we model a stream striking a flat wall. For this flow, we have

$$\begin{aligned} W(z) &= Az^2, && (4.104)\\ &= A(x + iy)^2, && (4.105)\\ &= A((x^2 - y^2) + i(2xy)), && (4.106)\\ \phi &= A(x^2 - y^2), \qquad \psi = A(2xy). && (4.107) \end{aligned}$$

So the streamlines are hyperbolas. For the velocity field, we take

$$\frac{dW}{dz} = 2Az = 2A(x + iy) = u - iv, \qquad (4.108)$$

$$u = 2Ax, \qquad v = -2Ay. \qquad (4.109)$$

This flow actually represents flow in a corner formed by a right angle or flow striking a flat plate, or the impingement of two streams. For $n = 2$, streamlines are sketched in in Figure 4.6.

4.3.6 Doublets

We can form what is known as a doublet flow by considering the superposition of a source and sink and let the two approach each other. Consider a source and sink of equal and opposite strength straddling the y axis, each separated from the origin by a distance ϵ as sketched in Figure 4.7. The complex velocity potential is

$$\begin{aligned} W(z) &= \frac{Q}{2\pi}\ln(z + \epsilon) - \frac{Q}{2\pi}\ln(z - \epsilon), && (4.110)\\ &= \frac{Q}{2\pi}\ln\left(\frac{z + \epsilon}{z - \epsilon}\right). && (4.111) \end{aligned}$$

It can be shown by synthetic division that as $\epsilon \to 0$, that

$$\frac{z + \epsilon}{z - \epsilon} = 1 + \epsilon\frac{2}{z} + \epsilon^2\frac{2}{z^2} + \ldots. \qquad (4.112)$$

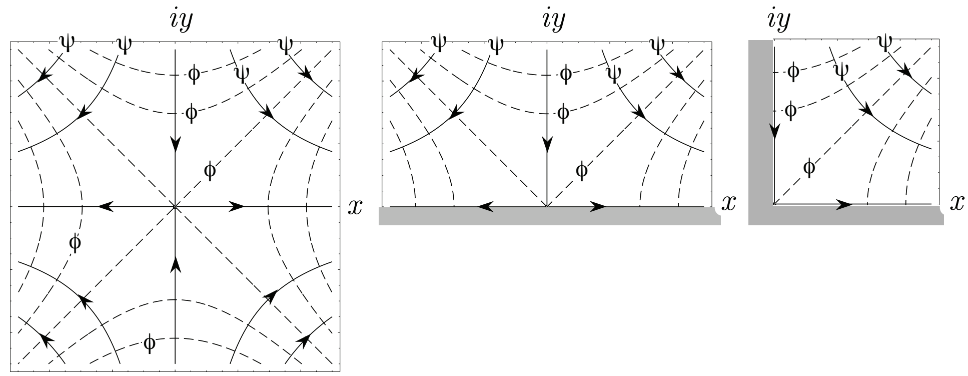

Figure 4.6: Sketch for impingement flow, stagnation flow, and flow in a corner, $n = 2$.

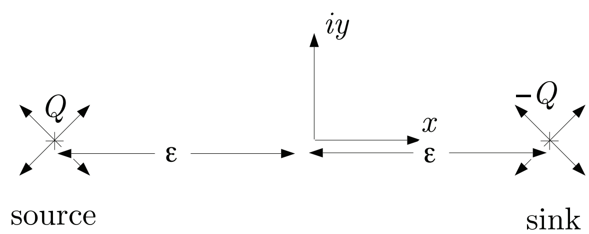

Figure 4.7: Source sink pair.

So the potential approaches

$$W(z) \sim \frac{Q}{2\pi} \ln\left(1 + \epsilon\frac{2}{z} + \epsilon^2\frac{2}{z^2} + \dots\right). \tag{4.113}$$

Now since $\ln(1+x) \to x$ as $x \to 0$, we get for small ϵ that

$$W(z) \sim \frac{Q}{2\pi}\epsilon\frac{2}{z} \sim \frac{Q\epsilon}{\pi z}. \tag{4.114}$$

Now if we require that

$$\lim_{\epsilon\to 0}\frac{Q\epsilon}{\pi} \to \mu, \tag{4.115}$$

we have

$$W(z) = \frac{\mu}{z} = \frac{\mu}{x+iy}\frac{x-iy}{x-iy} = \frac{\mu(x-iy)}{x^2+y^2}. \tag{4.116}$$

So

$$\phi(x,y) = \mu\frac{x}{x^2+y^2}, \qquad \psi(x,y) = -\mu\frac{y}{x^2+y^2}. \tag{4.117}$$

In polar coordinates, we then say

$$\phi = \mu\frac{\cos\theta}{r}, \qquad \psi = -\mu\frac{\sin\theta}{r}. \tag{4.118}$$

Streamlines and equipotential lines for a doublet are plotted in Figure 4.8.

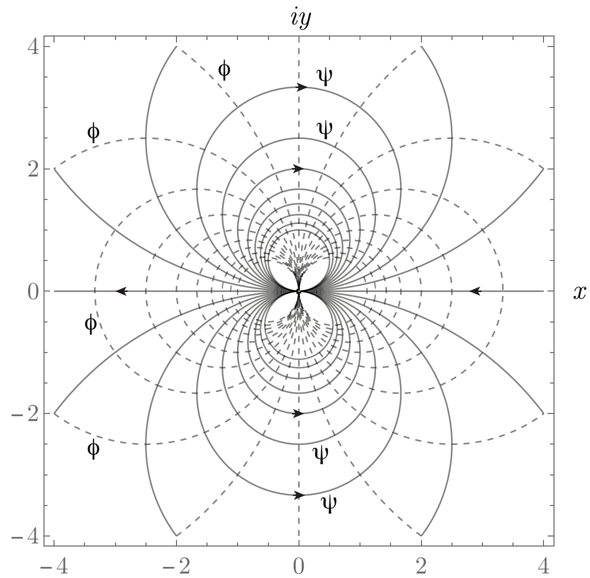

Figure 4.8: Streamlines and equipotential lines for a doublet.

4.3.7 Rankine half body

Now consider the superposition of a uniform stream and a source, which we define to be a Rankine half body:

$$W(z) = Uz + \frac{Q}{2\pi}\ln z, \qquad \text{with} \qquad U, Q \in \mathbb{R}^1, \tag{4.119}$$

$$= Ure^{i\theta} + \frac{Q}{2\pi}(\ln r + i\theta), \tag{4.120}$$

$$= Ur(\cos\theta + i\sin\theta) + \frac{Q}{2\pi}(\ln r + i\theta), \tag{4.121}$$

$$= \left(Ur\cos\theta + \frac{Q}{2\pi}\ln r\right) + i\left(Ur\sin\theta + \frac{Q}{2\pi}\theta\right). \tag{4.122}$$

So

$$\phi = Ur\cos\theta + \frac{Q}{2\pi}\ln r, \qquad \psi = Ur\sin\theta + \frac{Q}{2\pi}\theta. \tag{4.123}$$

Streamlines for a Rankine half body are plotted in Figure 4.9. Now for the Rankine half body, it is clear that there is a stagnation point somewhere on the x axis, along $\theta = \pi$. With the velocity given by

$$\frac{dW}{dz} = U + \frac{Q}{2\pi z} = u - iv, \tag{4.124}$$

we get

$$U + \frac{Q}{2\pi}\frac{1}{r}e^{-i\theta} = u - iv, \tag{4.125}$$

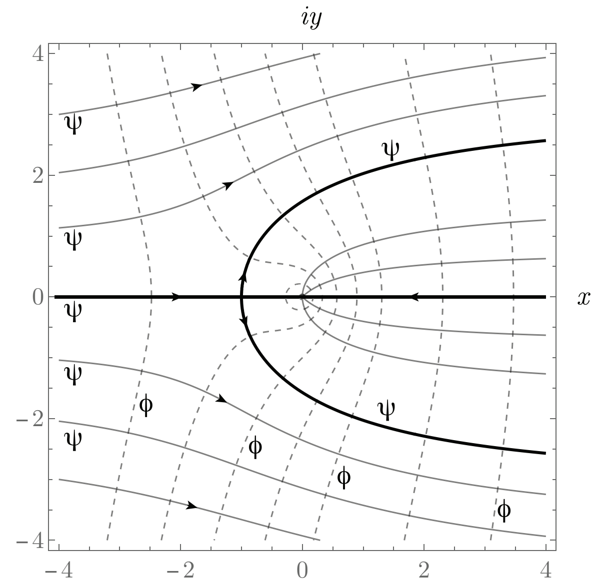

Figure 4.9: Streamlines for a Rankine half body.

$$U + \frac{Q}{2\pi}\frac{1}{r}(\cos\theta - i\sin\theta) \;=\; u - iv, \tag{4.126}$$

$$u = U + \frac{Q}{2\pi r}\cos\theta, \qquad v \;=\; \frac{Q}{2\pi r}\sin\theta. \tag{4.127}$$

When $\theta = \pi$, we get $u = 0$ when;

$$0 \;=\; U + \frac{Q}{2\pi r}(-1), \tag{4.128}$$

$$r \;=\; \frac{Q}{2\pi U}. \tag{4.129}$$

4.3.8 Flow over a cylinder

We can model flow past a cylinder without circulation by superposing a uniform flow with a doublet. Defining $a^2 = \mu/U$, we write

$$W(z) \;=\; Uz + \frac{\mu}{z} = U\left(z + \frac{a^2}{z}\right), \tag{4.130}$$

$$= U\left(re^{i\theta} + \frac{a^2}{re^{i\theta}}\right), \tag{4.131}$$

$$= U\left(r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)\right), \tag{4.132}$$

$$= U\left(\left(r\cos\theta + \frac{a^2}{r}\cos\theta\right) + i\left(r\sin\theta - \frac{a^2}{r}\sin\theta\right)\right), \tag{4.133}$$

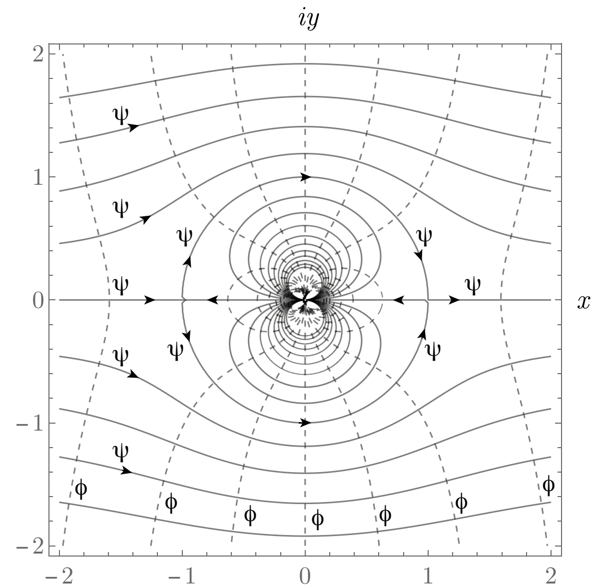

Figure 4.10: Streamlines and equipotential lines for flow over a cylinder without circulation.

$$= Ur\left(\cos\theta\left(1+\frac{a^2}{r^2}\right)+i\sin\theta\left(1-\frac{a^2}{r^2}\right)\right). \tag{4.134}$$

So

$$\phi = Ur\cos\theta\left(1+\frac{a^2}{r^2}\right), \qquad \psi = Ur\sin\theta\left(1-\frac{a^2}{r^2}\right). \tag{4.135}$$

Now on $r = a$, we have $\psi = 0$. Since the stream function is constant here, the curve $r = a$, a circle, must be a streamline through which no mass can pass.

A sketch of the streamlines and equipotential lines is plotted in Figure 4.10.

For the velocities, we have

$$v_r = \frac{\partial\phi}{\partial r} = U\cos\theta\left(1+\frac{a^2}{r^2}\right)+Ur\cos\theta\left(-2\frac{a^2}{r^3}\right), \tag{4.136}$$

$$= U\cos\theta\left(1-\frac{a^2}{r^2}\right), \tag{4.137}$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1+\frac{a^2}{r^2}\right). \tag{4.138}$$

So on $r = a$, we have $v_r = 0$, and $v_\theta = -2U\sin\theta$. Thus on the surface, we have

$$(\nabla\phi)^T\cdot\nabla\phi = 4U^2\sin^2\theta. \tag{4.139}$$

Bernoulli's equation for a steady flow with $p \to p_\infty$ as $r \to \infty$ then gives

$$\frac{p}{\rho}+\frac{1}{2}(\nabla\phi)^T\cdot\nabla\phi = \frac{p_\infty}{\rho}+\frac{U^2}{2}, \tag{4.140}$$

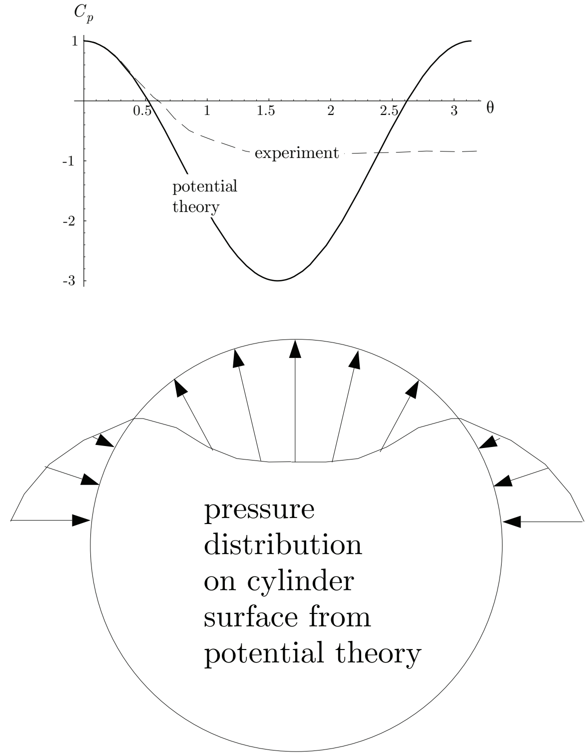

Figure 4.11: Pressure distribution for ideal flow over a cylinder without circulation.

$$p = p_\infty + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta). \tag{4.141}$$

The pressure coefficient C_p, defined below, then is

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta. \tag{4.142}$$

A sketch of the pressure distribution, both predicted and experimentally observed, is plotted in Figure 4.11. We note that the potential theory predicts the pressure well on the front surface of the cylinder, but not so well on the back surface. This is because in most real fluids, a phenomenon known as flow separation manifests itself in regions of negative pressure gradients. Correct modeling of separation events requires a re-introduction of viscous stresses. A potential theory cannot predict separation.

Example 4.1

For a cylinder of radius c at rest in an accelerating potential flow field with a far field velocity of $U = a + bt$, find the pressure on the stagnation point of the cylinder.

The velocity potential and velocities for this flow are

$$\phi(r,\theta,t) = (a+bt)r\cos\theta\left(1+\frac{c^2}{r^2}\right), \tag{4.143}$$

$$v_r = \frac{\partial\phi}{\partial r} = (a+bt)\cos\theta\left(1-\frac{c^2}{r^2}\right), \tag{4.144}$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -(a+bt)\sin\theta\left(1+\frac{c^2}{r^2}\right), \tag{4.145}$$

$$\frac{1}{2}(\nabla\phi)^T\cdot\nabla\phi = \frac{1}{2}(a+bt)^2\left(\cos^2\theta\left(1-\frac{c^2}{r^2}\right)^2+\sin^2\theta\left(1+\frac{c^2}{r^2}\right)^2\right), \tag{4.146}$$

$$= \frac{1}{2}(a+bt)^2\left(1+\frac{c^4}{r^4}+\frac{2c^2}{r^2}\left(\sin^2\theta-\cos^2\theta\right)\right). \tag{4.147}$$

Also, since the flow is unsteady, we will need $\partial\phi/\partial t$:

$$\frac{\partial\phi}{\partial t} = br\cos\theta\left(1+\frac{c^2}{r^2}\right). \tag{4.148}$$

Now we note in the limit as $r\to\infty$ that

$$\frac{\partial\phi}{\partial t}\to br\cos\theta, \qquad \frac{1}{2}(\nabla\phi)^T\cdot\nabla\phi\to\frac{1}{2}(a+bt)^2. \tag{4.149}$$

We also note that on the surface of the cylinder

$$v_r(r=c,\theta,t)=0. \tag{4.150}$$

Bernoulli's equation gives us

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^T\cdot\nabla\phi+\frac{p}{\rho}=f(t). \tag{4.151}$$

We use the far field behavior to evaluate $f(t)$:

$$br\cos\theta+\frac{1}{2}(a+bt)^2+\frac{p}{\rho}=f(t). \tag{4.152}$$

Now if we make the non-intuitive choice of $f(t)=\frac{1}{2}(a+bt)^2+p_o/\rho$, we get

$$br\cos\theta+\frac{1}{2}(a+bt)^2+\frac{p}{\rho}=\frac{1}{2}(a+bt)^2+\frac{p_o}{\rho}. \tag{4.153}$$

So

$$p=p_o-\rho br\cos\theta=p_o-\rho bx. \tag{4.154}$$

Note that since the flow at infinity is accelerating, there must be a far-field pressure gradient to induce this acceleration. Consider the x momentum equation in the far field

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial x}, \tag{4.155}$$

$$\rho(b) = -\rho(-b). \tag{4.156}$$

So for the pressure field, we have

$$br\cos\theta\left(1+\frac{c^2}{r^2}\right)+\frac{1}{2}(a+bt)^2\left(1+\frac{c^4}{r^4}+\frac{2c^2}{r^2}(\sin^2\theta-\cos^2\theta)\right)+\frac{p}{\rho}=\frac{1}{2}(a+bt)^2+\frac{p_o}{\rho}, \tag{4.157}$$

which reduces to

$$p(r,\theta,t) = p_o - \rho b r \cos\theta \left(1 + \frac{c^2}{r^2}\right) - \frac{1}{2}\rho(a+bt)^2 \left(\frac{c^4}{r^4} + \frac{2c^2}{r^2}(\sin^2\theta - \cos^2\theta)\right). \tag{4.158}$$

For the stagnation point, we evaluate as

$$p(c,\pi,t) = p_o - \rho b c(-1)\,(1+1) - \frac{1}{2}\rho(a+bt)^2\,(1 + 2(1)(0-1))\,, \tag{4.159}$$

$$= p_o + \frac{1}{2}\rho(a+bt)^2 + 2\rho b c. \tag{4.160}$$

The first two terms would be predicted by a naive extension of the steady Bernoulli's equation. The final term however is not intuitive and is a purely unsteady effect.

4.4 More complex variable theory

There are more basic ways to describe the force on bodies using complex variables directly. We shall give those methods, but first a discussion of the motivating complex variable theory is necessary.

4.4.1 Contour integrals

Consider the closed contour integral of a complex function in the complex plane. For such integrals, we have a useful theory which we will not prove, but will demonstrate here. Consider contour integrals enclosing the origin with a circle in the complex plane for four functions. The contour in each is $C : z = \hat{R}e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. For such a contour $dz = i\hat{R}e^{i\theta}\ d\theta$.

4.4.1.1 Simple pole

We describe a simple pole with the complex potential

$$W(z) = \frac{a}{z}. \tag{4.161}$$

and the contour integral is

$$\oint_C W(z)\ dz = \oint_C \frac{a}{z}\ dz = \int_{\theta=0}^{\theta=2\pi} \frac{a}{\hat{R}e^{i\theta}} i\hat{R}e^{i\theta}\ d\theta, \tag{4.162}$$

$$= ai\int_0^{2\pi} d\theta = 2\pi i a. \tag{4.163}$$

4.4.1.2 Constant potential

We describe a constant with the complex potential

$$W(z) = b. \tag{4.164}$$

and the contour integral is

$$\oint_C W(z)\, dz = \oint_C b\, dz = \int_{\theta=0}^{\theta=2\pi} bi\hat{R}e^{i\theta}\, d\theta, \tag{4.165}$$

$$= \frac{bi\hat{R}}{i}e^{i\theta}\Big|_0^{2\pi} = 0. \tag{4.166}$$

since $e^{0i} = e^{2\pi i} = 1$.

4.4.1.3 Uniform flow

We describe a constant with the complex potential

$$W(z) = cz. \tag{4.167}$$

and the contour integral is

$$\oint_C W(z)\, dz = \oint_C cz\, dz = \int_{\theta=0}^{\theta=2\pi} c\hat{R}e^{i\theta} i\hat{R}e^{i\theta}\, d\theta, \tag{4.168}$$

$$= ic\hat{R}^2 \int_0^{2\pi} e^{2\pi\theta}\, d\theta = \frac{ic\hat{R}^2}{2i}e^{2i\theta}\Big|_0^{2\pi} = 0. \tag{4.169}$$

since $e^{0i} = e^{4\pi i} = 1$.

4.4.1.4 Quadrupole

A quadrupole potential is described by

$$W(z) = \frac{k}{z^2} \tag{4.170}$$

Taking the contour integral, we find

$$\oint_C \frac{k}{z^2}\, dz = k\int_0^{2\pi} \frac{i\hat{R}e^{i\theta}}{\hat{R}^2 e^{2i\theta}}\, d\theta, \tag{4.171}$$

$$= \frac{ki}{\hat{R}}\int_0^{2\pi} e^{-i\theta}\, d\theta = \frac{ki}{\hat{R}}\frac{1}{-i}e^{-i\theta}\Big|_0^{2\pi} = 0. \tag{4.172}$$

So the only non-zero contour integral is for functions of the form $W(z) = a/z$. We find all polynomial powers of z have a zero contour integral about the origin for arbitrary contours except this special one.

4.4.2 Laurent series

Now it can be shown that any function can be expanded, much as for a Taylor series, as a *Laurent series*:[4]

$$W(z) = \ldots + C_{-2}(z-z_o)^{-2} + C_{-1}(z-z_o)^{-1} + C_0(z-z_o)^0 + C_1(z-z_o)^1 + C_2(z-z_o)^2 + \ldots . \tag{4.173}$$

In compact summation notation, we can say

$$W(z) = \sum_{n=-\infty}^{n=\infty} C_n (z-z_o)^n. \tag{4.174}$$

Taking the contour integral of both sides we get

$$\oint_C W(z)\ dz = \oint_C \sum_{n=-\infty}^{n=\infty} C_n (z-z_o)^n\ dz, \tag{4.175}$$

$$= \sum_{n=-\infty}^{n=\infty} C_n \oint_C (z-z_o)^n\ dz, \tag{4.176}$$

$$\text{this has value } 2\pi i \text{ only when } n=-1, \text{ so} \tag{4.177}$$

$$= C_{-1} 2\pi i \tag{4.178}$$

Here C_{-1} is known as the *residue* of the Laurent series. In general we have the Cauchy integral theorem which holds that if $W(z)$ is analytic within and on a closed curve C except for a finite number of singular points, then

$$\oint_C W(z) dz = 2\pi i \sum \text{residues}. \tag{4.179}$$

The constants C_n can be shown to be found by evaluating the contour integral

$$C_n = \frac{1}{2\pi i} \oint_C \frac{W(z)}{(z-z_o)^{n+1}} dz, \tag{4.180}$$

where C is any closed contour which has z_o in its interior.

4.5 Pressure distribution for steady flow

For steady, irrotational, incompressible flow with no body force present, we have the Bernoulli equation:

$$\frac{p}{\rho} + \frac{1}{2}(\nabla\phi)^T \cdot \nabla\phi = \frac{p_\infty}{\rho} + \frac{1}{2}U_\infty^2. \tag{4.181}$$

[4]Pierre Alphonse Laurent, 1813-1854, Parisian engineer who worked on port expansion in Le Harve, submitted his work on Laurent series for a Grand Prize in 1842, with the recommendation of Cauchy, but was rejected because of a late submission.

We can write this in terms of the complex potential in a simple fashion. First, recall that

$$(\nabla\phi)^T \cdot \nabla\phi = u^2 + v^2. \tag{4.182}$$

We also have $dW/dz = u - iv$, so $\overline{dW/dz} = u + iv$. Consequently,

$$\frac{dW}{dz}\overline{\frac{dW}{dz}} = u^2 + v^2 = (\nabla\phi)^T \cdot \nabla\phi. \tag{4.183}$$

So we get the pressure field from Bernoulli's equation to be

$$p = p_\infty + \frac{1}{2}\rho\left(U_\infty^2 - \frac{dW}{dz}\overline{\frac{dW}{dz}}\right). \tag{4.184}$$

The pressure coefficient C_p is

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{1}{U_\infty^2}\frac{dW}{dz}\overline{\frac{dW}{dz}}. \tag{4.185}$$

4.6 Blasius force theorem

For steady flows, we can find the net contribution of a pressure force on an arbitrary shaped solid body with the Blasius[5] force theorem.

Consider the geometry sketched in Figure 4.12. The surface of the arbitrarily shaped body is described by S_b, and C is a closed contour containing S_b. First consider the linear momenta equation for steady flow, no body forces, and no viscous forces,

$$\begin{aligned}
\rho\left(\mathbf{v}^T \cdot \nabla\right)\mathbf{v} &= -\nabla p, \qquad \text{add mass to get conservative form,} && (4.186)\\
\left(\nabla^T \cdot (\rho\mathbf{v}\mathbf{v}^T)\right)^T &= -\nabla p, \qquad \text{integrate over } V, && (4.187)\\
\int_V \left(\nabla^T \cdot (\rho\mathbf{v}\mathbf{v}^T)\right)^T dV &= -\int_V \nabla p \; dV, \qquad \text{use Gauss,} && (4.188)\\
\int_S \rho\mathbf{v}(\mathbf{v}^T \cdot \mathbf{n}) \; dS &= -\int_S p\mathbf{n} \; dS. && (4.189)
\end{aligned}$$

Now the surface integral here is really a line integral with unit depth b, $dS = b\; ds$. Moreover the surface enclosing the *fluid* has an inner contour S_b and an outer contour C. Now on C, which we prescribe, we will know $x(s)$ and $y(s)$, where s is arc length. So on C we also get the unit tangent $\boldsymbol{\alpha}$ and unit outward normal $\mathbf{n}$:

$$\boldsymbol{\alpha} = \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \end{pmatrix}, \qquad \text{on } C. \tag{4.190}$$

[5]Paul Richard Heinrich Blasius, 1883-1970, student of Ludwig Prandtl and long time teacher at the technical college of Hamburg whose 1907 Ph.D. thesis gave mathematical description of similarity solution to the boundary layer problem.

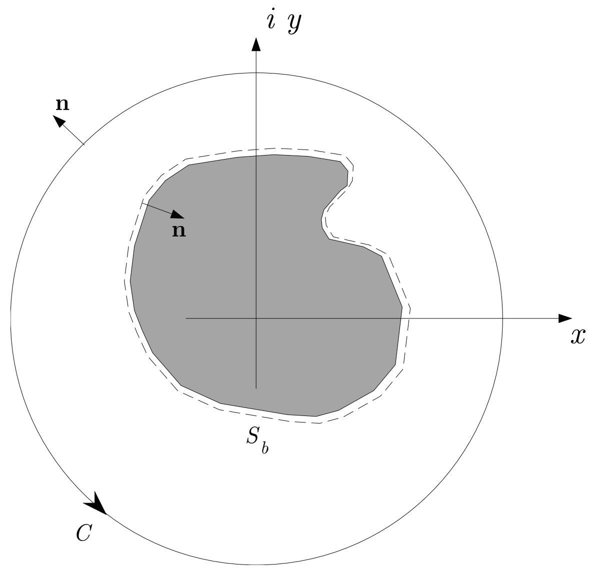

Figure 4.12: Potential flow about arbitrarily shaped two-dimensional body with fluid control volume indicated.

Moreover, on S_b we have, since it is a solid surface

$$\mathbf{v}^T \cdot \mathbf{n} = 0, \qquad \text{on } C. \tag{4.191}$$

Now let the force on the body due to fluid pressure be $\mathbf{F}$:

$$\int_{S_b} p\mathbf{n}\ dS = \mathbf{F}. \tag{4.192}$$

Now return to our linear momentum equation

$$\int_S \rho \mathbf{v}\mathbf{v}^T \cdot \mathbf{n}\ dS = -\int_S p\mathbf{n}\ dS, \qquad \text{break this up,} \tag{4.193}$$

$$\oint_{S_b} \rho \mathbf{v} \underbrace{\mathbf{v}^T \cdot \mathbf{n}}_{=0}\ dS + \oint_C \rho \mathbf{v}\mathbf{v}^T \cdot \mathbf{n}\ dS = -\underbrace{\oint_{S_b} p\mathbf{n}\ dS}_{=\mathbf{F}} - \oint_C p\mathbf{n}\ dS, \tag{4.194}$$

$$\oint_C \rho \mathbf{v}\mathbf{v}^T \cdot \mathbf{n}\ dS = -\mathbf{F} - \oint_C p\mathbf{n}\ dS. \tag{4.195}$$

We can break this into x and y components:

$$\oint_C \rho u \underbrace{\left(u\frac{dy}{ds} - v\frac{dx}{ds}\right)}_{\mathbf{v}^T\cdot\mathbf{n}} b\ ds = -\oint_C p\frac{dy}{ds} b\ ds - F_x, \tag{4.196}$$

$$\oint_C \rho v \underbrace{\left(u\frac{dy}{ds} - v\frac{dx}{ds}\right)}_{\mathbf{v}^T\cdot\mathbf{n}} b\ ds = \oint_C p\frac{dx}{ds} b\ ds - F_y. \tag{4.197}$$

Solving for F_x and F_y per unit depth, we get

$$\frac{F_x}{b} = \oint_C -p\ dy - \rho u^2\ dy + \rho uv\ dx, \tag{4.198}$$

$$\frac{F_y}{b} = \oint_C p\ dx + \rho v^2\ dx - \rho uv\ dy. \tag{4.199}$$

Now Bernoulli gives us $p = K - (1/2)\rho(u^2 + v^2)$, where K is some constant. So the x force per unit depth becomes

$$\frac{F_x}{b} = \oint_C -K\ dy + \frac{1}{2}\rho(u^2 + v^2)\ dy - \rho u^2\ dy + \rho uv\ dx, \tag{4.200}$$

since the integral over a closed contour of a constant K is zero, (4.201)

$$= \oint_C \frac{1}{2}\rho(-u^2 + v^2)\ dy + \rho uv\ dx, \tag{4.202}$$

$$= \frac{1}{2}\rho \oint_C (-u^2 + v^2)\ dy + 2uv\ dx. \tag{4.203}$$

Similarly for the y direction, we get

$$\frac{F_y}{b} = \oint_C K\ dx - \frac{1}{2}\rho(u^2 + v^2)\ dy + \rho v^2\ dx - \rho uv\ dy, \tag{4.204}$$

$$= \frac{1}{2}\rho \oint_C (-u^2 + v^2)\ dx - 2uv\ dx. \tag{4.205}$$

Now consider the group of terms $\frac{F_x - iF_y}{b}$:

$$\frac{F_x - iF_y}{b} = \frac{1}{2}\rho \oint_C (-u^2 + v^2)\ dy + 2uv\ dx - (-u^2 + v^2)i\ dx + 2uvi\ dy, \tag{4.206}$$

$$= \frac{1}{2}\rho \oint_C (i(u^2 - v^2) + 2uv)\ dx + ((-u^2 + v^2) + 2uvi)dy, \tag{4.207}$$

$$= \frac{1}{2}\rho \oint_C (i(u^2 - v^2) + 2uv)\ dx + (i(u^2 - v^2) + 2uv)i\ dy, \tag{4.208}$$

$$= \frac{1}{2}\rho \oint_C (i(u^2 - v^2) + 2uv)(dx + i\ dy), \tag{4.209}$$

$$= \frac{1}{2}\rho \oint_C i(u - iv)^2(dx + i\ dy), \tag{4.210}$$

$$= \frac{1}{2}\rho i \oint_C \left(\frac{dW}{dz}\right)^2\ dz. \tag{4.211}$$

So if we have the complex potential, we can easily get the force on a body.

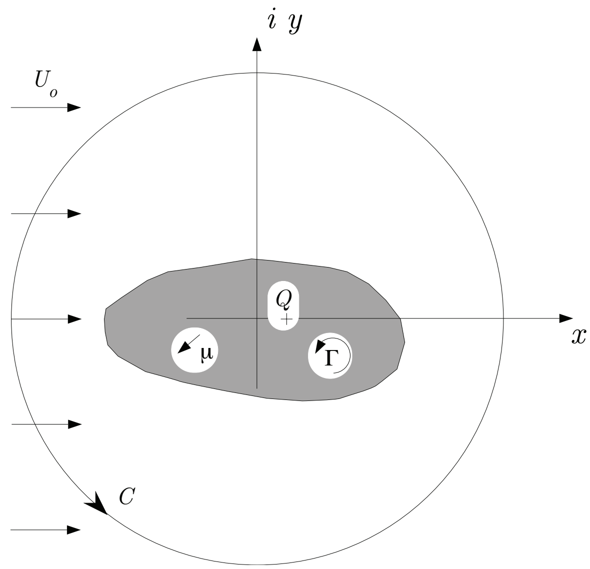

Figure 4.13: Potential flow about arbitrarily shaped two-dimensional body with distribution of sources, sinks, vortices, and dipoles.

4.7 Kutta-Zhukovsky lift theorem

Consider the geometry sketched in Figure 4.13. Here we consider a flow with a freestream constant velocity of U_o. We take an arbitrary body shape to enclose a distribution of canceling source sink pairs, doublets, point vortices, quadruples, and any other non-mass adding potential flow term. This combination gives rise to some surface which is a streamline.

Now far from the body surface a contour sees all of these features as effectively concentrated at the origin. Then, the potential can be written as

$$W(z) \sim \underbrace{Uz}_{\text{uniform flow}} + \underbrace{\frac{Q}{2\pi}\ln z - \frac{Q}{2\pi}\ln z}_{\text{canceling source sink pair}} + \underbrace{\frac{i\Gamma}{2\pi}\ln z}_{\text{clockwise! vortex}} + \underbrace{\frac{\mu}{z}}_{\text{doublet}} + \dots \tag{4.212}$$

Note that the sign convention for Γ has been violated here, by tradition. Now let us take D to be the so-called drag force per unit depth and L to be the so-called lift force per unit depth, so in terms of F_x and F_y, we have

$$\frac{F_x}{b} = D, \qquad \frac{F_y}{b} = L. \tag{4.213}$$

Now by the Blasius force theorem, we have

$$D - iL = \frac{1}{2}\rho i \oint_C \left(\frac{dW}{dz}\right)^2 dz, \tag{4.214}$$

$$= \frac{1}{2}\rho i \oint_C \left(U + \frac{i\Gamma}{2\pi z} - \frac{\mu}{z^2} + \dots\right)^2 dz, \tag{4.215}$$

$$= \frac{1}{2}\rho i \oint_C \left(U^2 + \frac{i\Gamma U}{\pi z} - \frac{1}{z^2}\left(\frac{\Gamma^2}{4\pi^2} + 2U\mu\right) + \dots \right) dz. \tag{4.216}$$

Now the Cauchy integral theorem gives is the contour integral is $2\pi i \sum$ residues. Here the residue is $i\Gamma U/\pi$. So we get

$$D - iL = \frac{1}{2}\rho i \left(2\pi i \left(\frac{i\Gamma U}{\pi}\right)\right), \tag{4.217}$$

$$= -i\rho\Gamma U. \tag{4.218}$$

So we see that

$$D = 0, \tag{4.219}$$

$$L = \rho U \Gamma. \tag{4.220}$$

Note that

- Γ is associated with *clockwise* circulation here. This is something of a tradition in aerodynamics.
- Since for airfoils $\Gamma \sim U$, we get the lift force $L \sim \rho U^2$,
- For steady inviscid flow, there is no drag. Consideration of either unsteady or viscous effects would lead to a non-zero x component of force.

Example 4.2

Consider the flow over a cylinder of radius a with clockwise circulation Γ.

To do so, we can superpose a point vortex onto the potential for flow over a cylinder in the following fashion:

$$W(z) = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\ln\left(\frac{z}{a}\right). \tag{4.221}$$

Breaking this up as before into real and complex parts, we get

$$W(z) = \left(Ur\cos\theta\left(1 + \frac{a^2}{r^2}\right)\right) + i\left(Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right)\right) + \frac{i\Gamma}{2\pi}\left(\ln\left(\frac{r}{a}\right) + i\theta\right). \tag{4.222}$$

So, we find

$$\psi = \Im(W(z)) = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right). \tag{4.223}$$

On $r = a$, we find that $\psi = 0$, so the addition of the circulation in the way we have proposed maintains the cylinder surface to be a streamline. It is important to note that this is valid for *arbitrary* Γ. That is the potential flow solution for flow over a cylinder is *non-unique*. In aerodynamics, this is used to advantage to add just enough circulation to enforce the so-called *Kutta condition*.[6] The Kutta condition is an experimentally observed fact that for a steady flow, the trailing edge of an airfoil is a stagnation point.

The Kutta-Zhukovsky[7] lift theorem tells us whenever we add circulation, that a lift force $L = \rho U \Gamma$ is induced. This is consistent with the phenomena observed in baseball that the "fastball" rises. The fastball leaves the pitcher's hand traveling towards the batter and rotating towards the pitcher. The induced aerodynamic force is opposite to the force of gravity.

Let us get the lift force the hard way and verify the Kutta-Zhukovsky theorem. We can easily get the velocity field from the velocity potential:

$$\phi = \Re(W(z)) = Ur\cos\theta\left(1+\frac{a^2}{r^2}\right) - \frac{\Gamma\theta}{2\pi}, \tag{4.224}$$

$$v_r = \frac{\partial \phi}{\partial r} = Ur\cos\theta\left(-\frac{2a^2}{r^3}\right) + U\cos\theta\left(1+\frac{a^2}{r^2}\right), \tag{4.225}$$

$$v_r|_{r=a} = U\cos\theta\left(-\frac{2a^3}{a^3}+1+\frac{a^2}{a^2}\right) = 0, \tag{4.226}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{r}\left(-Ur\sin\theta\left(1+\frac{a^2}{r^2}\right) - \frac{\Gamma}{2\pi}\right), \tag{4.227}$$

$$v_\theta|_{r=a} = -U\sin\theta\left(1+\frac{a^2}{a^2}\right) - \frac{\Gamma}{2\pi a}, \tag{4.228}$$

$$= -2U\sin\theta - \frac{\Gamma}{2\pi a}. \tag{4.229}$$

We get the pressure on the cylinder surface from Bernoulli's equation:

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho(\nabla\phi)^T\cdot\nabla\phi, \tag{4.230}$$

$$= p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2. \tag{4.231}$$

Now for a small element of the cylinder at $r = a$, the surface area is $dA = br\ d\theta = ba\ d\theta$. This is sketched in Figure 4.14. We also note that the x and y forces depend on the orientation of the element, given by θ. Elementary trigonometry shows that the elemental x and y forces per depth are

$$\frac{dF_x}{b} = -p\cos\theta a\ d\theta, \tag{4.232}$$

$$\frac{dF_y}{b} = -p\sin\theta a\ d\theta. \tag{4.233}$$

So integrating over the entire cylinder, we obtain,

$$\frac{F_x}{b} = \int_0^{2\pi} -\left(p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2\right)\cos\theta a\ d\theta, \tag{4.234}$$

$$\frac{F_y}{b} = \int_0^{2\pi} -\left(p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2\right)\sin\theta a\ d\theta. \tag{4.235}$$

[6]Martin Wilhelm Kutta, 1867-1944, Silesian-born German mechanician, studied at Breslau, taught mainly at Stuttgart, co-developer of Runge-Kutta method for integrating ordinary differential equations.

[7]Nikolai Egorovich Zhukovsky, 1847-1921, Russian applied mathematician and mechanician, father of Russian aviation, purchased glider from Lilienthal, developed lift theorem independently of Kutta, organized Central Aerohydrodynamic Institute in 1918.

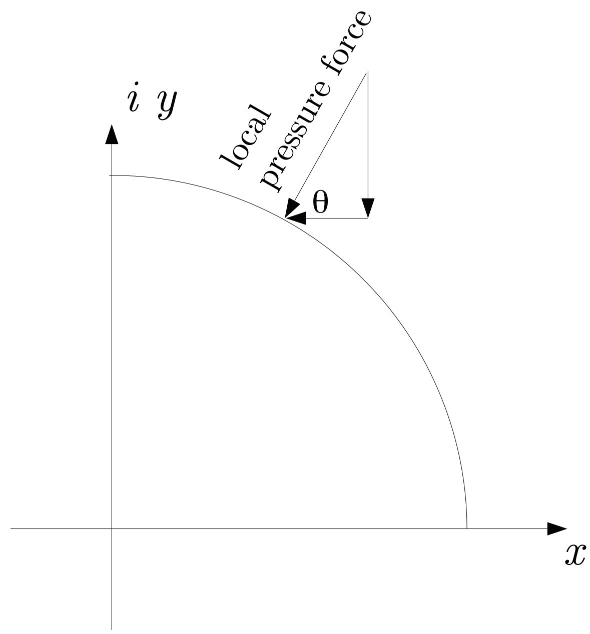

Figure 4.14: Pressure force on a differential area element of cylindrical surface.

Integration via computer algebra gives

$$\frac{F_x}{b} = 0, \quad (4.236)$$

$$\frac{F_y}{b} = \rho U \Gamma. \quad (4.237)$$

This is identical to the result we expect from the Kutta-Zhukovsky lift theorem.

4.8 Conformal mapping

Conformal mapping is a technique by which we can render results obtained for simple flows, such as those over a cylinder, applicable to flows over more complicated geometries. We will not consider these in any detail here, but the reader should refer to texts on potential flow for a full explanation. In short, one relies on a coordinate transformation to map the complicated geometry in an ordinary space into a simple geometry in a warped geometric space. In the warped space, on can obtain pressure fields in terms of the warped coordinates, then transform them back into ordinary space to get the actual pressure field.

Chapter 5

Viscous incompressible laminar flow

see Panton, Chapter 7, 11
see Yih, Chapter 7

Here we consider a few standard problems in viscous incompressible laminar flow. For this entire chapter, we will make the following assumptions:

- the flow is incompressible,
- body forces are negligible, and
- the fluid properties, c, μ and k, are constants.

5.1 Fully developed, one dimensional solutions

The first type of solution we will consider is known as a one-dimensional *fully developed* solution. These are commonly considered in first courses in fluid mechanics and heat transfer. The flows here are essentially one-dimensional, but not absolutely, as they were in the chapter on one-dimensional compressible flow. In this section, we will further enforce that

- the flow is time-independent, $\partial_o = 0$,
- the velocity and temperature gradients in the x and z direction are zero, $\partial \mathbf{v}/\partial x = 0$, $\partial \mathbf{v}/\partial z = 0$, $\partial T/\partial x = 0$, $\partial T/\partial z = 0$.

We will see that these assumptions give rise to flows with a non-zero x velocity u which varies in the y direction, and that other velocities v, and w, will be zero.

5.1.1 Pressure gradient driven flow in a slot

Consider the flow sketched in Figure 5.1. Here we have a large reservoir of fluid with a long narrow slot located around $y = 0$. We take the length of the slot in the z direction,

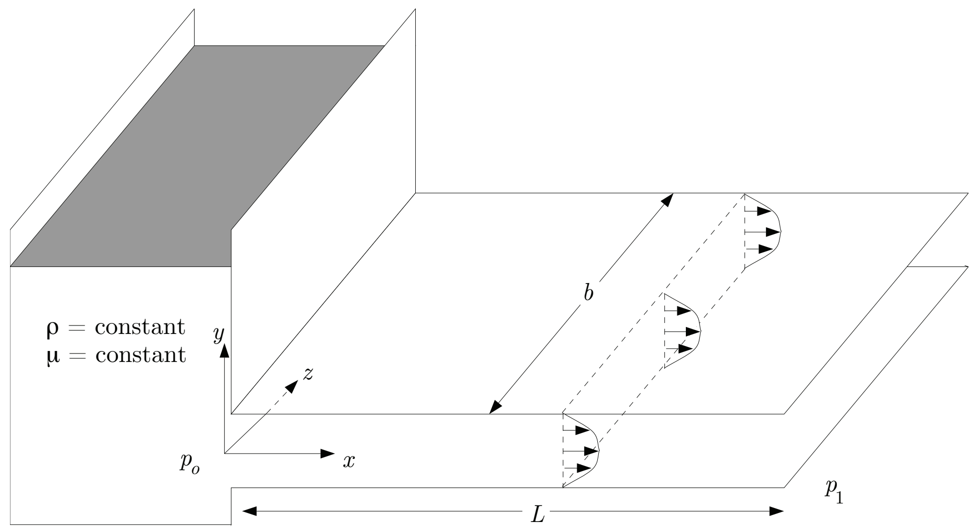

Figure 5.1: Pressure gradient driven flow in a slot.

b, to be long relative to the slot width in the y direction h. Attached to the slot are two parallel plates, separated by distance in the y direction h. The length of the plates in the x direction is L. We take $L >> h$. Because of gravity forces, which we neglect in the slot, the pressure at the entrance of the slot p_o is higher than atmospheric. At the end of the slot, the fluid expels to the atmosphere which is at p_1. Hence, there is a pressure gradient in the x direction, which drives the flow in the slot. We will see that the flow is resisted by viscous stresses. An analogous flow in a circular duct is defined as a Hagen-Poiseuille[1,2] flow.

Near $x = 0$, the flow accelerates in what is known as the *entrance length*. If L is sufficiently long, we observe that the fluid particles no longer accelerate after traveling in the slot. It is at this point where the viscous shear forces exactly balance the pressure forces to give rise to the fully developed velocity field.

For this flow, let us make the additional assumptions that

- there is no imposed pressure gradient in the z direction, and
- the walls are held at a constant temperature, T_o.

Incorporating some of these assumptions, we write the incompressible constant property Navier-Stokes equations as

$$\partial_i v_i = 0, \tag{5.1}$$

$$\rho \partial_o v_i + \rho v_j \partial_j v_i = -\partial_i p + \mu \partial_j \partial_j v_i, \tag{5.2}$$

$$\rho c \partial_o T + \rho c v_j \partial_j T = k \partial_i \partial_i T + 2\mu \partial_{(i} v_{j)} \partial_{(i} v_{j)}. \tag{5.3}$$

[1]Gotthilf Ludwig Hagen, 1797-1884, German engineer who measured velocity of water in small diameter tubes.

[2]Jean Louis Poiseuille, 1799-1869, French physician who repeated experiments of Hagen for simulated blood flow.

Here we have five equations in five unknowns, v_i, p, and T.

As for all incompressible flows with constant properties, we can get the velocity field by only considering the mass and momenta equations; velocity is only coupled one way to the energy equation.

The mass equation, recalling that gradients in x and z are zero, gives us

$$\underbrace{\frac{\partial}{\partial x}}_{=0} u + \frac{\partial}{\partial y} v + \underbrace{\frac{\partial}{\partial z}}_{=0} w = 0. \tag{5.4}$$

So the mass equation gives us

$$\frac{\partial v}{\partial y} = 0. \tag{5.5}$$

Now, from our assumptions of steady and fully developed flow, we know that v cannot be a function of x, z, or t. So the partial becomes a total derivative, and mass conservation holds that $dv/dy = 0$. Integrating, we find that $v(y) = C$. The constant C must be zero, since we must satisfy a no-slip boundary condition at either wall that $v(y = h/2) = v(y = -h/2) = 0$. Hence, mass conservation, coupled with the no slip boundary condition gives us

$$v = 0. \tag{5.6}$$

Now consider the x momentum equation:

$$\rho \underbrace{\frac{\partial}{\partial t}}_{=0} u + \rho u \underbrace{\frac{\partial}{\partial x}}_{=0} u + \rho \underbrace{v}_{=0} \frac{\partial}{\partial y} u + \rho w \underbrace{\frac{\partial}{\partial z}}_{=0} u = -\frac{\partial p}{\partial x} + \mu \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} u + \frac{\partial^2}{\partial y^2} u + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} u \right), \tag{5.7}$$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}. \tag{5.8}$$

We note for this fully developed flow that the acceleration, that is the material derivative of velocity, is formally zero, and the equation gives rise to a balance of pressure and viscous surface forces.

For the y momentum equation, we get

$$\rho \underbrace{\frac{\partial}{\partial t}}_{=0} \underbrace{v}_{=0} + \rho u \underbrace{\frac{\partial}{\partial x}}_{=0} \underbrace{v}_{=0} + \rho \underbrace{v}_{=0} \frac{\partial}{\partial y} \underbrace{v}_{=0} + \rho w \underbrace{\frac{\partial}{\partial z}}_{=0} \underbrace{v}_{=0} = -\frac{\partial p}{\partial y} \tag{5.9}$$

$$+\mu \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} v + \frac{\partial^2}{\partial y^2} \underbrace{v}_{=0} + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} v \right),$$

$$0 = \frac{\partial p}{\partial y}. \tag{5.10}$$

Hence, $p = p(x, z)$, but since we have assumed there is no pressure gradient in the z direction, we have at most that

$$p = p(x). \tag{5.11}$$

For the z momentum equation we get:

$$\rho \underbrace{\frac{\partial}{\partial t}}_{=0} w + \rho u \underbrace{\frac{\partial}{\partial x}}_{=0} w + \rho \underbrace{v}_{=0} \frac{\partial}{\partial y} w + \rho w \underbrace{\frac{\partial}{\partial z}}_{=0} w = - \underbrace{\frac{\partial p}{\partial z}}_{=0} + \mu \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} w + \frac{\partial^2}{\partial y^2} w + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} w \right), \tag{5.12}$$

$$0 = \frac{\partial^2 w}{\partial y^2}. \tag{5.13}$$

Solution of this partial differential equation gives us

$$w = f(x, z)y + g(x, z). \tag{5.14}$$

Now to satisfy no-slip, we must have $w = 0$ at $y = \pm h/2$. This leads us to two linear equations for f and g:

$$\begin{pmatrix} \frac{h}{2} & 1 \\ -\frac{h}{2} & 1 \end{pmatrix} \begin{pmatrix} f(x, z) \\ g(x, z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.15}$$

Since the determinant of the coefficient matrix, $h/2 + h/2 = h$, is non-zero, the only solution is the trivial solution $f(x, z) = g(x, z) = 0$. Hence,

$$w = 0. \tag{5.16}$$

Next consider how the energy equation reduces:

$$\rho c \underbrace{\frac{\partial}{\partial t}}_{=0} T + \rho c \left(u \underbrace{\frac{\partial}{\partial x}}_{=0} T + \underbrace{v}_{=0} \frac{\partial}{\partial y} T + \underbrace{w}_{=0} \underbrace{\frac{\partial}{\partial z}}_{=0} T \right) = k \left(\underbrace{\frac{\partial^2}{\partial x^2}}_{=0} T + \frac{\partial^2}{\partial y^2} T + \underbrace{\frac{\partial^2}{\partial z^2}}_{=0} T \right) + 2\mu \partial_{(i} v_{j)} \partial_{(i} v_{j)}, \tag{5.17}$$

$$0 = k \frac{\partial^2 T}{\partial y^2} + 2\mu \partial_{(i} v_{j)} \partial_{(i} v_{j)}. \tag{5.18}$$

Note that there is no tendency for a particle's temperature to increase. There is a balance between thermal energy generated by viscous dissipation and that conducted away by thermal diffusion. Thus the energy path is 1) viscous work is done to generate thermal energy, 2) thermal energy diffuses throughout the channel and out the boundary. Now consider the viscous dissipation term for this flow.

$$\partial_i v_j = \begin{pmatrix} \underbrace{\partial_1}_{=0} v_1 & \underbrace{\partial_1}_{=0}\underbrace{v_2}_{=0} & \underbrace{\partial_1}_{=0}\underbrace{v_3}_{=0} \\ \partial_2 v_1 & \partial_2 \underbrace{v_2}_{=0} & \partial_2 \underbrace{v_3}_{=0} \\ \underbrace{\partial_3}_{=0} v_1 & \underbrace{\partial_3}_{=0}\underbrace{v_2}_{=0} & \underbrace{\partial_3}_{=0}\underbrace{v_3}_{=0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \partial_2 v_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.19}$$

$$\partial_{(i} v_{j)} = \begin{pmatrix} 0 & \frac{1}{2}\left(\partial_2 v_1 + \underbrace{\partial_1 v_2}_{=0}\right) & 0 \\ \frac{1}{2}\left(\partial_2 v_1 + \underbrace{\partial_1 v_2}_{=0}\right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial u}{\partial y} & 0 \\ \frac{1}{2}\frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.20}$$

Further,

$$\partial_{(i} v_{j)} \partial_{(i} v_{j)} = \left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2 + \left(\frac{1}{2}\frac{\partial u}{\partial y}\right)^2 = \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2. \tag{5.21}$$

So the energy equation becomes finally

$$0 = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2. \tag{5.22}$$

At this point we have the x momentum and energy equations as the only two which seem to have any substance.

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \tag{5.23}$$

$$0 = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2. \tag{5.24}$$

This looks like two equations in three unknowns. One peculiarity of incompressible equations is that there is always some side condition, which ultimately hinges on the mass equation, which really gives a third equation. Without going into details, it involves for general flows solving a Poisson[3] equation for pressure which is of the form $\nabla^2 p = f(u, v)$. Note that this involves second derivatives of pressure. Here we can obtain a simple form of this general equation by taking the partial derivative with respect to x of the x momentum equation:

$$0 = -\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial}{\partial x}\frac{\partial^2 u}{\partial y^2}, \tag{5.25}$$

$$0 = -\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2}{\partial y^2}\underbrace{\frac{\partial u}{\partial x}}_{=0}. \tag{5.26}$$

[3]Siméon Denis Poisson, 1781-1840, French mathematician taught by Laplace, Lagrange, and Legendre, studied partial differential equations, potential theory, elasticity, and electrodynamics.

The viscous term above is zero because of our assumption of fully developed flow. Moreover, since $p = p(x)$ only, we then get

$$\frac{d^2p}{dx^2} = 0, \qquad p(0) = p_o, \; p(L) = p_1, \tag{5.27}$$

which has a solution showing the pressure field must be linear in x:

$$p(x) = p_o - \frac{p_o - p_1}{L} x, \tag{5.28}$$

$$\frac{dp}{dx} = -\frac{p_o - p_1}{L}. \tag{5.29}$$

Now, since u is at most a function of y, we can convert partial derivatives to ordinary derivatives, and write the x momentum equation and energy equation as two ordinary differential equations in two unknowns with appropriate boundary conditions at the wall $y = \pm h/2$:

$$\frac{d^2u}{dy^2} = -\frac{p_o - p_1}{\mu L}, \qquad u\left(\frac{h}{2}\right) = 0, \quad u\left(-\frac{h}{2}\right) = 0, \tag{5.30}$$

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2, \qquad T\left(\frac{h}{2}\right) = T_o, \quad T\left(-\frac{h}{2}\right) = T_o. \tag{5.31}$$

We could solve these equations directly, but instead let us first cast them in dimensionless form. This will give our results some universality and efficiency. Moreover, it will reveal more fundamental groups of terms which govern the fluid behavior. Let us select scales such that dimensionless variables, denoted by a * subscript, are as follows

$$y_* = \frac{y}{h}, \qquad T_* = \frac{T - T_o}{T_o}, \qquad u_* = \frac{u}{u_c}. \tag{5.32}$$

We have yet to determine the characteristic velocity u_c. Note that the dimensionless temperature has been chosen to render it zero at the boundaries. With these choices, the x momentum equation becomes

$$\frac{u_c}{h^2}\frac{d^2u_*}{dy_*^2} = -\frac{p_o - p_1}{\mu L}, \tag{5.33}$$

$$\frac{d^2u_*}{dy_*^2} = -\frac{(p_o - p_1)h^2}{\mu L u_c}, \tag{5.34}$$

$$u_c u_*(x_* h = h/2) = u_c u_*(x_* h = -h/2) = 0, \tag{5.35}$$

$$u_*(x_* = 1/2) = u_*(x_* = -1/2) = 0. \tag{5.36}$$

Let us now choose the characteristic velocity to render the x momentum equation to have a simple form:

$$u_c \equiv \frac{(p_o - p_1)h^2}{\mu L}. \tag{5.37}$$

Now scale the energy equation:

$$\frac{T_o}{h^2}\frac{d^2T_*}{dy_*^2} = -\frac{\mu u_c^2}{kh^2}\left(\frac{du_*}{dy_*}\right)^2, \tag{5.38}$$

$$\frac{d^2T_*}{dy_*^2} = -\frac{\mu u_c^2}{kT_o}\left(\frac{du_*}{dy_*}\right)^2, \tag{5.39}$$

$$= -\frac{\mu c}{k}\frac{u_c^2}{cT_o}\left(\frac{du_*}{dy_*}\right)^2, \tag{5.40}$$

$$= -Pr Ec\left(\frac{du_*}{dy_*}\right)^2, \tag{5.41}$$

$$T_*\left(-\frac{1}{2}\right) = T_*\left(\frac{1}{2}\right) = 0. \tag{5.42}$$

Here we have grouped terms so that the Prandtl number $Pr = \mu c/k$, explicitly appears. Further, we have defined the Eckert[4] number Ec as

$$Ec = \frac{u_c^2}{cT_o} = \frac{\left(\frac{(p_o-p_1)h^2}{\mu L}\right)^2}{cT_o}. \tag{5.43}$$

In summary our dimensionless differential equations and boundary conditions are

$$\frac{d^2u_*}{dy_*^2} = -1, \qquad u\left(\pm\frac{1}{2}\right) = 0, \tag{5.44}$$

$$\frac{d^2T_*}{dy_*^2} = -PrEc\left(\frac{du_*}{dy_*}\right)^2, \qquad T_*\left(\pm\frac{1}{2}\right) = 0. \tag{5.45}$$

These boundary conditions are homogeneous; hence, they do not contribute to a non-trivial solution. The pressure gradient is an inhomogeneous forcing term in the momentum equation, and the viscous dissipation is a forcing term in the energy equation.

The solution for the velocity field which satisfies the differential equation and boundary conditions is quadratic in y_* and is

$$u_* = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - y_*^2\right). \tag{5.46}$$

Note that the maximum velocity occurs at $y_* = 0$ and has value

$$u_{*max} = \frac{1}{8}. \tag{5.47}$$

[4]Ernst R. G. Eckert, 1904-2004, scholar of convective heat transfer.

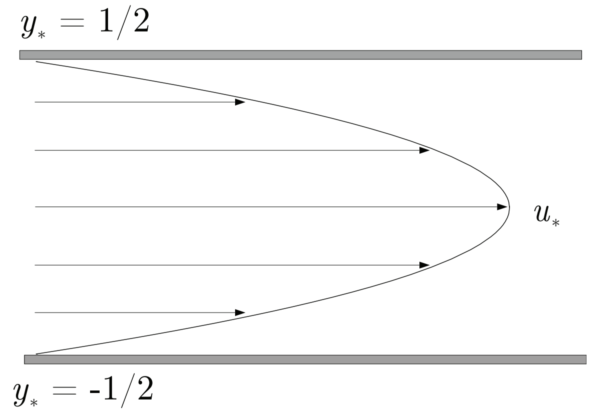

Figure 5.2: Velocity profile for pressure gradient driven flow in a slot.

The mean velocity is found through integrating the velocity field to arrive at

$$
\begin{aligned}
u_{*mean} &= \int_{-1/2}^{1/2} u_*(y_*)\ dy_*, &(5.48)\\
&= \int_{-1/2}^{1/2} \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - y_*^2\right) dy_*, &(5.49)\\
&= \frac{1}{2}\left(\frac{1}{4}y_* - \frac{1}{3}y_*^3\right)\bigg|_{-1/2}^{1/2}, &(5.50)\\
&= \frac{1}{12}. &(5.51)
\end{aligned}
$$

Note that we could have scaled the velocity field in such a fashion that either the maximum or the mean velocity was unity. The scaling we chose gave rise to a non-unity value of both. In dimensional terms we could say

$$
\frac{u}{\frac{(p_o-p_1)h^2}{\mu L}} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - \left(\frac{y}{h}\right)^2\right). \qquad (5.52)
$$

The velocity profile is sketched in Figure 5.2.

Now let us get the temperature field.

$$
\begin{aligned}
\frac{d^2T_*}{dy_*^2} &= -PrEc\left(\frac{d}{dy_*}\left(\frac{1}{2}\left(\left(\frac{1}{2}\right)^2 - y_*^2\right)\right)\right)^2, &(5.53)\\
&= -PrEc\ (-y_*)^2, &(5.54)\\
&= -PrEc\ y_*^2, &(5.55)\\
\frac{dT_*}{dy_*} &= -\frac{1}{3}PrEc\ y_*^3 + C_1, &(5.56)
\end{aligned}
$$

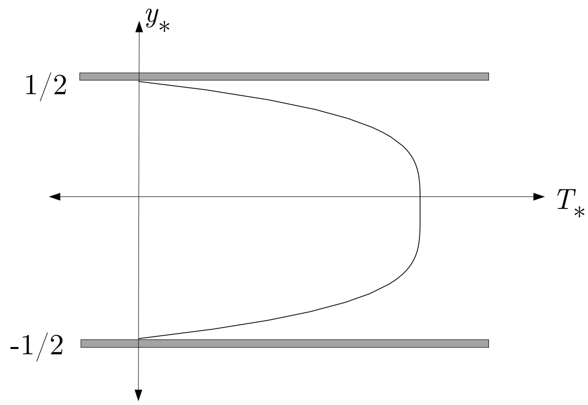

Figure 5.3: Temperature profile for pressure gradient driven flow in a slot.

$$
\begin{aligned}
T_* &= -\frac{1}{12} Pr Ec\, y_*^4 + C_1 y_* + C_2, && (5.57)\\
0 &= -\frac{1}{12} Pr Ec\, \frac{1}{16} + C_1 \frac{1}{2} + C_2, \qquad y_* = \frac{1}{2}, && (5.58)\\
0 &= -\frac{1}{12} Pr Ec\, \frac{1}{16} - C_1 \frac{1}{2} + C_2, \qquad y_* = -\frac{1}{2}, && (5.59)\\
C_1 &= 0, \qquad C_2 = \frac{Pr Ec}{192}. && (5.60)
\end{aligned}
$$

Regrouping, we find that

$$T_* = \frac{Pr Ec}{12} \left(\left(\frac{1}{2} \right)^4 - y_*^4 \right). \qquad (5.61)$$

In terms of dimensional quantities, we can say

$$\frac{T - T_o}{T_o} = \frac{(p_o - p_1)^2 h^4}{12 \mu L^2 k T_o} \left(\left(\frac{1}{2} \right)^4 - \left(\frac{y}{h} \right)^4 \right). \qquad (5.62)$$

The temperature profile is sketched in Figure 5.3.

From knowledge of the velocity and temperature field, we can calculate other quantities of interest. Let us calculate the field of shear stress and heat flux, and then evaluate both at the wall.

First for the shear stress, recall that in dimensional form we have

$$
\begin{aligned}
\tau_{ij} &= 2\mu \partial_{(i} v_{j)} + \lambda \underbrace{\partial_k v_k}_{=0} \delta_{ij}, && (5.63)\\
&= 2\mu \partial_{(i} v_{j)}. && (5.64)
\end{aligned}
$$

We have already seen the only non-zero components of the symmetric part of the velocity gradient tensor are the 12 and 21 components. Thus the 21 stress component is

$$\tau_{21} = 2\mu\partial_{(2}v_{1)} = 2\mu\left(\frac{\partial_2 v_1 + \underbrace{\partial_1 v_2}_{=0}}{2}\right), \tag{5.65}$$

$$= \mu\partial_2 v_1. \tag{5.66}$$

In (x, y) space, we then say here that

$$\tau_{yx} = \mu\frac{du}{dy}. \tag{5.67}$$

Note this is a stress on the y (tangential) face which points in the x direction; hence, it is certainly a shearing stress. In dimensionless terms, we can define a characteristic shear stress τ_c, so that the scale shear is $\tau_* = \tau_{yx}/\tau_c$. Thus, our equation for shear becomes

$$\tau_c\tau_* = \frac{\mu u_c}{h}\frac{du_*}{dy_*}. \tag{5.68}$$

Now take

$$\tau_c \equiv \frac{\mu u_c}{h} = \frac{\mu(p_o - p_1)h^2}{h\mu L} = (p_o - p_1)\left(\frac{h}{L}\right). \tag{5.69}$$

With this definition, we get

$$\tau_* = \frac{du_*}{dy_*}. \tag{5.70}$$

Evaluating for the velocity profile of the pressure gradient driven flow, we find

$$\tau_* = -y_*. \tag{5.71}$$

The stress is zero at the centerline $y_* = 0$ and has maximum magnitude of $1/2$ at either wall, $y_* = \pm 1/2$. In dimensional terms, the wall shear stress τ_w is

$$\tau_w = -\frac{1}{2}(p_o - p_1)\left(\frac{h}{L}\right). \tag{5.72}$$

Note that the wall shear stress is governed by the pressure difference and not the viscosity. However, the viscosity plays a determining role in selecting the maximum fluid velocity. The shear profile is sketched in Figure 5.4.

Next, let us calculate the heat flux vector. Recall that, for this flow, with no x or z variation of T, we have the heat flux vector as

$$q_y = -k\frac{\partial T}{\partial y}. \tag{5.73}$$

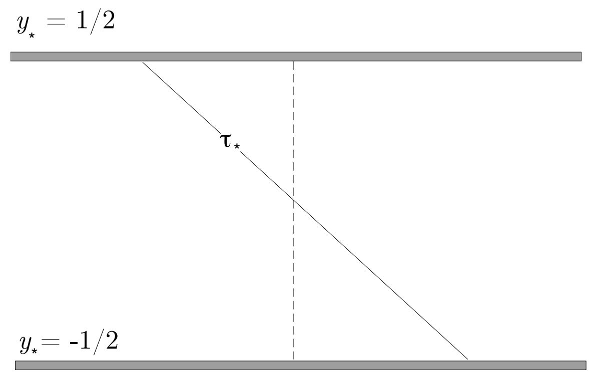

Figure 5.4: Shear stress profile for pressure gradient driven flow in a slot.

Now define scale the heat flux by a characteristic heat flux q_c, to be determined, to obtain a dimensionless heat flux:

$$q_* = \frac{q_y}{q_c}. \tag{5.74}$$

So,

$$q_c q_* = -\frac{kT_o}{h}\frac{dT_*}{dy_*}, \tag{5.75}$$

$$q_* = -\frac{kT_o}{hq_c}\frac{dT_*}{dt_*}. \tag{5.76}$$

Let $q_c \equiv kT_o/h$, so

$$q_* = -\frac{dT_*}{dy_*}, \tag{5.77}$$

$$q_* = \frac{1}{3}PrEc\ y_*^3. \tag{5.78}$$

For our flow, we have a cubic variation of the heat flux vector. There is no heat flux at the centerline, which corresponds to this being a region of no shear. The magnitude of the heat flux is maximum at the wall, the region of maximum shear. At the upper wall, we have

$$q_*|_{y_*=1/2} = \frac{1}{24}PrEc. \tag{5.79}$$

The heat flux profile is sketched in Figure 5.5. In dimensional terms we have

$$\frac{q_w}{\frac{kT_o}{h}} = \frac{1}{24}\frac{(p_o - p_1)^2 h^4}{\mu L^2 kT_o}, \tag{5.80}$$

$$q_w = \frac{1}{24}\frac{(p_o - p_1)^2 h^3}{\mu L^2}. \tag{5.81}$$

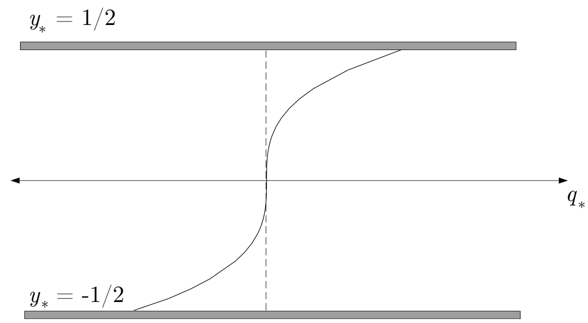

Figure 5.5: Heat flux profile for pressure gradient driven flow in a slot.

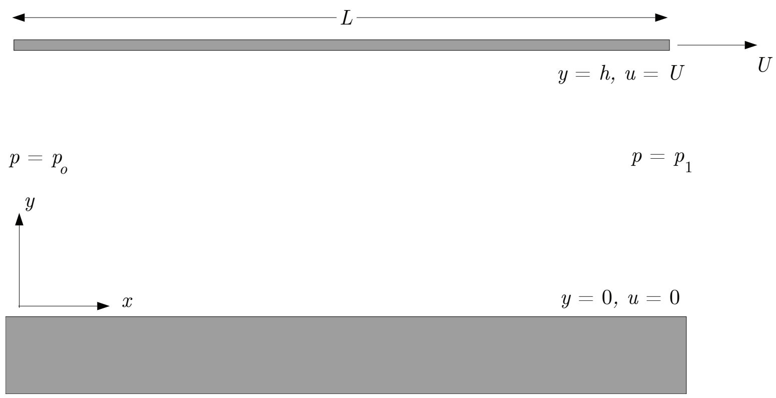

Figure 5.6: Configuration for Couette flow with pressure gradient.

5.1.2 Couette flow with pressure gradient

We next consider Couette flow with a pressure gradient. Couette flow implies that there is a moving plate at one boundary and a fixed plate at the other. It is a common experimental configuration, and used often to actually determine a fluid's viscosity. Here we will take the same assumptions as for pressure gradient driven flow in a slot, expect for the boundary condition at the upper surface, which we will require to have a constant velocity U. We will also shift the coordinates so that $y = 0$ matches the lower plate surface and $y = h$ matches the upper plate surface. The configuration for this flow is shown in Figure 5.6.

Our equations governing this flow are

$$\frac{d^2u}{dy^2} = -\frac{p_o - p_1}{\mu L}, \qquad u(0) = 0, \quad u(h) = U, \tag{5.82}$$

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2, \qquad T(0) = T_o, \quad T(h) = T_o. \tag{5.83}$$

Once again in momentum, there is no acceleration, and viscous stresses balance shear stresses. In energy, there is no energy increase, and generation of thermal energy due to viscous work is balanced by diffusion of the thermal energy, ultimately out of the system through the boundaries. Here there are inhomogeneities in both the forcing terms and the boundary conditions. In terms of work, both the pressure gradient and the pulling of the plate induce work.

Once again let us scale the equations. This time, we have a natural velocity scale, U, the upper plate velocity. So take

$$y_* = \frac{y}{h}, \qquad T_* = \frac{T - T_o}{T_o}, \qquad u_* = \frac{u}{U}. \tag{5.84}$$

The momentum equation becomes

$$\frac{U}{h^2}\frac{d^2 u_*}{dy_*^2} = -\frac{p_o - p_1}{\mu L}, \tag{5.85}$$

$$\frac{d^2 u_*}{dy_*^2} = -\frac{(p_o - p_1)h^2}{\mu U L}. \tag{5.86}$$

With dimensionless pressure gradient

$$\mathcal{P} \equiv \frac{(p_o - p_1)h^2}{\mu U L}, \tag{5.87}$$

we get

$$\frac{d^2 u_*}{dy_*^2} = -\mathcal{P}, \tag{5.88}$$

$$u_*(0) = 0, \quad u_*(1) = 1. \tag{5.89}$$

$$\tag{5.90}$$

This has solution

$$u_* = -\frac{1}{2}\mathcal{P} y_*^2 + C_1 y_* + C_2. \tag{5.91}$$

Applying the boundary conditions, we get

$$0 = -\frac{1}{2}\mathcal{P}(0)^2 + C_1(0) + C_2, \tag{5.92}$$

$$0 = C_2, \tag{5.93}$$

$$1 = -\frac{1}{2}\mathcal{P}(1)^2 + C_1(1), \tag{5.94}$$

$$C_1 = 1 + \frac{1}{2}\mathcal{P}, \tag{5.95}$$

$$u_* = -\frac{1}{2}\mathcal{P} y_*^2 + \left(1 + \frac{1}{2}\mathcal{P}\right) y_*, \tag{5.96}$$

$$u_* = \underbrace{\frac{1}{2}\mathcal{P} y_*(1 - y_*) +}_{\text{pressure effect}} \underbrace{y_*}_{\text{Couette effect}}. \tag{5.97}$$

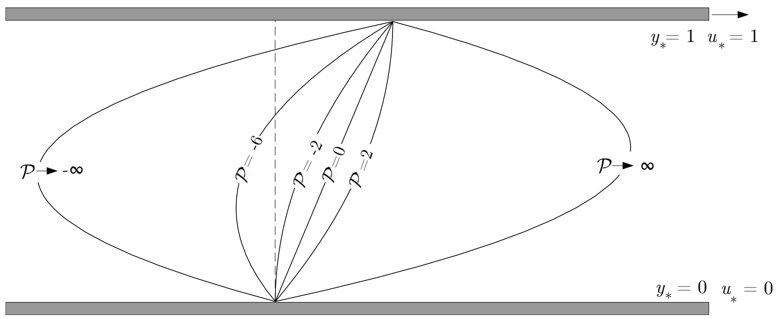

Figure 5.7: Velocity profiles for various values of $\mathcal{P}$ for Couette flow with pressure gradient.

We see that the pressure gradient generates a velocity profile that is quadratic in y_*. This is distinguished from the Couette effect, that is the effect of the upper plate's motion, which gives a linear profile. Because our governing equation here is linear, it is appropriate to think of these as superposed solutions. Velocity profiles for various values of $\mathcal{P}$ are shown in Figure 5.7.

Let us now calculate the shear stress profile. With $\tau = \mu(du/dy)$, and taking $\tau_* = \tau/\tau_c$, we get

$$\tau_c \tau_* = \frac{\mu U}{h}\frac{du_*}{dy_*}, \tag{5.98}$$

$$\tau_* = \frac{\mu U}{h\tau_c}\frac{du_*}{dy_*}, \tag{5.99}$$

$$\text{taking} \quad \tau_c \equiv \frac{\mu U}{h}, \tag{5.100}$$

$$\tau_* = \frac{du_*}{dy_*}, \qquad \text{so here,} \tag{5.101}$$

$$\tau_* = -\mathcal{P}y_* + \frac{1}{2}\mathcal{P} + 1, \qquad \text{and} \tag{5.102}$$

$$\tau_*\big|_{y_*=0} = \frac{1}{2}\mathcal{P} + 1, \tag{5.103}$$

$$\tau_*\big|_{y_*=1} = -\frac{1}{2}\mathcal{P} + 1. \tag{5.104}$$

The wall shear has a pressure gradient effect and a Couette effect as well. In fact we can select a pressure gradient to balance the Couette effect at one or the other wall, but not both.

We can also calculate the dimensionless volume flow rate Q_*, which for incompressible flow, is directly proportional to the mass flux. Ignoring how the scaling would be done, we arrive at

$$Q_* = \int_0^1 u_* dy_*, \tag{5.105}$$

$$= \int_0^1 \left(-\frac{1}{2}\mathcal{P}y_*^2 + \left(1+\frac{1}{2}\mathcal{P}\right) y_*\right) dy_*, \tag{5.106}$$

$$= \left(-\frac{1}{6}\mathcal{P}y_*^3\right)\Big|_0^1 + \left(1+\frac{1}{2}\mathcal{P}\right)\frac{y_*^2}{2}\Big|_0^1, \tag{5.107}$$

$$= -\frac{\mathcal{P}}{6} + \left(1+\frac{1}{2}\mathcal{P}\right)\frac{1}{2}, \tag{5.108}$$

$$= \frac{\mathcal{P}}{12} + \frac{1}{2}. \tag{5.109}$$

Again there is a pressure gradient contribution and a Couette contribution, and we could select $\mathcal{P}$ to give no net volume flow rate.

We can summarize some of the special cases as follows

- $\mathcal{P} \to -\infty$: $u_* = (1/2)\mathcal{P}y_*(1-y_*)$; $\tau_* = \mathcal{P}(1/2 - y_*)$, $Q_* = \mathcal{P}/12$. Here the fluid flows in the opposite direction as driven by the plate because of the large pressure gradient.
- $\mathcal{P} = -6$. Here we get no net mass flow and $u_* = 3y_*^2 - 2y_*$, $\tau_* = 2y_*$, $Q_* = 0$.
- $\mathcal{P} = -2$. Here we get no shear at the bottom wall and $u_* = y_*^2$, $\tau_* = 2y_*$, $Q_* = 1/3$.
- $\mathcal{P} = 0$. Here we have no pressure gradient and $u_* = y_*$, $\tau_* = 1$, $Q_* = 1/2$.
- $\mathcal{P} = 2$. Here we get no shear at the top wall and $u_* = -y_*^2 + 2y_*$, $\tau* = -2y_* + 2$, $Q_* = 2/3$.
- $\mathcal{P} \to \infty$: $u_* = (1/2)\mathcal{P}y_*(1-y_*)$; $\tau_* = \mathcal{P}(1/2 - y_*)$, $Q_* = \mathcal{P}/12$. Here the fluid flows in the same direction as driven by the plate.

We now consider the heat transfer problem. Scaling, we get

$$\frac{T_o}{h^2}\frac{d^2T_*}{dy_*^2} = -\frac{\mu U^2}{kh^2}\left(\frac{du_*}{dy_*}\right)^2, \qquad T_*(0) = T_*(1) = 0, \tag{5.110}$$

$$\frac{d^2T_*}{dy_*^2} = -\frac{\mu U^2}{kT_o}\left(\frac{du_*}{dy_*}\right)^2, \tag{5.111}$$

$$= -\frac{\mu c}{k}\frac{U^2}{cT_o}\left(\frac{du_*}{dy_*}\right)^2, \tag{5.112}$$

$$= -Pr Ec\left(\frac{du_*}{dy_*}\right)^2, \tag{5.113}$$

$$= -Pr Ec\, \tau_*^2, \tag{5.114}$$

$$
\begin{aligned}
&= -PrEc\left(-\mathcal{P}y_* + \frac{1}{2}\mathcal{P} + 1\right)^2, && (5.115)\\
&= -PrEc\left(\mathcal{P}^2 y_*^2 - 2\mathcal{P}\left(\frac{1}{2}\mathcal{P}+1\right)y_* + \left(1+\frac{1}{2}\mathcal{P}\right)^2\right), && (5.116)\\
\frac{dT_*}{dy_*} &= -PrEc\left(\frac{\mathcal{P}^2}{3}y_*^3 - \mathcal{P}\left(\frac{1}{2}\mathcal{P}+1\right)y_*^2 + \left(1+\frac{1}{2}\mathcal{P}\right)^2 y_*\right) + C_1, && (5.117)\\
T_* &= -PrEc\left(\frac{\mathcal{P}^2}{12}y_*^4 - \frac{\mathcal{P}}{3}\left(\frac{1}{2}\mathcal{P}+1\right)y_*^3 + \frac{1}{2}\left(1+\frac{1}{2}\mathcal{P}\right)^2 y_*^2\right)\\
&\quad + C_1 y_* + C_2, && (5.118)\\
T_*(0) &= 0 = C_2, && (5.119)\\
T_*(1) &= 0 = -PrEc\left(\frac{\mathcal{P}^2}{12} - \frac{\mathcal{P}}{3}\left(\frac{1}{2}\mathcal{P}+1\right) + \frac{1}{2}\left(1+\frac{1}{2}\mathcal{P}\right)^2\right) + C_1, && (5.120)\\
C_1 &= PrEc\left(\frac{1}{2} + \frac{\mathcal{P}}{6} + \frac{\mathcal{P}^2}{24}\right), && (5.121)\\
T_* &= -PrEc\left(\frac{\mathcal{P}^2}{12}y_*^4 - \frac{\mathcal{P}}{3}\left(\frac{1}{2}\mathcal{P}+1\right)y_*^3 + \frac{1}{2}\left(1+\frac{1}{2}\mathcal{P}\right)^2 y_*^2\right)\\
&\quad + PrEc\left(\frac{1}{2} + \frac{\mathcal{P}}{6} + \frac{\mathcal{P}^2}{24}\right)y_*. && (5.122)
\end{aligned}
$$

Factoring, we can write the temperature profile as

$$
T_* = \frac{PrEc}{24} y_*(1-y_*)(12 + 4\mathcal{P} + \mathcal{P}^2 - 8\mathcal{P}y_* - 2\mathcal{P}^2 y_* + 2\mathcal{P}^2 y_*^2). \quad (5.123)
$$

For the wall heat transfer, recall $q_y = -k(dT/dy)$. Scaling, we get

$$
\begin{aligned}
q_c q_* &= -\frac{kT_o}{h}\frac{dT_*}{dy_*}, && (5.124)\\
q_* &= -\frac{kT_o}{hq_c}\frac{dT_*}{dy_*}, && (5.125)\\
\text{choosing}\quad q_c &\equiv \frac{kT_o}{h}, && (5.126)\\
q_* &= -\frac{dT_*}{dy_*}. && (5.127)
\end{aligned}
$$

So

$$
q_* = PrEc\left(\frac{\mathcal{P}^2}{3}y_*^3 - \mathcal{P}\left(\frac{1}{2}\mathcal{P}+1\right)y_*^2 + \left(1+\frac{1}{2}\mathcal{P}\right)^2 y_* - \frac{1}{2} - \frac{\mathcal{P}}{6} - \frac{\mathcal{P}^2}{24}\right). \quad (5.128)
$$

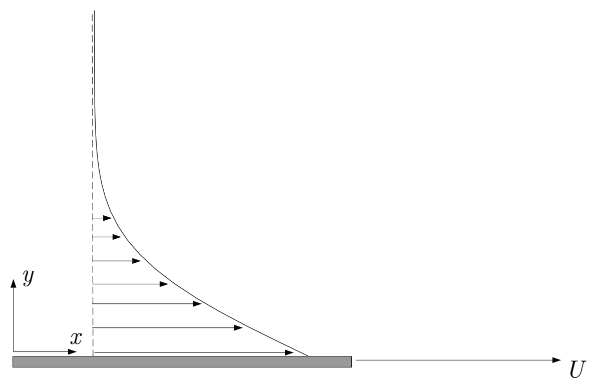

Figure 5.8: Schematic for Stokes' first problem of a suddenly accelerated plate diffusing linear momentum into a fluid at rest.

At the bottom wall $y_* = 0$, we get for the heat transfer vector

$$q_*|_{y_*=0} = -PrEc\left(\frac{1}{2} + \frac{\mathcal{P}}{6} + \frac{\mathcal{P}^2}{24}\right). \tag{5.129}$$

5.2 Similarity solutions

In this section, we will consider problems which can be addressed by what is known as a similarity transformation. The problems themselves will be fundamental ones which have variation in either time and one spatial coordinate, or with two spatial coordinates. This is in contrast with solutions of the previous section which varied only with one spatial coordinate.

Since two coordinates are involved, we must resort to solving partial differential equations. The similarity transformation actually reveals a hidden symmetry of the partial differential equations by defining a new independent variable, which is a grouping of the original independent variables, under which the partial differential equations transform into ordinary differential equations. We then solve the resulting ordinary differential equations by standard techniques.

5.2.1 Stokes' first problem

The first problem we will consider which uses a similarity transformation is known as Stokes' first problem, as Stokes addressed it in his original work which developed the Navier-Stokes equations in the mid-nineteenth century.[5] The problem is described as follows, and is sketched in Figure 5.8. Consider a flat plate of infinite extent lying at rest for $t < 0$ on the $y = 0$ plane in $x - y - z$ space. In the volume described by $y > 0$ exists a fluid of semi-infinite extent which is at rest at time $t < 0$. At $t = 0$, the flat plate is suddenly accelerated to

[5]Stokes, G. G., 1851, "On the effect of the internal friction of fluids on the motion of pendulums," *Transactions of the Cambridge Philosophical Society*, 9(2): 8-106.

a constant velocity of U, entirely in the x direction. Because the no-slip condition is satisfied for the viscous flow, this induces the fluid at the plate surface to acquire an instantaneous velocity of $u(0) = U$. Because of diffusion of linear x momentum via tangential viscous shear forces, the fluid in the region above the plate begins to acquire a positive velocity in the x direction as well. We will use the Navier-Stokes equations to quantify this behavior. Let us make identical assumptions as we did in the previous section, except that 1) we will not neglect time derivatives, as they are an obviously important feature of the flow, and 2) we will assume all pressure gradients are zero; hence the fluid has a constant pressure.

Under these assumptions, the x momentum equation,

$$\rho\frac{\partial}{\partial t}u + \rho u \underbrace{\frac{\partial}{\partial x}u}_{=0} + \rho \underbrace{v}_{=0}\frac{\partial}{\partial y}u + \rho w \underbrace{\frac{\partial}{\partial z}u}_{=0} = -\underbrace{\frac{\partial p}{\partial x}}_{=0} + \mu\left(\underbrace{\frac{\partial^2}{\partial x^2}u}_{=0} + \frac{\partial^2}{\partial y^2}u + \underbrace{\frac{\partial^2}{\partial z^2}u}_{=0}\right), \tag{5.130}$$

is the only relevant component of linear momenta, and reduces to

$$\underbrace{\rho\frac{\partial u}{\partial t}}_{\text{(mass)(acceleration)}} = \underbrace{\mu\frac{\partial^2 u}{\partial y^2}}_{\text{shear force}}. \tag{5.131}$$

The energy equation reduces as follows

$$\rho c\frac{\partial}{\partial t}T + \rho c\left(u\underbrace{\frac{\partial}{\partial x}T}_{=0} + \underbrace{v}_{=0}\frac{\partial}{\partial y}T + \underbrace{w}_{=0}\underbrace{\frac{\partial}{\partial z}T}_{=0}\right) = k\left(\underbrace{\frac{\partial^2}{\partial x^2}T}_{=0} + \frac{\partial^2}{\partial y^2}T + \underbrace{\frac{\partial^2}{\partial z^2}T}_{=0}\right) + 2\mu\partial_{(i}v_{j)}\partial_{(i}v_{j)}, \tag{5.132}$$

$$\underbrace{\rho c\frac{\partial T}{\partial t}}_{\text{energy increase}} = \underbrace{k\frac{\partial^2 T}{\partial y^2}}_{\text{thermal diffusion}} + \underbrace{\mu\left(\frac{\partial u}{\partial y}\right)^2}_{\text{viscous work source}} \tag{5.133}$$

Let us first consider the x momentum equation. Recalling the momentum diffusivity definition $\nu = \mu/\rho$, we get the following partial differential equation, initial and boundary conditions:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{5.134}$$

$$u(y,0) = 0, \qquad u(0,t) = U, \qquad u(\infty,t) = 0. \tag{5.135}$$

Now let us scale the equations. Choose

$$u_* = \frac{u}{U}, \qquad t_* = \frac{t}{t_c}, \qquad y_* = \frac{y}{y_c}. \tag{5.136}$$

We have yet to choose characteristic length, (y_c), and time, (t_c), scales. The equations become

$$\frac{U}{t_c}\frac{\partial u_*}{\partial t_*} = \frac{\nu U}{y_c^2}\frac{\partial^2 u_*}{\partial y_*^2}, \tag{5.137}$$

$$\frac{\partial u_*}{\partial t_*} = \frac{\nu t_c}{y_c^2}\frac{\partial^2 u_*}{\partial y_*^2}. \tag{5.138}$$

Wasting no time, we choose

$$y_c \equiv \frac{\nu}{U} = \frac{\mu}{\rho U}. \tag{5.139}$$

Noting the SI units, we see $\mu/(\rho U)$ has units of length: $\frac{\text{N s}}{\text{m}^2}\frac{\text{m}^3}{\text{kg}}\frac{\text{s}}{\text{m}} = \frac{\text{kg m}}{\text{s}^2}\frac{\text{s}}{\text{m}^2}\frac{\text{m}^3}{\text{kg}}\frac{\text{s}}{\text{m}} = \text{m}$. With this choice, we get

$$\frac{\nu t_c}{y_c^2} = \frac{\nu t_c U^2}{\nu^2} = \frac{t_c U^2}{\nu}. \tag{5.140}$$

This suggests we choose

$$t_c = \frac{\nu}{U^2}. \tag{5.141}$$

With all of these choices the complete system can be written as

$$\frac{\partial u_*}{\partial t_*} = \frac{\partial^2 u_*}{\partial y_*^2}, \tag{5.142}$$

$$u_*(y_*,0) = 0, \qquad u_*(0,t_*) = 1, \qquad u_*(\infty,t_*) = 0. \tag{5.143}$$

Now for *self-similarity*, we seek transformation which reduce this partial differential equation, as well as its initial and boundary conditions, into an ordinary differential equation with suitable boundary conditions. If this transformation does not exist, no similarity solution exists. In this, but not all cases, the transformation does exist.

Let us first consider a general transformation from a y_*, t_* coordinate system to a new $\eta_*, \hat{t}_*$ coordinate system. We assume then a general transformation

$$\eta_* = \eta_*(y_*, t_*), \tag{5.144}$$

$$\hat{t}_* = \hat{t}_*(y_*, t_*). \tag{5.145}$$

We assume then that a general variable ψ_* which is a function of y_* and t_* also has the same value at the transformed point $\eta_*, \hat{t}_*$:

$$\psi_*(y_*, t_*) = \psi_*(\eta_*, \hat{t}_*). \tag{5.146}$$

The chain rule then gives expressions for derivatives:

$$\left.\frac{\partial \psi_*}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \psi_*}{\partial \eta_*}\right|_{\hat{t}_*}\left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} + \left.\frac{\partial \psi_*}{\partial \hat{t}_*}\right|_{\eta_*}\left.\frac{\partial \hat{t}_*}{\partial t_*}\right|_{y_*}, \tag{5.147}$$

$$\left.\frac{\partial \psi_*}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \psi_*}{\partial \eta_*}\right|_{\hat{t}_*}\left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*} + \left.\frac{\partial \psi_*}{\partial \hat{t}_*}\right|_{\eta_*}\left.\frac{\partial \hat{t}_*}{\partial y_*}\right|_{t_*}. \tag{5.148}$$

Now we will restrict ourselves to the transformation

$$\hat{t}_* = t_*, \tag{5.149}$$

so we have $\left.\frac{\partial \hat{t}_*}{\partial t_*}\right|_{y_*} = 1$ and $\left.\frac{\partial \hat{t}_*}{\partial y_*}\right|_{t_*} = 0$, so our rules for differentiation reduce to

$$\left.\frac{\partial \psi_*}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \psi_*}{\partial \eta_*}\right|_{\hat{t}_*} \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} + \left.\frac{\partial \psi_*}{\partial \hat{t}_*}\right|_{\eta_*}, \tag{5.150}$$

$$\left.\frac{\partial \psi_*}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \psi_*}{\partial \eta_*}\right|_{\hat{t}_*} \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*}. \tag{5.151}$$

The next assumption is key for a similarity solution to exist. We restrict ourselves to transformations for which $\psi_* = \psi_*(\eta_*)$. That is we allow no dependence of ψ_* on $\hat{t}_*$. Hence we must require that $\left.\frac{\partial \psi_*}{\partial \hat{t}_*}\right|_{\eta_*} = 0$. Moreover, partial derivatives of ψ_* become total derivatives, giving us a final form of transformations for the derivatives

$$\left.\frac{\partial \psi_*}{\partial t_*}\right|_{y_*} = \frac{d\psi_*}{d\eta_*} \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*}, \tag{5.152}$$

$$\left.\frac{\partial \psi_*}{\partial y_*}\right|_{t_*} = \frac{d\psi_*}{d\eta_*} \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*}. \tag{5.153}$$

In terms of operators we can say

$$\left.\frac{\partial}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} \frac{d}{d\eta_*}, \tag{5.154}$$

$$\left.\frac{\partial}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*} \frac{d}{d\eta_*}. \tag{5.155}$$

Now returning to Stokes' first problem, let us assume that a similarity solution exists of the form $u_*(y_*, t_*) = u_*(\eta_*)$. It is not always possible to find a similarity variable η_*. One of the more robust ways to find a similarity variable, if it exists, comes from group theory,[6] and is explained in detail in the recent monograph by Cantwell. Group theory, which is too

[6]Group theory has a long history in mathematics and physics. Its complicated origins generally include attribution to Évariste Galois, 1811-1832, a somewhat romantic figure, as well as Niels Henrick Abel, 1802-1829, the Norwegian mathematician. Critical developments were formalized by Marius Sophus Lie, 1842-1899, another Norwegian mathematician, in what today is known as Lie group theory. A modern variant, known as "renormalization group" (RNG) theory is an area for active research. The 1982 Nobel prize in physics went to Kenneth Geddes Wilson, 1936-, of Cornell University and The Ohio State University, for use of RNG in studying phase transitions, first done in the 1970s. The award citation refers to the possibilities of using RNG in studying the great unsolved problem of turbulence, a modern area of research in which Steven Alan Orszag, 1943-2011, made many contributions.

Quoting from the useful Eric Weisstein's World of Mathematics, available online at `http://mathworld.wolfram.com/Group.html`, "A group G is a finite or infinite set of elements together detailed to explicate in full here, relies on a generalized *symmetry* of equations to find simpler forms. In the same sense that a snowflake, subjected to rotations of $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$, or 2π, is transformed into a form which is indistinguishable from its original form, we seek transformations of the variables in our partial differential equation which map the equation into a form which is indistinguishable from the original. When systems are subject to such transformations, known as group operators, they are said to exhibit symmetry.

Let us subject our governing partial differential equation along with initial and boundary conditions to a particularly simple type of transformation, a simple stretching of space, time, and velocity:

$$\tilde{t} = e^a t_*, \qquad \tilde{y} = e^b y_*, \qquad \tilde{u} = e^c u_*. \tag{5.156}$$

Here the "$\sim$" variables are stretched variables, and a, b, and c are constant parameters. The exponential will be seen to be a convenience, which is not absolutely necessary. Note that for $a \in (-\infty, \infty)$, $b \in (-\infty, \infty)$, $c \in (-\infty, \infty)$, that $e^a \in (0, \infty)$, $e^b \in (0, \infty)$, $e^c \in (0, \infty)$. So the stretching does not change the direction of the variable; that is it is not a reflecting transformation. We note that with this stretching, the domain of the problem remains unchanged; that is $t_* \in [0, \infty)$ maps into $\tilde{t} \in [0, \infty)$; $y_* \in [0, \infty)$ maps into $\tilde{y} \in [0, \infty)$. The range is also unchanged if we allow $u_* \in [0, \infty)$, which maps into $\tilde{u} \in [0, \infty)$. Direct substitution of the transformation shows that in the stretched space, the system becomes

$$e^{a-c}\frac{\partial \tilde{u}}{\partial \tilde{t}} = e^{2b-c}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}, \tag{5.157}$$

$$e^{-c}\tilde{u}(\tilde{y}, 0) = 0, \qquad e^{-c}\tilde{u}(0, \tilde{t}) = 1, \qquad e^{-c}\tilde{u}(\infty, \tilde{t}) = 0. \tag{5.158}$$

In order that the stretching transformation map the system into a form indistinguishable from the original, that is for the transformation to exhibit symmetry, we must take

$$c = 0, \qquad a = 2b. \tag{5.159}$$

with a binary operation which together satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property. The operation with respect to which a group is defined is often called the 'group operation,' and a set is said to be a group 'under' this operation. Elements A, B, $C, \ldots$ with binary operations A and B denoted AB form a group if

1. Closure: If A and B are two elements in G, then the product AB is also in G.
2. Associativity: The defined multiplication is associative, i.e. for all $A, B, C \in G$, $(AB)C = A(BC)$.
3. Identity: There is an identity element I (a.k.a. $\mathbf{1}$, E, or e) such that $IA = AI = A$ for every element $A \in G$.
4. Inverse: There must be an inverse or reciprocal of each element. Therefore, the set must contain an element $B = A^{-1}$ such that $AA^{-1} = A^{-1}A = I$ for each element of G.

..., A map between two groups which preserves the identity and the group operation is called a homomorphism. If a homomorphism has an inverse which is also a homomorphism, then it is called an isomorphism and the two groups are called isomorphic. Two groups which are isomorphic to each other are considered to be 'the same' when viewed as abstract groups." For example, the group of 90 degree rotations of a square are isomorphic.

So our symmetry transformation is

$$\tilde{t} = e^{2b} t_*, \qquad \tilde{y} = e^{b} y_*, \qquad \tilde{u} = u_*, \tag{5.160}$$

giving in transformed space

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}, \tag{5.161}$$

$$\tilde{u}(\tilde{y}, 0) = 0, \qquad \tilde{u}(0, \tilde{t}) = 1, \qquad \tilde{u}(\infty, \tilde{t}) = 0. \tag{5.162}$$

Now both the original and transformed systems are the same, and the remaining stretching parameter b does not enter directly into either formulation, so we cannot expect it in the solution of either form. That is we expect a solution to be independent of the stretching parameter b. This can be achieved if we take both u_* and $\tilde{u}$ to be functions of special combinations of the independent variables, combinations that are formed such that b does not appear. Eliminating b via

$$e^b = \frac{\tilde{y}}{y_*}, \tag{5.163}$$

we get

$$\frac{\tilde{t}}{t_*} = \left(\frac{\tilde{y}}{y_*}\right)^2, \tag{5.164}$$

or after rearrangement

$$\frac{y_*}{\sqrt{t_*}} = \frac{\tilde{y}}{\sqrt{\tilde{t}}}. \tag{5.165}$$

We thus expect $u_* = u_*\left(y_*/\sqrt{t_*}\right)$ or equivalently $\tilde{u} = \tilde{u}\left(\tilde{y}/\sqrt{\tilde{t}}\right)$. This form also allows $u_* = u_*\left(\alpha y_*/\sqrt{t_*}\right)$, where α is any constant. Let us then define our similarity variable η_* as

$$\eta_* = \frac{y_*}{2\sqrt{t_*}}. \tag{5.166}$$

Here the factor of 1/2 is simply a convenience adopted so that the solution takes on a traditional form. We would find that any constant in the similarity transformation would induce a self-similar result.

Let us rewrite the differential equation, boundary, and initial conditions ($\partial u_*/\partial t_* = \partial^2 u_*/\partial y_*^2$, $u_*(y_*, 0) = 0$, $u_*(0, t_*) = 1$, $u_*(\infty, t_*) = 0$), in terms of the similarity variable η_*. We first must use the chain rule to get expressions for the derivatives. Applying the general results just developed, we get

$$\frac{\partial u_*}{\partial t_*} = \frac{\partial \eta_*}{\partial t_*}\frac{du_*}{d\eta_*} = -\frac{1}{2}\frac{y_*}{2} t_*^{-3/2}\frac{du_*}{d\eta_*} = -\frac{\eta_*}{2t_*}\frac{du_*}{d\eta_*}, \tag{5.167}$$

$$\frac{\partial u_*}{\partial y_*} = \frac{\partial \eta_*}{\partial y_*}\frac{du_*}{d\eta_*} = \frac{1}{2\sqrt{t_*}}\frac{du_*}{d\eta_*}, \tag{5.168}$$

$$\frac{\partial^2 u_*}{\partial y_*^2} = \frac{\partial}{\partial y_*}\left(\frac{\partial u_*}{\partial y_*}\right) = \frac{\partial}{\partial y_*}\left(\frac{1}{2\sqrt{t_*}}\frac{du_*}{d\eta_*}\right), \tag{5.169}$$

$$= \frac{1}{2\sqrt{t_*}}\frac{\partial}{\partial y_*}\left(\frac{du_*}{d\eta_*}\right) = \frac{1}{2\sqrt{t_*}}\left(\frac{1}{2\sqrt{t_*}}\frac{d^2u_*}{d\eta_*^2}\right) = \frac{1}{4t_*}\frac{d^2u_*}{d\eta_*^2}. \tag{5.170}$$

Thus, applying these rules to our governing linear momenta equation, we recover

$$-\frac{\eta_*}{2t_*}\frac{du_*}{d\eta_*} = \frac{1}{4t_*}\frac{d^2u_*}{d\eta_*^2}, \tag{5.171}$$

$$\frac{d^2u_*}{d\eta_*^2} + 2\eta_*\frac{du_*}{d\eta_*} = 0. \tag{5.172}$$

Note our governing equation has a singularity at $t_* = 0$. As it appears on both sides of the equation, we cancel it on both sides, but we shall see that this point is associated with special behavior of the similarity solution. The important result is that the reduced equation has dependency on η_* only. If this did not occur, we could not have a similarity solution.

Now consider the initial and boundary conditions. They transform as follows:

$$y_* = 0, \Longrightarrow \eta_* = 0, \tag{5.173}$$

$$y_* \to \infty, \Longrightarrow \eta_* \to \infty, \tag{5.174}$$

$$t_* \to 0, \Longrightarrow \eta_* \to \infty. \tag{5.175}$$

Note that the three important points for t_* and y_* collapse into two corresponding points in η_*. This is also necessary for the similarity solution to exist. Consequently, our conditions in η_* space reduce to

$$u_*(0) = 1, \qquad \text{no slip}, \tag{5.176}$$

$$u_*(\infty) = 0, \qquad \text{initial and far-field}. \tag{5.177}$$

We solve the second order differential equation by the method of reduction of order, noticing that it is really two first order equations in disguise:

$$\frac{d}{d\eta_*}\left(\frac{du_*}{d\eta_*}\right) + 2\eta_*\left(\frac{du_*}{d\eta_*}\right) = 0. \tag{5.178}$$

$$\text{multiplying by the integrating factor} \qquad e^{\eta_*^2}, \tag{5.179}$$

$$e^{\eta_*^2}\frac{d}{d\eta_*}\left(\frac{du_*}{d\eta_*}\right) + 2\eta_* e^{\eta_*^2}\left(\frac{du_*}{d\eta_*}\right) = 0. \tag{5.180}$$

$$\frac{d}{d\eta_*}\left(e^{\eta_*^2}\frac{du_*}{d\eta_*}\right) = 0, \tag{5.181}$$

$$e^{\eta_*^2}\frac{du_*}{d\eta_*} = A, \tag{5.182}$$

$$\frac{du_*}{d\eta_*} = Ae^{-\eta_*^2}, \tag{5.183}$$

$$u_* = B + A\int_0^{\eta_*} e^{-s^2}\, ds. \tag{5.184}$$

Now applying the condition $u_* = 1$ at $\eta_* = 0$ gives

$$1 = B + A \underbrace{\int_0^0 e^{-s^2}\, ds}_{=0}, \tag{5.185}$$

$$B = 1. \tag{5.186}$$

So we have

$$u_* = 1 + A \int_0^{\eta_*} e^{-s^2}\, ds. \tag{5.187}$$

Now applying the condition $u_* = 0$ at $\eta_* \to \infty$, we get

$$0 = 1 + A \underbrace{\int_0^\infty e^{-s^2}\, ds}_{=\sqrt{\pi}/2}, \tag{5.188}$$

$$0 = 1 + A\frac{\sqrt{\pi}}{2}, \tag{5.189}$$

$$A = -\frac{2}{\sqrt{\pi}}. \tag{5.190}$$

Though not immediately obvious, it can be shown by a simple variable transformation to a polar coordinate system that the above integral from 0 to ∞ has the value $\sqrt{\pi}/2$. It is not surprising that this integral has finite value over the semi-infinite domain as the integrand is bounded between zero and one, and decays rapidly to zero as $s \to \infty$. Consequently, the velocity profile can be written as

$$u_*(\eta_*) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta_*} e^{-s^2}\, ds, \tag{5.191}$$

$$u_*(y_*, t_*) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{y_*}{2\sqrt{t_*}}} e^{-s^2}\, ds, \tag{5.192}$$

$$u_*(y_*, t_*) = \operatorname{erfc}\left(\frac{y_*}{2\sqrt{t_*}}\right). \tag{5.193}$$

In the last form above, we have introduced the so-called error function complement, "erfc." Plots for the velocity profile in terms of both η_* and y_*, t_* are given in Figure 5.9. We see that in similarity space, the curve is a single curve that in which u_* has a value of unity at $\eta_* = 0$ and has nearly relaxed to zero when $\eta_* = 1$. In dimensionless physical space, we see that at early time, there is a thin momentum layer near the surface. At later time more momentum is present in the fluid. We can say in fact that momentum is diffusing into the fluid.

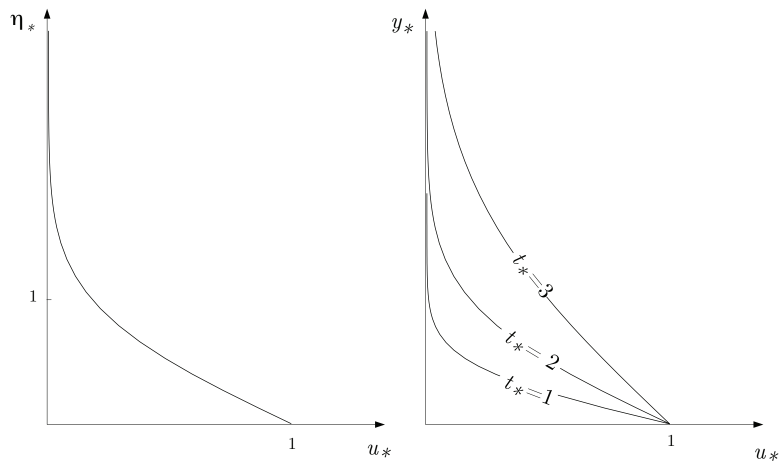

Figure 5.9: Sketch of velocity field solution for Stokes' first problem in both similarity coordinate η_* and primitive coordinates y_*, t_*.

We define the momentum diffusion length as the length for which significant momentum has diffused into the fluid. This is well estimated by taking $\eta_* = 1$. In terms of physical variables, we have

$$\frac{y_*}{2\sqrt{t_*}} = 1, \tag{5.194}$$

$$y_* = 2\sqrt{t_*}, \tag{5.195}$$

$$\frac{y}{\frac{\nu}{U}} = 2\sqrt{\frac{t}{\frac{\nu}{U^2}}}, \tag{5.196}$$

$$y = \frac{2\nu}{U}\sqrt{\frac{U^2 t}{\nu}}, \tag{5.197}$$

$$y = 2\sqrt{\nu t}. \tag{5.198}$$

We can in fact define this as a boundary layer thickness. That is to say the momentum boundary layer thickness in Stokes' first problem grows at a rate proportional to the square root of momentum diffusivity and time. This class of result is a hallmark of all diffusion processes, be it mass, momentum, or energy.

Taking standard properties of air, we find after one minute that its boundary layer thickness is 0.01 m. For oil after one minute, we get a thickness of 0.002 m.

We next consider the shear stress field. For this problem, the shear stress reduces to simply

$$\tau = \mu \frac{\partial u}{\partial y}. \tag{5.199}$$

Scaling as before by a characteristic stress τ_c, we get

$$\tau_* \tau_c = \frac{\mu U}{\frac{\nu}{U}} \frac{\partial u_*}{\partial y_*}, \tag{5.200}$$

$$\tau_* = \frac{\mu U^2}{\nu}\frac{1}{\tau_c}\frac{\partial u_*}{\partial y_*}. \tag{5.201}$$

Taking $\tau_c = \mu U^2/\nu = \mu U^2/(\mu/\rho) = \rho U^2$, we get

$$\begin{aligned}
\tau_* &= \frac{\partial u_*}{\partial y_*} = \frac{1}{2\sqrt{t_*}}\frac{du_*}{d\eta_*}, && (5.202)\\
&= \frac{1}{2\sqrt{t_*}}\left(-\frac{2}{\sqrt{\pi}}e^{-\eta_*^2}\right), && (5.203)\\
&= -\frac{1}{\sqrt{\pi t_*}}e^{-\eta_*^2}, && (5.204)\\
&= -\frac{1}{\sqrt{\pi t_*}}\exp\left(-\frac{y_*}{2\sqrt{t_*}}\right)^2. && (5.205)
\end{aligned}$$

Now at the wall, $y_* = 0$, and we get

$$\tau_*|_{y_*=0} = -\frac{1}{\sqrt{\pi t_*}}. \tag{5.206}$$

So the shear stress does not have a similarity solution, but is directly related to time variation. The equation holds that the stress is infinite at $t_* = 0$, and decreases as time increases. This is because the velocity gradient flattens as time progresses. It can also be shown that while the stress is unbounded at a single point in time, that the impulse over a finite time span is finite, even when the time span includes $t_* = 0$. It can also be shown that the flow corresponds to a pulse of vorticity being introduced at the wall, which subsequently diffuses into the fluid.

In dimensional terms, we can say

$$\begin{aligned}
\frac{\tau}{\rho U^2} &= -\frac{1}{\sqrt{\pi\frac{U^2 t}{\nu}}}, && (5.207)\\
\tau &= -\frac{\rho U^2}{\sqrt{\pi}\frac{U}{\sqrt{\nu}}\sqrt{t}}, && (5.208)\\
&= -\frac{\rho U\sqrt{\frac{\mu}{\rho}}}{\sqrt{\pi t}}, && (5.209)\\
&= -\frac{U\sqrt{\rho\mu}}{\sqrt{\pi t}}. && (5.210)
\end{aligned}$$

Now let us consider the heat transfer problem. Recall the governing equation, initial and boundary conditions are

$$\begin{aligned}
\rho c\frac{\partial T}{\partial t} &= k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2, && (5.211)\\
T(y,0) &= T_o, \qquad T(0,t) = T_o, \qquad T(\infty,t) = T_o. && (5.212)
\end{aligned}$$

We will adopt the same time t_c an length y_c scales as before. Take the dimensionless temperature to be

$$T_* = \frac{T - T_o}{T_o}. \tag{5.213}$$

So we get

$$\frac{\rho c T_o}{t_c}\frac{\partial T_*}{\partial t_*} = \frac{kT_o}{y_c^2}\frac{\partial^2 T_*}{\partial y_*^2} + \frac{\mu U^2}{y_c^2}\left(\frac{\partial u_*}{\partial y_*}\right)^2, \tag{5.214}$$

$$\frac{\partial T_*}{\partial t_*} = \frac{kT_o}{y_c^2}\frac{t_c}{\rho c T_o}\frac{\partial^2 T_*}{\partial y_*^2} + \frac{\mu U^2}{y_c^2}\frac{t_c}{\rho c T_o}\left(\frac{\partial u_*}{\partial y_*}\right)^2, \tag{5.215}$$

$$\text{now} \quad \frac{k}{y_c^2}\frac{t_c}{\rho c} = \frac{kU^2}{\nu^2}\frac{\nu}{U^2}\frac{1}{\rho c} = \frac{k}{\rho c \nu} = \frac{k}{\mu c} = \frac{1}{Pr}, \tag{5.216}$$

$$\frac{\mu T^2}{y_c^2}\frac{t_c}{\rho c T_o} = \frac{\mu U^2 U^2}{\nu^2}\frac{\nu}{U^2}\frac{1}{\rho c T_o} = \frac{\mu U^2}{\frac{\mu}{\rho}\rho c T_o} = \frac{U^2}{cT_o} = Ec. \tag{5.217}$$

So we have in dimensionless form

$$\frac{\partial T_*}{\partial t_*} = \frac{1}{Pr}\frac{\partial^2 T_*}{\partial y_*^2} + Ec\left(\frac{\partial u_*}{\partial y_*}\right)^2, \tag{5.218}$$

$$T_*(y_*, 0) = 0, \qquad T_*(0, t_*) = 0, \qquad T_*(\infty, t_*) = 0. \tag{5.219}$$

Notice that the only driving inhomogeneity is the viscous work. Now we know from our solution of the linear momentum equation that

$$\frac{\partial u_*}{\partial y*} = -\frac{1}{\sqrt{\pi t_*}}\exp\left(-\frac{y_*^2}{4t_*}\right). \tag{5.220}$$

So we can rewrite the equation for temperature variation as

$$\frac{\partial T_*}{\partial t_*} = \frac{1}{Pr}\frac{\partial^2 T_*}{\partial y_*^2} + \frac{Ec}{\pi t_*}\exp\left(-\frac{y_*^2}{2t_*}\right), \tag{5.221}$$

$$T_*(y_*, 0) = 0, \qquad T_*(0, t_*) = 0, \qquad T_*(\infty, t_*) = 0. \tag{5.222}$$

Before considering the general solution, let us consider some limiting cases.

- $Ec \to 0$

  In the limit as $Ec \to 0$, we get a trivial solution, $T_*(y_*, t_*) = 0$.

- $Pr \to \infty$

  Recalling that the Prandtl number is the ratio of momentum diffusivity to thermal diffusivity, this limit corresponds to materials for which momentum diffusivity is much greater than thermal diffusivity. For example for SAE 30 oil, the Prandtl number is around 3500. Naively assuming that we can simply neglect conduction, we write the energy equation in this limit as

$$\frac{\partial T_*}{\partial t_*} = \frac{Ec}{\pi t_*} \exp\left(-\frac{y_*^2}{2t_*}\right). \tag{5.223}$$

and with $T_* = T_*(\eta_*)$ and $\eta_* = y_*/(2\sqrt{t_*})$, we get the transformed partial time derivative to be

$$\frac{\partial T_*}{\partial t_*} = -\frac{\eta_*}{2t_*}\frac{dT_*}{d\eta_*}. \tag{5.224}$$

So the governing equation reduces to

$$-\frac{\eta_*}{2t_*}\frac{dT_*}{d\eta_*} = \frac{Ec}{\pi t_*} e^{-2\eta_*^2}, \tag{5.225}$$

$$\frac{dT_*}{d\eta_*} = -\frac{2Ec}{\pi}\frac{1}{\eta_*} e^{-2\eta_*^2}, \tag{5.226}$$

$$T_* = \frac{2Ec}{\pi}\int_{\eta_*}^{\infty} \frac{1}{s} e^{-2s^2}\, ds. \tag{5.227}$$

We cannot satisfy both boundary conditions; the equation has been solved so as to satisfy the boundary condition in the far field of $T_*(\infty) = 0$.

Unfortunately, we notice that we cannot satisfy the boundary condition at $\eta_* = 0$. We simply do not have enough degrees of freedom. In actuality, what we have found is an outer solution, and to match the boundary condition at 0, we would have to reintroduce conduction, which has a higher derivative.

First let us see how the outer solution behaves near $\eta_* = 0$. Expanding the differential equation in a Taylor series about $\eta_* = 0$ and solving gives

$$\frac{dT_*}{d\eta_*} = -\frac{2Ec}{\pi}\left(\frac{1}{\eta_*} - 2\eta_* + 2\eta_*^3 + \ldots\right), \tag{5.228}$$

$$T_* = -\frac{2Ec}{\pi}\left(\ln \eta_* - \eta_*^2 + \frac{1}{2}\eta_*^4 + \ldots\right). \tag{5.229}$$

It turns out that solving the inner layer problem and the matching is of about the same difficulty as solving the full general problem, so we will defer this until later in this section.

- $Pr \to 0$

In this limit, we get

$$\frac{\partial^2 T_*}{\partial y_*^2} = 0. \tag{5.230}$$

The solution which satisfies the boundary conditions is

$$T_* = 0. \tag{5.231}$$

In this limit, momentum diffuses slowly relative to energy. So we can interpret the results as follows. In the boundary layer, momentum is generated in a thin layer. Viscous dissipation in this layer gives rise to a local change in temperature in the layer which rapidly diffuses throughout the entire flow. The effect of smearing a localized finite thermal energy input over a semi-infinite domain has a negligible influence on the temperature of the global domain.

So let us bring back diffusion and study solutions for finite Prandtl number. Our governing equation in similarity variables then becomes

$$-\frac{\eta_*}{2t_*}\frac{dT_*}{d\eta_*} = \frac{1}{Pr}\frac{1}{4t_*}\frac{d^2T_*}{d\eta_*^2} + \frac{Ec}{\pi t_*}e^{-2\eta_*^2}, \tag{5.232}$$

$$-2\eta_*\frac{dT_*}{d\eta_*} = \frac{1}{Pr}\frac{d^2T_*}{d\eta_*^2} + \frac{4Ec}{\pi}e^{-2\eta_*^2}, \tag{5.233}$$

$$\frac{d^2T_*}{d\eta_*^2} + 2Pr\ \eta_*\frac{dT_*}{d\eta_*} = -\frac{4}{\pi}EcPr\ e^{-2\eta_*^2}, \tag{5.234}$$

$$T_*(0) = 0, \qquad T_*(\infty) = 0. \tag{5.235}$$

The second order differential equation is really two first order differential equations in disguise. There is an integrating factor of $e^{Pr\ \eta_*^2}$. Multiplying by the integrating factor and operating on the system, we find

$$e^{Pr\ \eta_*^2}\frac{d^2T_*}{d\eta_*^2} + 2Pr\ \eta_* e^{Pr\ \eta_*^2}\frac{dT_*}{d\eta_*} = -\frac{4}{\pi}EcPr\ e^{(Pr-2)\eta_*^2}, \tag{5.236}$$

$$\frac{d}{d\eta_*}\left(e^{Pr\ \eta_*^2}\frac{dT_*}{d\eta_*}\right) = -\frac{4}{\pi}EcPr\ e^{(Pr-2)\eta_*^2}, \tag{5.237}$$

$$e^{Pr\ \eta_*^2}\frac{dT_*}{d\eta_*} = -\frac{4}{\pi}EcPr\int_0^{\eta_*} e^{(Pr-2)s^2}ds + C_1, \tag{5.238}$$

$$\frac{dT_*}{d\eta_*} = -\frac{4}{\pi}EcPr\ e^{-Pr\ \eta_*^2}\int_0^{\eta_*} e^{(Pr-2)s^2}\ ds + C_1 e^{-Pr\eta_*^2}, \tag{5.239}$$

$$T_* = -\frac{4}{\pi}EcPr\int_0^{\eta_*} e^{-Pr\ p^2}\int_0^{p} e^{(Pr-2)s^2}\ ds\ dp + C_1\int_0^{\eta_*} e^{-Pr\ s^2}\ ds + C_2. \tag{5.240}$$

The boundary condition $T_*(0) = 0$ gives us $C_2 = 0$. The boundary condition at ∞ gives us then

$$0 = -\frac{4}{\pi}EcPr\int_0^{\infty} e^{-Pr\ p^2}\int_0^{p} e^{(Pr-2)s^2}\ ds\ dp + C_1\underbrace{\int_0^{\infty} e^{-Pr\ s^2}\ ds}_{\frac{1}{2}\sqrt{\frac{\pi}{Pr}}}. \tag{5.241}$$

(5.242)

Therefore, we get

$$\frac{4}{\pi}EcPr\int_0^\infty e^{-Pr\ p^2}\int_0^p e^{(Pr-2)s^2}ds\ dp = \frac{C_1}{2}\sqrt{\frac{\pi}{Pr}}, \tag{5.243}$$

$$C_1 = \frac{8}{\pi^{3/2}}EcPr^{3/2}\int_0^\infty e^{-Pr\ p^2}\int_0^p e^{(Pr-2)s^2}\ ds\ dp. \tag{5.244}$$

So finally, we have for the temperature profile

$$\begin{aligned} T_*(\eta_*) &= -\frac{4}{\pi}EcPr\int_0^{\eta_*} e^{-Pr\ p^2}\int_0^p e^{(Pr-2)s^2}\ ds\ dp \\ &\quad + \left(\frac{8}{\pi^{3/2}}EcPr^{3/2}\int_0^\infty e^{-Pr\ p^2}\int_0^p e^{(Pr-2)s^2}\ ds\ dp\right)\int_0^{\eta_*} e^{-Pr\ s^2}\ ds. \end{aligned} \tag{5.245}$$

This simplifies somewhat to

$$-\frac{2EcPr}{\sqrt{\pi(2-Pr)}}\left(\int_0^\eta e^{-Pr\ p^2}\text{erf}\left(\sqrt{2-Pr}\ p\right)\ dp - \text{erf}\left(\sqrt{Pr}\ \eta\right)\int_0^\infty e^{-Pr\ p^2}\text{erf}\left(\sqrt{2-Pr}\ p\right)\ dp\right). \tag{5.246}$$

This analysis simplifies considerably in the limit of $Pr = 1$, that is when momentum and energy diffuse at the same rate. This is a close to reality for many gases. In this case, the temperature profile becomes

$$T_*(\eta_*) = -\frac{4}{\pi}Ec\int_0^{\eta_*} e^{-p^2}\int_0^p e^{-s^2}\ ds\ dp + C_1\int_0^{\eta_*} e^{-s^2}\ ds. \tag{5.247}$$

Now if $h(p) = \int_0^p e^{-s^2}\ ds$, we get $dh/dp = e^{-p^2}$. Using this, we can rewrite the temperature profile as

$$\begin{aligned} T_*(\eta_*) &= -\frac{4}{\pi}Ec\int_0^{\eta_*} h(p)\frac{dh}{dp}\ dp + C_1\int_0^{\eta_*} e^{-s^2}\ ds, && (5.248)\\ &= -\frac{4Ec}{\pi}\int_0^{\eta_*} d\left(\frac{h^2}{2}\right) + C_1\int_0^{\eta_*} e^{-s^2}\ ds, && (5.249)\\ &= -\frac{4Ec}{\pi}\left(\frac{1}{2}\right)\left(\int_0^{\eta_*} e^{-s^2}ds\right)^2 + C_1\int_0^{\eta_*} e^{-s^2}\ ds, && (5.250)\\ &= \left(-\frac{2Ec}{\pi}\int_0^{\eta_*} e^{-s^2}ds + C_1\right)\int_0^{\eta_*} e^{-s^2}\ ds. && (5.251) \end{aligned}$$

Now for $T(\infty) = 0$, we get

$$\begin{aligned} 0 &= \left(-\frac{2Ec}{\pi}\int_0^\infty e^{-s^2}ds + C_1\right)\int_0^\infty e^{-s^2}\ ds, && (5.252)\\ 0 &= \left(-\frac{2Ec}{\pi}\frac{\sqrt{\pi}}{2} + C_1\right)\frac{\sqrt{\pi}}{2}, && (5.253)\\ C_1 &= \frac{Ec}{\sqrt{\pi}}. && (5.254) \end{aligned}$$

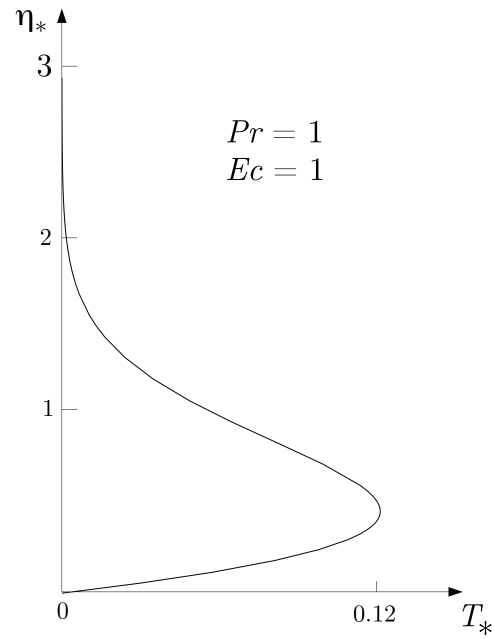

Figure 5.10: Plot of temperature field for Stokes' first problem for $Pr = 1$, $Ec = 1$.

So the temperature profile can be expressed as

$$T_*(\eta_*) = \frac{Ec}{\sqrt{\pi}} \left(\int_0^{\eta_*} e^{-s^2} \, ds \right) \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta_*} e^{-s^2} \, ds \right). \tag{5.255}$$

We notice that we can write this directly in terms of the velocity as

$$T_*(\eta_*) = \frac{Ec}{2} u_*(\eta_*) \left(1 - u_*(\eta_*)\right). \tag{5.256}$$

This is a consequence of what is known as *Reynolds' analogy* which holds for $Pr = 1$ that the temperature field can be directly related to the velocity field. The temperature field for Stokes' first problem for $Pr = 1$, $Ec = 1$ is plotted in Figure 5.10.

5.2.2 Blasius boundary layer

We next consider the well known problem of the flow of a viscous fluid over a flat plate. This problem forms the foundation for a variety of viscous flows over more complicated geometries. It also illustrates some important features of viscous flow physics, as well as giving the original motivating problem for the mathematical technique of matched asymptotic expansions. Here we will consider, as sketched in Figure 5.11, the incompressible flow of viscous fluid of constant viscosity and thermal conductivity over a flat plate. In the far field, the fluid will be a uniform stream with constant velocity. At the plate surface, the no-slip condition must

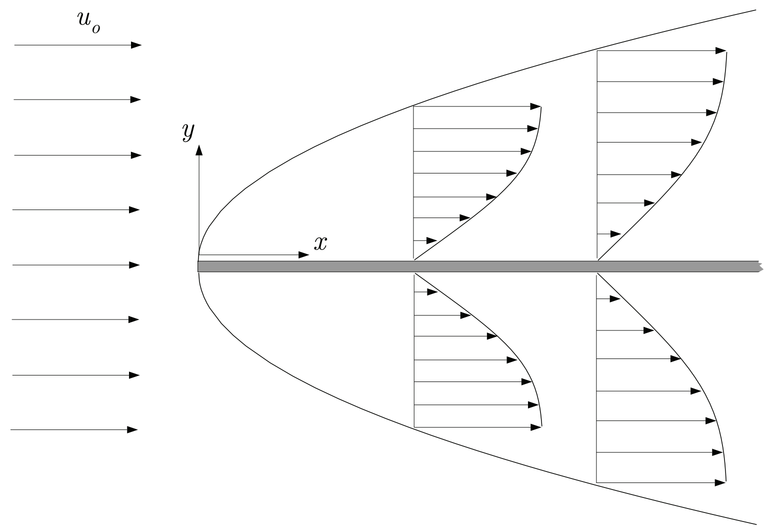

Figure 5.11: Schematic for flat plate boundary layer problem.

be enforced, which will give rise to a zone of adjustment where the fluid's velocity changes from zero at the plate surface to its freestream value. This zone is called the *boundary layer*.

Considering first the velocity field, we find, assuming the flow is steady as well, that the dimensionless two-dimensional Navier-Stokes equations are as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5.257}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{5.258}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{5.259}$$

The dimensionless boundary conditions are

$$u(x, y \to \infty) = 1, \tag{5.260}$$

$$v(x, y \to \infty) = 0, \tag{5.261}$$

$$p(x, y \to \infty) = 0, \tag{5.262}$$

$$u(x, 0) = 0, \tag{5.263}$$

$$v(x, 0) = 0. \tag{5.264}$$

In this section, we are dispensing with the $*$'s and assuming all variables are dimensionless. In fact we have assumed a scaling of the following form, where *dim* is a subscript denoting a dimensional variable.

$$u = \frac{u_{dim}}{u_o}, \quad v = \frac{v_{dim}}{u_o}, \quad x = \frac{x_{dim}}{L}, \quad y = \frac{y_{dim}}{L}, \quad p = \frac{p_{dim} - p_o}{\rho u_o^2}. \tag{5.265}$$

Note for our flat plate of semi-infinite extent, we do not have a natural length scale. This suggests that we may find a similarity solution which removes the effect of L.

Now let us consider that for $Re \to \infty$, we have an outer solution of $u = 1$ to be valid for most of the flow field sufficiently far away from the plate surface. In fact the solution $u = 1$, $v = 0$, $p = 0$, satisfies all of the governing equations and boundary conditions except for the no slip condition at $y = 0$. Because in the limit as $Re \to \infty$, we effectively ignore the high order derivatives found in the viscous terms, we cannot expect to satisfy all boundary conditions for the full problem. We call this the *outer solution*, which is also an inviscid solution to the equations, allowing for a slip condition at the boundary.

Let us *rescale* our equations near the plate surface $y = 0$ to

- bring back the effect of the viscous terms,
- bring back the no-slip condition, and
- match our inviscid outer solution to a viscous inner solution.

This is *the* first example of the use of the method of matched asymptotic expansions as introduced by Prandtl and his student Blasius in the early twentieth century.

With some difficulty, we could show how to choose the scaling, let us simply adopt a scaling and show that it indeed achieves our desired end. So let us take a scaled y distance and velocity, denoted by a ˜ superscript, to be

$$\tilde{v} = \sqrt{Re}\, v, \qquad \tilde{y} = \sqrt{Re}\, y. \tag{5.266}$$

With this scaling, assuming the Reynolds number is large, when we examine small y or v, we are examining an order unity $\tilde{y}$ or $\tilde{v}$. Our equations rescale as

$$\frac{\partial u}{\partial x} + \frac{1/\sqrt{Re}}{1/\sqrt{Re}}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{5.267}$$

$$u\frac{\partial u}{\partial x} + \frac{1/\sqrt{Re}}{1/\sqrt{Re}}\tilde{v}\frac{\partial u}{\partial \tilde{y}} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + Re\frac{\partial^2 u}{\partial \tilde{y}^2}\right), \tag{5.268}$$

$$\frac{1}{\sqrt{Re}}u\frac{\partial \tilde{v}}{\partial x} + \frac{(1/\sqrt{Re})(1/\sqrt{Re})}{1/\sqrt{Re}}\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{1}{1/\sqrt{Re}}\frac{\partial p}{\partial \tilde{y}} + \frac{1}{Re}\left(\frac{1}{\sqrt{Re}}\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{1/\sqrt{Re}}{1/Re}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right). \tag{5.269}$$

Simplifying, this reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{5.270}$$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \tilde{y}^2}, \tag{5.271}$$

$$u\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -Re\frac{\partial p}{\partial \tilde{y}} + \frac{1}{Re}\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}. \tag{5.272}$$

Now in the limit as $Re \to \infty$, the rescaled equations reduce to the well known *boundary layer equations*:

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{5.273}$$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial \tilde{y}^2}, \tag{5.274}$$

$$0 = \frac{\partial p}{\partial \tilde{y}}. \tag{5.275}$$

To match the outer solution, we need the boundary conditions which are

$$u(x, \tilde{y} \to \infty) = 1, \tag{5.276}$$

$$\tilde{v}(x, \tilde{y} \to \infty) = 0, \tag{5.277}$$

$$p(x, \tilde{y} \to \infty) = 0, \tag{5.278}$$

$$u(x, 0) = 0, \tag{5.279}$$

$$\tilde{v}(x, 0) = 0. \tag{5.280}$$

The $\tilde{y}$ momentum equation gives us

$$p = p(x). \tag{5.281}$$

In general, we can consider this to be an imposed pressure gradient which is supplied by the outer inviscid solution. For general flows, that pressure gradient dp/dx will be non-zero. For the Blasius problem, we will choose to study problems for which there is *no pressure gradient*. That is we take

$$p(x) = 0, \qquad \text{for Blasius flat plate boundary layer.} \tag{5.282}$$

So called Falkner-Skan solutions consider flows over curved plates, for which the outer inviscid solution does not have a constant pressure. This ultimately affects the behavior of the fluid in the boundary layer, giving results which differ in important features from our Blasius problem.

With our assumptions, the Blasius problem reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{5.283}$$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \tilde{y}} = \frac{\partial^2 u}{\partial \tilde{y}^2}. \tag{5.284}$$

The boundary conditions are now only on velocity and are

$$u(x, \tilde{y} \to \infty) = 1, \tag{5.285}$$

$$\tilde{v}(x, \tilde{y} \to \infty) = 0, \tag{5.286}$$

$$u(x, 0) = 0, \tag{5.287}$$

$$\tilde{v}(x, 0) = 0. \tag{5.288}$$

Now to simplify, we invoke the stream function ψ, which allows us to satisfy continuity automatically and eliminate u and $\tilde{v}$ at the expense of raising the order of the differential equation. So taking

$$u = \frac{\partial \psi}{\partial \tilde{y}}, \qquad \tilde{v} = -\frac{\partial \psi}{\partial x}, \tag{5.289}$$

we find that mass conservation reduces to $\frac{\partial^2 \psi}{\partial x \partial \tilde{y}} - \frac{\partial^2 \psi}{\partial \tilde{y} \partial x} = 0$. The x momentum equation and associated boundary conditions become

$$\frac{\partial \psi}{\partial \tilde{y}} \frac{\partial^2 \psi}{\partial x \partial \tilde{y}} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial \tilde{y}^2} = \frac{\partial^3 \psi}{\partial \tilde{y}^3}, \tag{5.290}$$

$$\frac{\partial \psi}{\partial \tilde{y}}(x, \tilde{y} \to \infty) = 1, \tag{5.291}$$

$$\frac{\partial \psi}{\partial x}(x, \tilde{y} \to \infty) = 0, \tag{5.292}$$

$$\frac{\partial \psi}{\partial \tilde{y}}(x, 0) = 0, \tag{5.293}$$

$$\frac{\partial \psi}{\partial x}(x, 0) = 0. \tag{5.294}$$

Let us try stretching all the variables of this system to see if there are stretching transformations under which the system exhibits symmetry; that is we seek a stretching transformation under which the system is invariant. Take

$$\hat{x} = e^a x, \qquad \hat{y} = e^b \tilde{y}, \qquad \hat{\psi} = e^c \psi. \tag{5.295}$$

Under this transformation, the x-momentum equation and boundary conditions transform to

$$e^{a+2b-2c} \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} - e^{a+2b-2c} \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} = e^{3b-c} \frac{\partial^3 \hat{\psi}}{\partial \hat{y}^3}, \tag{5.296}$$

$$e^{b-c} \frac{\partial \hat{\psi}}{\partial \hat{y}}(\hat{x}, \hat{y} \to \infty) = 1, \tag{5.297}$$

$$e^{a-c} \frac{\partial \hat{\psi}}{\partial \hat{x}}(\hat{x}, \hat{y} \to \infty) = 0, \tag{5.298}$$

$$e^{b-c} \frac{\partial \hat{\psi}}{\partial \hat{y}}(\hat{x}, 0) = 0, \tag{5.299}$$

$$e^{a-c} \frac{\partial \hat{\psi}}{\partial \hat{x}}(\hat{x}, 0) = 0. \tag{5.300}$$

If we demand $b = c$ and $a = 2c$, then the transformation is invariant, yielding

$$\frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} - \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} = \frac{\partial^3 \hat{\psi}}{\partial \hat{y}^3}, \tag{5.301}$$

$$\frac{\partial \hat{\psi}}{\partial \hat{y}}(\hat{x}, \hat{y} \to \infty) = 1, \tag{5.302}$$

$$\frac{\partial \hat{\psi}}{\partial \hat{x}}(\hat{x}, \hat{y} \to \infty) = 0, \tag{5.303}$$

$$\frac{\partial \hat{\psi}}{\partial \hat{y}}(\hat{x}, 0) = 0, \tag{5.304}$$

$$\frac{\partial \hat{\psi}}{\partial \hat{x}}(\hat{x}, 0) = 0. \tag{5.305}$$

Now our transformation is reduced to

$$\hat{x} = e^{2c}x, \qquad \hat{y} = e^{c}\tilde{y}, \qquad \hat{\psi} = e^{c}\psi. \tag{5.306}$$

Since c does not appear explicitly in either the original equation set nor the transformed equation set, the solution must not depend on this stretching. Eliminating c from the transformation by $e^c = \sqrt{\hat{x}/x}$ we find that

$$\frac{\hat{y}}{\tilde{y}} = \sqrt{\frac{\hat{x}}{x}}, \qquad \frac{\hat{\psi}}{\psi} = \sqrt{\frac{\hat{x}}{x}}, \tag{5.307}$$

or

$$\frac{\hat{y}}{\sqrt{\hat{x}}} = \frac{\tilde{y}}{\sqrt{x}}, \qquad \frac{\hat{\psi}}{\sqrt{\hat{x}}} = \frac{\psi}{\sqrt{x}}. \tag{5.308}$$

Thus motivated, let us seek solutions of the form

$$\frac{\psi}{\sqrt{x}} = f\left(\frac{\tilde{y}}{\sqrt{x}}\right). \tag{5.309}$$

That is taking

$$\eta = \frac{\tilde{y}}{\sqrt{x}}, \tag{5.310}$$

we seek

$$\psi = \sqrt{x} f(\eta). \tag{5.311}$$

Let us check that our similarity variable is independent of L our unknown length scale.

$$\eta = \frac{\tilde{y}}{\sqrt{x}} = \frac{\sqrt{Re}\ y}{\sqrt{x}} = \frac{\sqrt{Re}\ y_{dim}/L}{\sqrt{x_{dim}/L}} = \sqrt{\frac{u_o L}{\nu}} \frac{y_{dim}}{L} \frac{\sqrt{L}}{\sqrt{x_{dim}}} = \sqrt{\frac{u_o}{\nu}} \frac{y_{dim}}{\sqrt{x_{dim}}}. \tag{5.312}$$

So indeed, our similarity variable is independent of any arbitrary length scale we happen to have chosen.

With our similarity transformation, we have

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2}\tilde{y}x^{-3/2} = -\frac{1}{2}\frac{\eta}{x}, \tag{5.313}$$

$$\frac{\partial \eta}{\partial \tilde{y}} = \frac{1}{\sqrt{x}}. \tag{5.314}$$

Now we need expressions for $\partial\psi/\partial x$, $\partial\psi/\partial\tilde{y}$, $\partial^2\psi/\partial x\partial\tilde{y}$, $\partial^2\psi/\partial\tilde{y}^2$, and $\partial^3\psi/\partial\tilde{y}^3$. First, consider the partial derivatives of the stream function ψ. Operating on each partial derivative, we find

$$\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x}f(\eta)\right) = \sqrt{x}\frac{df}{d\eta}\frac{\partial\eta}{\partial x} + \frac{1}{2}\frac{1}{\sqrt{x}}f, \tag{5.315}$$

$$= \sqrt{x}\left(-\frac{1}{2}\right)\frac{\eta}{x}\frac{df}{d\eta} + \frac{1}{2}\frac{1}{\sqrt{x}}f = \frac{1}{2\sqrt{x}}\left(f - \eta\frac{df}{d\eta}\right), \tag{5.316}$$

$$\text{so} \qquad \tilde{v} = -\frac{1}{2\sqrt{x}}\left(f - \eta\frac{df}{d\eta}\right). \tag{5.317}$$

$$\frac{\partial\psi}{\partial\tilde{y}} = \frac{\partial}{\partial\tilde{y}}\left(\sqrt{x}f(\eta)\right) = \sqrt{x}\frac{\partial}{\partial\tilde{y}}(f(\eta)) = \sqrt{x}\frac{df}{d\eta}\frac{\partial\eta}{\partial\tilde{y}} = \sqrt{x}\frac{df}{d\eta}\frac{1}{\sqrt{x}} = \frac{df}{d\eta}, \tag{5.318}$$

$$\text{so} \qquad u = \frac{\partial\psi}{\partial\tilde{y}} = \frac{df}{d\eta}. \tag{5.319}$$

$$\frac{\partial^2\psi}{\partial x\partial\tilde{y}} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial\tilde{y}}\right) = \frac{\partial}{\partial x}\left(\frac{df}{d\eta}\right) = \frac{d^2f}{d\eta^2}\frac{\partial\eta}{\partial x} = -\frac{1}{2x}\eta\frac{d^2f}{d\eta^2}. \tag{5.320}$$

$$\frac{\partial^2\psi}{\partial\tilde{y}^2} = \frac{\partial}{\partial\tilde{y}}\left(\frac{\partial\psi}{\partial\tilde{y}}\right) = \frac{\partial}{\partial\tilde{y}}\left(\frac{df}{d\eta}\right) = \frac{d^2f}{d\eta^2}\frac{\partial\eta}{\partial\tilde{y}} = \frac{1}{\sqrt{x}}\frac{d^2f}{d\eta^2}. \tag{5.321}$$

$$\frac{\partial^3\psi}{\partial\tilde{y}^3} = \frac{\partial}{\partial\tilde{y}}\left(\frac{\partial^2\psi}{\partial\tilde{y}^2}\right) = \frac{\partial}{\partial\tilde{y}}\left(\frac{1}{\sqrt{x}}\frac{d^2f}{d\eta^2}\right) = \frac{1}{\sqrt{x}}\frac{\partial}{\partial\tilde{y}}\left(\frac{d^2f}{d\eta^2}\right) = \frac{1}{\sqrt{x}}\frac{d^3f}{d\eta^3}\frac{\partial\eta}{\partial\tilde{y}}, \tag{5.322}$$

$$= \frac{1}{x}\frac{d^3f}{d\eta^3}. \tag{5.323}$$

Now we substitute each of these expressions into the x momentum equation and get

$$\frac{df}{d\eta}\left(-\frac{1}{2x}\eta\frac{d^2f}{d\eta^2}\right) - \frac{1}{2\sqrt{x}}\left(f - \eta\frac{df}{d\eta}\right)\frac{1}{\sqrt{x}}\frac{d^2f}{d\eta^2} = \frac{1}{x}\frac{d^3f}{d\eta^3}, \tag{5.324}$$

$$-\eta\frac{df}{d\eta}\frac{d^2f}{d\eta^2} - \left(f - \eta\frac{df}{d\eta}\right)\frac{d^2f}{d\eta^2} = 2\frac{d^3f}{d\eta^3}, \tag{5.325}$$

$$-f\frac{d^2f}{d\eta^2} = 2\frac{d^3f}{d\eta^3}, \tag{5.326}$$

$$\frac{d^3f}{d\eta^3} + \frac{1}{2}f\frac{d^2f}{d\eta^2} = 0. \tag{5.327}$$

This is a third order non-linear ordinary differential equation for $f(\eta)$. We need three boundary conditions. Now at the surface $\tilde{y} = 0$, we have $\eta = 0$. And as $\tilde{y} \to \infty$, we have $\eta \to \infty$. To satisfy the no-slip condition on u at the plate surface, we require

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = 0. \tag{5.328}$$

For no-slip on $\tilde{v}$, we require

$$\tilde{v}(0) = 0 = -\frac{1}{2\sqrt{x}}\left(f - \eta\frac{df}{d\eta}\right), \tag{5.329}$$

$$0 = f(0) - 0 \underbrace{\left.\frac{df}{d\eta}\right|_{\eta=0}}_{=0}, \tag{5.330}$$

$$f(0) = 0. \tag{5.331}$$

And to satisfy the freestream condition on u as $\eta \to \infty$, we need

$$\left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 1. \tag{5.332}$$

The most standard way to solve non-linear ordinary differential equations of this type is to reduce them to systems of first order ordinary differential equations and use some numerical technique, such as a Runge[7]-Kutta integration. We recall that Runge-Kutta techniques, as well as most other common techniques, require a well-defined set of initial conditions to predict the final state. To achieve the desired form, we define

$$g \equiv \frac{df}{d\eta}, \qquad h \equiv \frac{d^2 f}{d\eta^2}. \tag{5.333}$$

Thus the x momentum equation becomes

$$\frac{dh}{d\eta} + \frac{1}{2} f h = 0. \tag{5.334}$$

But this is one equation in three unknowns. We need to write our equations as a system of three first order equations, along with associated initial conditions. They are

$$\frac{df}{d\eta} = g, \qquad f(0) = 0, \tag{5.335}$$

$$\frac{dg}{d\eta} = h, \qquad g(0) = 0, \tag{5.336}$$

$$\frac{dh}{d\eta} = -\frac{1}{2} f h, \qquad h(0) = ?. \tag{5.337}$$

Everything is well-defined except we do not have an initial condition on h. We do however have a far-field condition on g which is $g(\infty) = 1$. One viable option we have for getting a final solution is to use a numerical trial and error procedure, guessing $h(0)$ until we find that $g(\infty) \to 1$. We will use a slightly more efficient method here, which only requires one guess.

To do this, let us first demonstrate the following lemma: If $F(\eta)$ is a solution to $d^3 f/d\eta^3 + \frac{1}{2} f(d^2 f/d\eta^2) = 0$, then $aF(a\eta)$ is also a solution. The proof is as follows. Take $w(\eta) =$

[7] Carl David Tolmè Runge, 1856-1927, German mathematician and physicist, close friend of Max Planck, studied spectral line elements of non-Hydrogen molecules, held chairs at Hanover and Göttingen, entertained grandchildren at age 70 by doing handstands.

$aF(a\eta)$. Then we have

$$w = aF(a\eta), \tag{5.338}$$

$$\frac{dw}{d\eta} = a^2\frac{dF(a\eta)}{d\eta}, \tag{5.339}$$

$$\frac{d^2w}{d\eta^2} = a^3\frac{d^2F(a\eta)}{d\eta^2}, \tag{5.340}$$

$$\frac{d^3w}{d\eta^3} = a^4\frac{d^3F(a\eta)}{d\eta^3}. \tag{5.341}$$

Substituting these expressions into the x momentum equation, we find

$$a^4\frac{d^3F(a\eta)}{d\eta^3} + \frac{1}{2}a^4F(a\eta)\frac{d^2F(a\eta)}{d\eta^2} = 0, \tag{5.342}$$

$$\frac{d^3F(a\eta)}{d\eta^3} + \frac{1}{2}F(a\eta)\frac{d^2F(a\eta)}{d\eta^2} = 0. \tag{5.343}$$

But we know this to be true as $F(a\eta)$ is a solution. Hence $aF(a\eta)$ is also a solution.

So to solve our non-linear system, let us first solve the following related system:

$$\frac{dF}{d\eta} = G, \qquad F(0) = 0, \tag{5.344}$$

$$\frac{dG}{d\eta} = H, \qquad G(0) = 0, \tag{5.345}$$

$$\frac{dH}{d\eta} = -\frac{1}{2}FH, \qquad H(0) = 1. \tag{5.346}$$

After one numerical integration, we find that with this guess for $H(0)$ that

$$G(\infty) = 2.08540918... \tag{5.347}$$

Now our numerical solution also gives us F, and so we know that $f = aF(a\eta)$ is also a solution. Moreover

$$\frac{df}{d\eta} = a^2\frac{dF(a\eta)}{d\eta}, \qquad \text{that is} \tag{5.348}$$

$$g(\eta) = a^2G(a\eta). \tag{5.349}$$

Now we want $g(\infty) = 1$, so take $1 = a^2G(\infty)$, so $a^2 = 1/G(\infty)$. So

$$a = \frac{1}{\sqrt{G(\infty)}}. \tag{5.350}$$

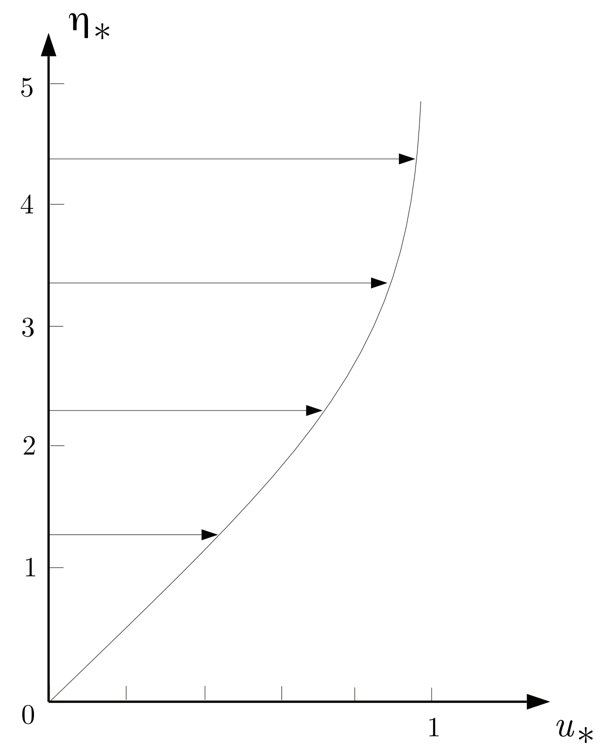

Figure 5.12: Velocity profile for Blasius boundary layer.

Now

$$\frac{d^2 f}{d\eta^2} = a^3 \frac{d^2 F(a\eta)}{d\eta^2}, \tag{5.351}$$

$$\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} = a^3 \left.\frac{d^2 F(a\eta)}{d\eta^2}\right|_{\eta=0}, \tag{5.352}$$

$$h(0) = a^3 H(0), \tag{5.353}$$

$$h(0) = a^3(1), \tag{5.354}$$

$$h(0) = a^3 = G^{-3/2}(\infty), \tag{5.355}$$

$$h(0) = (2.08540918...)^{-3/2} = 0.332057335... \tag{5.356}$$

This is the proper choice for the initial condition on h. Numerically integrating once more, we get the behavior of f, g, and h as functions of η which indeed satisfies the condition at ∞. A plot of $u = df/d\eta$ as a function of η is shown in Figure 5.12. From this plot, we see that when $\eta = 5$, the velocity has nearly acquired the freestream value of $u = 1$. In fact, examination of the numerical results shows that when $\eta = 4.9$, that the u component of velocity has 0.99 of its freestream value. As the velocity only reaches its freestream value at ∞, we define the *boundary layer thickness*, $\delta_{0.99}$, as that value of y_{dim} for which the velocity has 0.99 of its freestream value. Recalling that

$$\eta = \sqrt{\frac{u_o}{\nu}} \frac{y_{dim}}{\sqrt{x_{dim}}}, \tag{5.357}$$

we say that

$$4.9 = \sqrt{\frac{u_o}{\nu}} \frac{\delta_{0.99}}{\sqrt{x_{dim}}}. \tag{5.358}$$

Rearranging, we get

$$\frac{\delta_{0.99}}{x_{dim}} = 4.9\sqrt{\frac{\nu}{u_o x_{dim}}}, \tag{5.359}$$

$$= 4.9 Re_{x_{dim}}^{-1/2}. \tag{5.360}$$

Here we have taken a Reynolds number based on local distance to be

$$Re_{x_{dim}} = \frac{u_o x_{dim}}{\nu}. \tag{5.361}$$

This formula is valid for laminar flows, and has been seen to be valid for $Re_{x_{dim}} < 3 \times 10^6$. For greater lengths, there can be a transition to turbulent flow. For water flowing a 1 m/s and a downstream distance of 1 m, we find $\delta_{0.99} = 0.5\ cm$. For air under the same conditions, we find $\delta_{0.99} = 1.9\ cm$. We also note that the boundary layer grows with the square root of distance along the plate. We further note that higher kinematic viscosity leads to thicker boundary layers, while lower kinematic viscosity lead to thinner boundary layers.

Now let us determine the shear stress at the wall, and the viscous force acting on the wall. So let us find

$$\tau_w = \mu \left. \frac{\partial u_{dim}}{\partial y_{dim}} \right|_{y_{dim}=0}. \tag{5.362}$$

Consider

$$\frac{\partial u}{\partial \tilde{y}} = \frac{\partial^2 \psi}{\partial \tilde{y}^2} = \frac{1}{\sqrt{x}} \frac{d^2 f}{d\eta^2}, \tag{5.363}$$

$$\frac{\partial \left(\frac{u_{dim}}{u_o} \right)}{\partial \left(\sqrt{\frac{u_o L}{\nu}} \frac{y_{dim}}{L} \right)} = \frac{1}{\sqrt{\frac{x_{dim}}{L}}} \frac{d^2 f}{d\eta^2}, \tag{5.364}$$

$$\frac{\partial u_{dim}}{\partial y_{dim}} = u_o \sqrt{\frac{\rho u_o}{\mu}} \frac{1}{\sqrt{x_{dim}}} \frac{d^2 f}{d\eta^2}, \tag{5.365}$$

$$\tau = \mu \frac{\partial u_{dim}}{\partial y_{dim}} = u_o \sqrt{\frac{\rho u_o \mu}{x_{dim}}} \frac{d^2 f}{d\eta^2}, \tag{5.366}$$

$$\frac{\tau(0)}{\frac{1}{2}\rho u_o^2} = C_f = 2\sqrt{\frac{\mu}{\rho u_o x_{dim}}} \frac{d^2 f}{d\eta^2}(0), \tag{5.367}$$

$$C_f = 2(Re_{x_{dim}})^{-1/2}\frac{d^2 f}{d\eta^2}(0), \tag{5.368}$$

$$C_f = \frac{0.664...}{\sqrt{Re_{x_{dim}}}}. \tag{5.369}$$

We notice that at $x_{dim} = 0$ that the stress is infinite. This seeming problem is seen not to be one when we consider the actual viscous force on a finite length of plate. Consider a plate of length L and width b. Then the viscous force acting on the plate is

$$F = \int_0^L \tau \; dA, \tag{5.370}$$

$$= \int_0^L \tau(x_{dim}, 0) b \; dx_{dim}, \tag{5.371}$$

$$= b\int_0^L f''(0) u_o \sqrt{\rho u_o \mu} \frac{1}{\sqrt{x_{dim}}} \; dx_{dim}, \tag{5.372}$$

$$= b f''(0) u_o \sqrt{\rho u_o \mu} \int_0^L \frac{dx_{dim}}{\sqrt{x_{dim}}}, \tag{5.373}$$

$$= b f''(0) u_o \sqrt{\rho u_o \mu} \left(2\sqrt{x_{dim}}\right)_0^L, \tag{5.374}$$

$$= 2b f''(0) u_o \sqrt{\rho u_o \mu}\sqrt{L}, \tag{5.375}$$

$$\frac{F}{\frac{1}{2}\rho u_o^2 L b} = C_D = 4f''(0)\sqrt{\frac{\mu}{\rho u_o L}} = 4f''(0) Re_L^{-1/2} = 1.328 Re_L^{-1/2}. \tag{5.376}$$

Now let us consider the thermal boundary layer. Here we will take the boundary conditions so that the wall and far field are held at a constant fixed temperature $T_{dim} = T_o$. We need to do the scaling on the energy equation, so let us start with the steady incompressible two-dimensional dimensional energy equation:

$$\rho c_p \left(u_{dim}\frac{\partial T_{dim}}{\partial x_{dim}} + v_{dim}\frac{\partial T_{dim}}{\partial y_{dim}} \right) = k\left(\frac{\partial^2 T_{dim}}{\partial x_{dim}^2} + \frac{\partial^2 T_{dim}}{\partial y_{dim}^2} \right) + \mu\left(2\left(\frac{\partial u_{dim}}{\partial x_{dim}}\right)^2 + 2\left(\frac{\partial v_{dim}}{\partial y_{dim}}\right)^2 + \left(\frac{\partial u_{dim}}{\partial y_{dim}} + \frac{\partial v_{dim}}{\partial x_{dim}}\right)^2 \right). \tag{5.377}$$

Taking as before,

$$x = \frac{x_{dim}}{L}, \quad y = \frac{y_{dim}}{L}, \quad T = \frac{T_{dim} - T_o}{T_o}, \quad u = \frac{u_{dim}}{u_o}, \quad v = \frac{v_{dim}}{u_o}. \tag{5.378}$$

Making these substitutions, we get

$$\frac{\rho c_p u_o T_o}{L}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = \frac{k T_o}{L^2}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{5.379}$$

$$+\frac{\mu u_o^2}{L^2}\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right),$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p u_o L}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \qquad (5.380)$$

$$+\frac{\mu u_o}{\rho c_p L T_o}\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right). \qquad (5.381)$$

Now we have

$$\frac{k}{\rho c_p u_o L} = \frac{k}{c_p\mu}\frac{\mu}{\rho u_o L} = \frac{1}{Pr}\frac{1}{Re}, \qquad (5.382)$$

$$\frac{\mu u_o}{\rho c_p L T_o} = \frac{\mu}{\rho u_o L}\frac{u_o^2}{c_p T_o} = \frac{Ec}{Re}. \qquad (5.383)$$

So the dimensionless energy equation with boundary conditions can be written as

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \frac{1}{Pr\,Re}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \qquad (5.384)$$

$$+\frac{Ec}{Re}\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right),$$

$$T(x,0) = 0, \qquad T(x,\infty)=0. \qquad (5.385)$$

Now as $Re \to \infty$, we see that $T = 0$ is a solution that satisfies the energy equation and all boundary conditions. For finite Reynolds number, non-zero velocity gradients generate a temperature field. Once again, we rescale in the boundary layer using $\tilde{v} = \sqrt{Re}\ v$, and $\tilde{y} = \sqrt{Re}\ y$. This gives

$$u\frac{\partial T}{\partial x}+\frac{1}{\sqrt{Re}}\frac{1}{1/\sqrt{Re}}\tilde{v}\frac{\partial T}{\partial \tilde{y}} = \frac{1}{Pr\,Re}\left(\frac{\partial^2 T}{\partial x^2}+Re\frac{\partial^2 T}{\partial \tilde{y}^2}\right) \qquad (5.386)$$

$$+\frac{Ec}{Re}\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial \tilde{v}}{\partial \tilde{y}}\right)^2+\left(\sqrt{Re}\frac{\partial u}{\partial \tilde{y}}+\frac{1}{\sqrt{Re}}\frac{\partial \tilde{v}}{\partial x}\right)^2\right).$$

$$u\frac{\partial T}{\partial x}+\tilde{v}\frac{\partial T}{\partial \tilde{y}} = \frac{1}{Pr}\left(\frac{1}{Re}\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial \tilde{y}^2}\right) \qquad (5.387)$$

$$+Ec\left(\frac{2}{Re}\left(\frac{\partial u}{\partial x}\right)^2+\frac{2}{Re}\left(\frac{\partial \tilde{v}}{\partial \tilde{y}}\right)^2+\left(\frac{\partial u}{\partial \tilde{y}}+\frac{1}{Re}\frac{\partial \tilde{v}}{\partial x}\right)^2\right).$$

$$\text{as } Re \to \infty, \qquad (5.388)$$

$$u\frac{\partial T}{\partial x}+\tilde{v}\frac{\partial T}{\partial \tilde{y}} = \frac{1}{Pr}\frac{\partial^2 T}{\partial \tilde{y}^2}+Ec\left(\frac{\partial u}{\partial \tilde{y}}\right)^2. \qquad (5.389)$$

Now take $T = T(\eta)$ with $\eta = \tilde{y}/\sqrt{x}$ as well as $u = df/d\eta$, $\tilde{v} = -(1/(2\sqrt{x}))\,(f - \eta(df/d\eta))$ and $\partial u/\partial\tilde{y} = (1/\sqrt{x})(d^2f/d\eta^2)$. We also have for derivatives, that

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{dT}{d\eta}\left(-\frac{1}{2}\frac{\eta}{x}\right), \tag{5.390}$$

$$\frac{\partial T}{\partial \tilde{y}} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial \tilde{y}} = \frac{dT}{d\eta}\frac{1}{\sqrt{x}}, \tag{5.391}$$

$$\frac{\partial^2 T}{\partial \tilde{y}^2} = \frac{\partial}{\partial \tilde{y}}\left(\frac{\partial T}{\partial \tilde{y}}\right) = \frac{\partial}{\partial \tilde{y}}\left(\frac{1}{\sqrt{x}}\frac{dT}{d\eta}\right) = \frac{1}{\sqrt{x}}\frac{\partial}{\partial \tilde{y}}\frac{dT}{d\eta} = \frac{1}{x}\frac{d^2T}{d\eta^2}. \tag{5.392}$$

The energy equation is then rendered as

$$\frac{df}{d\eta}\left(-\frac{1}{2}\frac{\eta}{x}\frac{dT}{d\eta}\right) + \left(-\frac{1}{2\sqrt{x}}\left(f - \eta\frac{df}{d\eta}\right)\right)\frac{1}{\sqrt{x}}\frac{dT}{d\eta} = \frac{1}{Pr}\frac{1}{x}\frac{d^2T}{d\eta^2} + \frac{Ec}{x}\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{5.393}$$

$$-\frac{1}{2}\eta\frac{df}{d\eta}\frac{dT}{d\eta} - \frac{1}{2}\left(f - \eta\frac{df}{d\eta}\right)\frac{dT}{d\eta} = \frac{1}{Pr}\frac{d^2T}{d\eta^2} + Ec\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{5.394}$$

$$-\frac{1}{2}f\frac{dT}{d\eta} = \frac{1}{Pr}\frac{d^2T}{d\eta^2} + Ec\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{5.395}$$

$$\frac{d^2T}{d\eta^2} + \frac{1}{2}Pr\,f\frac{dT}{d\eta} = -Pr\,Ec\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{5.396}$$

$$T(0) = 0, \qquad T(\infty) = 0. \tag{5.397}$$

Now for $Ec \to 0$, we get $T = 0$ as a solution which satisfies the governing differential equation and boundary conditions. Let us consider a solution for non-trivial Ec, but for $Pr = 1$. We could extend this for general values of Pr as well. Here, following Reynolds analogy, when thermal diffusivity equals momentum diffusivity, we expect the temperature field to be directly related to the velocity field. For $Pr = 1$, the energy equation reduces to

$$\frac{d^2T}{d\eta^2} + \frac{1}{2}f\frac{dT}{d\eta} = -Ec\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{5.398}$$

$$T(0) = 0, \qquad T(\infty) = 0. \tag{5.399}$$

Here the integrating factor is

$$e^{\int_0^\eta \frac{1}{2}f(t)\;dt}. \tag{5.400}$$

Multiplying the energy equation by the integrating factor gives

$$e^{\int_0^\eta \frac{1}{2}f(t)\;dt}\frac{d^2T}{d\eta^2} + \frac{1}{2}f e^{\int_0^\eta \frac{1}{2}f(t)\;dt}\frac{dT}{d\eta} = -Ec\;e^{\int_0^\eta \frac{1}{2}f(t)\;dt}\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{5.401}$$

$$\frac{d}{d\eta}\left(e^{\int_0^\eta \frac{1}{2}f(t)\;dt}\frac{dT}{d\eta}\right) = -Ec\;e^{\int_0^\eta \frac{1}{2}f(t)\;dt}\left(\frac{d^2f}{d\eta^2}\right)^2. \tag{5.402}$$

Now from the x momentum equation, $f''' + \frac{1}{2}ff'' = 0$, we have

$$f = -2\frac{f'''}{f''}. \tag{5.403}$$

So we can rewrite the integrating factor as

$$e^{\int_0^\eta \frac{1}{2}f(t)\ dt} = e^{\int_0^\eta \frac{1}{2}\frac{(-2)f'''}{f''}\ dt} = e^{-\ln\left(\frac{f''(\eta)}{f''(0)}\right)} = \frac{f''(0)}{f''(\eta)}. \tag{5.404}$$

So the energy equation can be written as

$$\frac{d}{d\eta}\left(\frac{f''(0)}{f''(\eta)}\frac{dT}{d\eta}\right) = -Ec\left(\frac{f''(0)}{f''(\eta)}\right)\left(\frac{d^2f}{d\eta^2}\right)^2, \tag{5.405}$$

$$= -Ecf''(0)\frac{d^2f}{d\eta^2}, \tag{5.406}$$

$$\frac{f''(0)}{f''(\eta)}\frac{dT}{d\eta} = -Ecf''(0)\int_0^\eta \frac{d^2f}{ds^2}\ ds + C_1, \tag{5.407}$$

$$\frac{dT}{d\eta} = -Ec\frac{d^2f}{d\eta^2}\int_0^\eta \frac{d^2f}{ds^2}\ ds + C_1\frac{d^2f}{d\eta^2}, \tag{5.408}$$

$$= = -Ec\frac{d^2f}{d\eta^2}\left(\frac{df}{d\eta} - \underbrace{f'(0)}_{=0}\right) + C_1\frac{d^2f}{d\eta^2}, \tag{5.409}$$

$$= -Ec\frac{d^2f}{d\eta^2}\frac{df}{d\eta} + C_1\frac{d^2f}{d\eta^2}, \tag{5.410}$$

$$= -Ec\frac{d}{d\eta}\left(\frac{1}{2}\left(\frac{df}{d\eta}\right)^2\right) + C_1\frac{d^2f}{d\eta^2}, \tag{5.411}$$

$$T = -\frac{Ec}{2}\left(\frac{df}{d\eta}\right)^2 + C_1\frac{df}{d\eta} + C_2, \tag{5.412}$$

$$T(0) = 0 = -\frac{Ec}{2}(\underbrace{f'(0)}_{=0})^2 + C_1\underbrace{f'(0)}_{=0} + C_2, \tag{5.413}$$

$$C_2 = 0, \tag{5.414}$$

$$T(\infty) = 0 = -\frac{Ec}{2}(\underbrace{f'(\infty)}_{=1})^2 + C_1\underbrace{f'(\infty)}_{=1}, \tag{5.415}$$

$$C_1 = \frac{Ec}{2}, \tag{5.416}$$

$$T(\eta) = \frac{Ec}{2}\frac{df}{d\eta}\left(1 - \frac{df}{d\eta}\right), \tag{5.417}$$

$$T(\eta) = \frac{Ec}{2}u(\eta)(1 - u(\eta)). \tag{5.418}$$

A plot of the temperature profile for $Pr = 1$ and $Ec = 1$ is given in Figure 5.13.

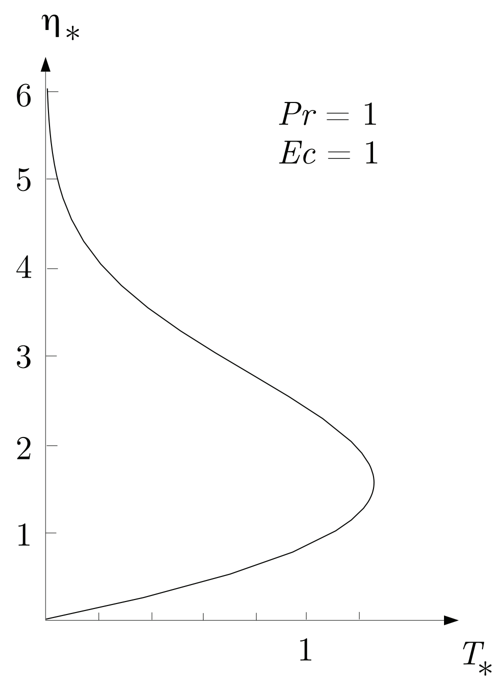

Figure 5.13: Temperature profile for Blasius boundary layer, $Ec = 1$, $Pr = 1$.

Bibliography

This bibliography focuses on books which are closely related to the material presented in this course in classical fluid mechanics, especially with regards to graduate level treatment of continuum mechanical principles applied to fluids, compressible flow, viscous flow, and vortex dynamics. It also has some general works of historic importance. It is by no means a comprehensive survey of works on fluid mechanics. Only a few works are given here which focus on such important topics as low Reynolds number flows, turbulence, bio-fluids, computational fluid dynamics, microfluids, molecular dynamics, magneto-hydrodynamics, geo-physical flows, rheology, astrophysical flows, as well as elementary undergraduate texts. That said, those which are listed are among the best that exist and would be useful to examine.

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