

TUTORIAL (5): INTRODUCTION TO STRAIN AND STRESS

Aims and Objectives

1. To give you an understanding of scalar, vector and tensor quantities, and also of vector calculus.
2. To introduce you to strain as preparation for insar crustal strain practical.
3. To introduce you to stress as preparation for lectures on faulting

You are required to actually solve any equations: this should allow you to concentrate on the relationship between mathematical structure and real physical quantities and processes. Although the practical is rather technical, the examples uses are drawn from geological systems. The tools developed here are essential for a range of problems in geodynamics.

Please have patience if it seems a bit easy and obvious to start with. On the other hand, you will be doing very well if you grasp all the details in one sitting. It should take about 3 hours.

1 Cartesian coordinates

You already know what Cartesian coordinates are; this is just a check on basic concepts notation. In Cartesian coordinates, we first define an origin at a fixed point in space, and then the directions of three mutually perpendicular coordinate axes. Here we will number these directions (1, 2, 3) instead of say, x, y, z. This turns out to be more convenient. It is handy to define it unit vectors corresponding to the coordinates directions:

$$\begin{aligned}\hat{e}_1 &= (1, 0, 0) \\ \hat{e}_2 &= (0, 1, 0) \\ \hat{e}_3 &= (0, 0, 1)\end{aligned}\tag{1, 2, 3}$$

The position of an object in this coordinate system may be represented by a vector joining the origin to the object. This is commonly represented using various favours of vector notation, such as \mathbf{x} , \underline{x} etc. However, when it comes to actually putting in numbers, we always deal with individual *components* of vectors, resolved into the coordinate directions:

$$\underline{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3\tag{4}$$

This is a pretty simple vector equation but is nonetheless a bit cumbersome. It is tidier to represent the vector simply as

$$(x_1, x_2, x_3)$$

or concisely but unambiguously, as

$$x_i$$

where the suffix i takes the values 1, 2 or 3, corresponding to the coordinate directions. We can use this compact *suffix notation* because we have already agreed what coordinate system to use.

EXERCISE 1: make a sketch showing Cartesian coordinate axes, unit vectors and an arbitrary position vector \underline{x} . Show projections of \underline{x} onto the axes and the components x_i .

2 Scalars, vectors and tensors

How do we represent physical quantities with the space we have now defined? This depends on the quantity. We will now develop a taxonomy based on two criteria:

- (i) Is the quantity constant, or a spatial variable, and,
- (ii) Is the quantity a scalar, vector or tensor? (later we will simply ask what the *tensor rank* of the quantity is).

2.1 Scalar constants

These are the simplest quantities. There are of course absolute constants like the speed of light c . In the humble world of geodynamics we are more likely to encounter quantities that could in principle vary, but which are effectively constant in a given scenario. For example, in heat flow problems we often regard the thermal conductivity K as a constant.

2.2 Scalar variables

These are quantities, such as temperature T , which can be represented by a single number at each point in space. To indicate that temperature is a scalar function of position (which is a vector), we use the notation $T(\underline{x})$ or $T(x_i)$. The difference between scalar constants and variables is most obviously when we start taking spatial derivatives, which are always zero for constants.

2.3 Vector constants

These are unusual, but an important example is acceleration due to gravity. (As discussed above, gravity is not an absolute constant, but is effectively constant in certain scenarios.) The magnitude of gravity is often indicated by the scalar g , but gravitational force of course acts in a particular direction so we should use a vector \underline{g} or g_i . If the 3-direction is chosen to be downwards, we can write:

$$\underline{g} = g \hat{e}_3 \tag{7}$$

or

$$g_i = (0, 0, g) \tag{8}$$

Note that the presence or absence of the suffix i is starting to take on some real meaning: scalars have no suffix, whereas vectors have a single suffix.

2.4 Vector variables

We have already been using one of these: the position vector \underline{x} or x_i . This is rather a special case because position is what other quantities vary with respect to (think about it). A more representative example is the displacement field for a deformed continuum, \underline{u} or u_i , which you will see in the section on strain. To indicate that displacement is a vector function of position we use the notation $\underline{u}(\underline{x})$ or $u_i(\underline{x})$. (We deliberately avoid the notation $u_i(x_i)$, for reasons that should become clear by the end of the tutorial).

2.5 Tensor constants

Vector quantities are, geometrically, one step more complicated than scalars. Consequently, the mathematical representation of vectors is more complex than that of scalars. In a similar way, some physical quantities are a further geometrical step more complicated than vector quantities, and are called *tensor quantities*. The mathematical representation of tensor quantities is also a step more complex.

We will start with *tensor constants* (with the usual caveat about non-absolute constants). A nice example is permeability, a property of rocks in aquifers or reservoirs through which water or oil can flow. Permeability relates two vector variables: the flow velocity of the fluid w_i and the pressure gradient, which we will call P_i' . If the rock is *isotropic*, we can get away with using a scalar permeability k , as follows:

$$\eta w_i = k P_i' \tag{9}$$

where η is the viscosity of the fluid (usually a scalar constant). This equation (which is called “Darcy’s Law”) is actually shorthand for three equations, one for each value of the suffix i :

$$\begin{aligned} \eta w_1 &= k P_1' \\ \eta w_2 &= k P_2' \\ \eta w_3 &= k P_3' \end{aligned} \tag{10, 11, 12}$$

Hence we are saying that each component of flow depends only on the corresponding component of pressure gradient, e.g., if the pressure gradient is entirely vertical, only vertical flow will occur. This is fine for a rock with a nice isotropic pore structure (like a sponge). If, however the rock has a system of aligned cracks, the flow will be deflected along the cracks even if the pressure gradient acts in some oblique direction. To describe this anisotropy, we

need components of permeability that relate, for example, the 1-component of flow to all three components of pressure gradients.

This produces an augmented version of the above equations:

$$\begin{aligned}\eta w_1 &= k_{11} P'_1 + k_{12} P'_2 + k_{13} P'_3 \\ \eta w_2 &= k_{21} P'_1 + k_{22} P'_2 + k_{23} P'_3 \\ \eta w_3 &= k_{31} P'_1 + k_{32} P'_2 + k_{33} P'_3\end{aligned}\tag{13, 14, 15}$$

The permeability k has sprouted two suffices! The first refers to the component of flow, the second to the component of pressure gradient. We refer to this new object as the *permeability tensor* k_{ij} , where the suffices i and j are independent and can each take the values 1, 2 or 3.

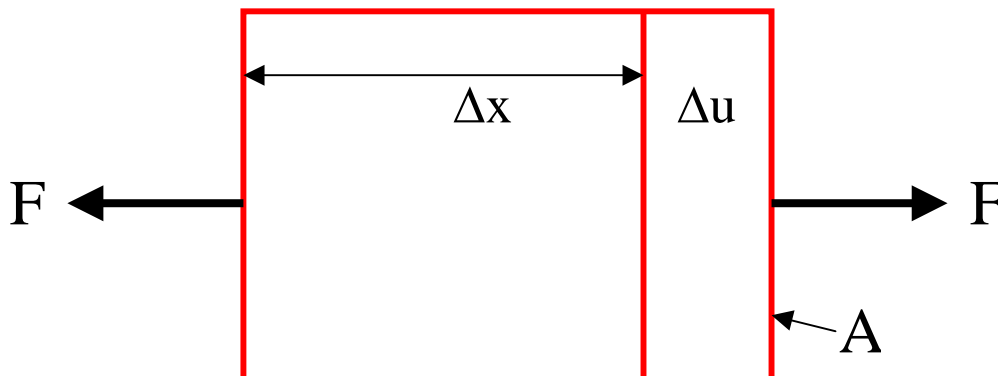
EXERCISE 2: In two dimensions (1 and 2) sketch coordinate axes and a square rock with some parallel cracks crossing it, oblique to the axes. Indicate how you would expect flow driven by a pressure gradient in the 1-direction to be deflected along the 2-direction by the cracks. What is the sign of k_{21} ? By considering a pressure gradient in the 2-direction, decide whether k_{12} has the same sign.

EXERCISE 3: Rewrite the 3 equations (13-15) as a single equation by grouping the k_{ij} terms into a 3 x 3 matrix.

Later we will see how this set of equations can be written more clearly and concisely. For the time being, note that anisotropic permeability is the rule rather than the exception in real rocks, and we *have* to use a tensor to describe it.

2.6 Tensor variables

Stress and strain, which are the bread and butter of much of geodynamics, are both represented by tensor variables. This is because both, like the permeability tensor k_{ij} , depend on the choice of two independent directions at each point in space.



You will already be familiar with the general notion of stress as force-per-unit-area (Nm^{-2} or Pa), and with the fact that forces is a vector $f(\underline{x})$. The second independent vector that comes into the definition of stress is the normal to an element of surface area, n_i . Suffice to say that to fully describe the forces acting within a continuum requires a tensor $\sigma_{ij}(\underline{x})$, which is the force acting in the i – direction on a surface with normal in the j – direction, at the position \underline{x} . The nine components of the stress tensor can be written as:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \quad (16)$$

In fact, the stress tensor is always *symmetric*, which means that $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$. This reduced the number of *independent* components of the stress tensor to six.

The strain tensor describes the deformation of a continuum is obtained by considering gradients in the displacement field $u_i(\underline{x})$ (a vector of course). The second independent vector arise from the fact that the gradient in displacement can be taken in any direction, i.e., $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$. The strain tensor $\varepsilon_{ij}(\underline{x})$ is related to the gradient in the j -direction, of displacement in the i -direction, at the position \underline{x} .

EXERCISE 4: Write out the components of the strain tensor ε_{ij} in the matrix form, using the fact that is symmetric (like σ_{ij}) to reduce the number of independent numbers from 9 to 6.

2.7 The rank of a tensor

You should now be aware that, when using the suffix notation, scalars have no suffix, vectors have one suffix and tensors have two suffices. In general, the number of suffices is called the *rank* of a tensor.

Scalars are also “ 0^{th} -rank” tensors.

Vectors are also “ 1^{st} -rank” tensors

Tensors like stress and strain are really “ 2^{nd} -rank tensors. (If you just say “Tensor”, it is implicit that you mean 2^{nd} -rank.)

There are also (just when you thought matters were in control) tensors of higher rank. We will work with a 3^{rd} -rank tensor later on (3 suffices). For the moment, it is reasonably easy to understand the need for a 4^{th} -rank tensor constant (4 suffices) in elasticity. To write down equations describing the deformation of an elastic solid due to certain applied forces, we must consider the relationship between stress and strain. Both of these are already 2^{nd} -rank tensors. Now, recall the permeability tensor, which was introduced to provide a general relationship between two vector variables. If each of the components of stress is to be independently related to each component of strain, we need equations like the following:

$$\sigma_{11} = C_{1111}e_{11} + C_{1112}e_{12} + C_{1113}e_{13} + C_{1121}e_{21} + C_{1122}e_{22} + C_{1123}e_{23} + C_{1131}e_{31} + C_{1132}e_{32} + C_{1133}e_{33} \quad (17)$$

This looks pretty intimidating, but all we have done is to permit a particular component of stress σ_{11} to depend on all nine components of strain. The elastic constants are components of a 4th-rank tensor C_{ijkl} , wherein the first pair of suffices (ij) identify the component of stress, and the second pair (kl) identify the components of strain. This equation is in fact a component of the general form of Hooke's law.

EXERCISE 5: Write out the component of Hooke's law relating σ_{12} to the nine components of strain.

The elastic modulus tensor has, at first sight, $3 \times 3 \times 3 \times 3 = 81$ independent components, but various symmetries reduce this to 21. This is still too many for most mortals to contemplate. Fortunately, if an elastic material is isotropic, further symmetries reduce the number of independent elastic moduli to two, which you may already know as the shear modulus (termed G or μ) and the bulk modulus, K . For an isotropic material, equation (17) reduces to:

$$\sigma_{11} = \left(K - \frac{2}{3} G \right) (e_{11} + e_{22} + e_{33}) + 2 G e_{11} \quad (18)$$

3. Manipulation of tensors

3.1 The summation convention

You would not have been very appreciative if Exercise 5 had asked for all nine of the equations of Hooke's law. You are not alone; a certain *A. Einstein* also realised that this was a waste of time and paper, and devised a compact notation for tensor equations. As you will see, the beauty of Einstein's *summation convention* is that it is easy to recover the full equation when you need it, with the components written out.

Look back to equations (13, 14 & 15), which represent Darcy's law for an anisotropic material. A more compact notation for the equations is:

$$\eta w_i = \sum_{j=1}^3 k_{ij} P_j \quad (19)$$

This equation says that, for a given value of i , the RHS consists of the sum of three terms, one for each value of j .

EXERCISE 6: By setting $i = 1$ in equation (19), reproduce equation (13).

Now for the touch of genius. We can make equation (19) one step more concise by leaving out the summation sign completely:

$$\eta w_i = k_{ij} P_j \quad (20)$$

How do you interpret this rather minimal-looking equation?

First, note that these are two terms containing quantities that are multiplied together: on the LHS we have ηw_i , and on the RHS we have $k_{ij} P_j$.

There are two rules you need to remember, which have to do with the number of times a particular suffix appears in each term. There are only two possibilities allowed: a suffix may appear once or twice.

Rule 1: If a suffix appears *once*, it is called a *free suffix*. All terms in the equation must have the same free suffix (or suffixes). (In equation (20), both terms have i as a free suffix.)

Rule 2: If a suffix appears *twice*, it is called a *summed suffix*. A summation sign for that suffix is then implicitly placed in front of the term. (In equation (20), the term on the RHS terms has j as a summed suffix.)

Applying these rules takes practise, but once you get the hang of it you will never look back. The idea may seem rather innocuous, but it greatly simplifies the manipulation of vectors and tensors. The beauty of equations like (20) is that you can see, at a glance, the general structure: flow and pressure gradient are related through the permeability. You can also quickly write out the gory details when you have to plug in some real numbers. The various kinds of vector notation also show the structure, but tend to be less clear on the details.

EXERCISE 7: Write down the scalar (or dot) product of two vectors \underline{a} and \underline{b} using the summation convention:

$$\underline{a} \bullet \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = ?? \quad (21)$$

EXERCISE 8: Write down the general expression for the terms in Hooke's law, of which equation (17) is an example. The relationship should involve σ_{ij} , e_{kl} and C_{ijkl} , and two summations. First use explicit summation (cf. equation (19)), then make the summation implicit (cf. equation (20)). Identify the free and summed indices in the equation you have written.

(At this point you may be realising why it is best to avoid writing down $u_i(x_i)$, when you wish to show that the vector \underline{u} is a function of position \underline{x} . It makes i look like a summed suffix, when it is not.)

3.2 The Kronecker delta

This useful 2nd-rank tensor is a constant and is defined as follows:

$$\delta_{ij} = \begin{pmatrix} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{pmatrix} \quad (22)$$

In matrix notation, δ_{ij} , is simply the identity matrix:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (23)$$

As an example of the effect of δ_{ij} , consider the term $g \delta_{i3}$. This will clearly be zero unless $i = 3$, regardless of the value of g . The term is therefore a vector of magnitude g in the 3-direction.

As a second example, consider $a_i \delta_{ij}$. This will be zero unless $i = j$, so effect is to replace i with j , i.e.,

$$a_i \delta_{ij} = a_j \tag{24}$$

EXERCISE 9: Show that if you set $k_{ij} = k \delta_{ij}$ in Darcy's law for a general, anisotropic material (20), you reproduce Darcy's law for an isotropic material (9). (This shows that a scalar permeability is *really* a tensor with a very simple structure.)

3.3. The 3rd-rank antisymmetric tensor

This is another "useful" tensor constant, this time of 3rd rank. It is defined as follows:

$$\varepsilon_{ijk} = \begin{pmatrix} 0, & \text{if any two } ijk \text{ are the same} \\ 1, & \text{if } ijk \text{ is an even permutation of } 123 \quad (\text{ie } 123, 312, 231) \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \quad (\text{ie } 132, 321, 213) \end{pmatrix}$$

Hence, only 6 of 27 components of ε_{ijk} are non-zero.

This tensor is used to produce vector (or cross) products when using suffix notation:

$$(\underline{a} \times \underline{b})_i = \varepsilon_{ijk} a_j b_k$$

We can obtain, for example, the $i = 1$ component of $\underline{a} \times \underline{b}$ by requiring that (for ε_{ijk} to be non-zero) j and k share the values 2 and 3 between them. If we set $j = 2$ and $k = 3$, ε_{ijk} will be positive and *vice versa*. Hence,

$$(\underline{a} \times \underline{b})_1 = a_2 b_3 - a_3 b_2 \tag{27}$$

4. Vector calculus

You should work through this section if you have already encountered an operator called ∇ , which is used in vector notation to represent various kinds of spatial derivative. In this section we will show how suffix notation and the summation convention can be used to make vector calculus more transparent.

In general, the differential operator ∇ is defined as a vector of partial derivatives:

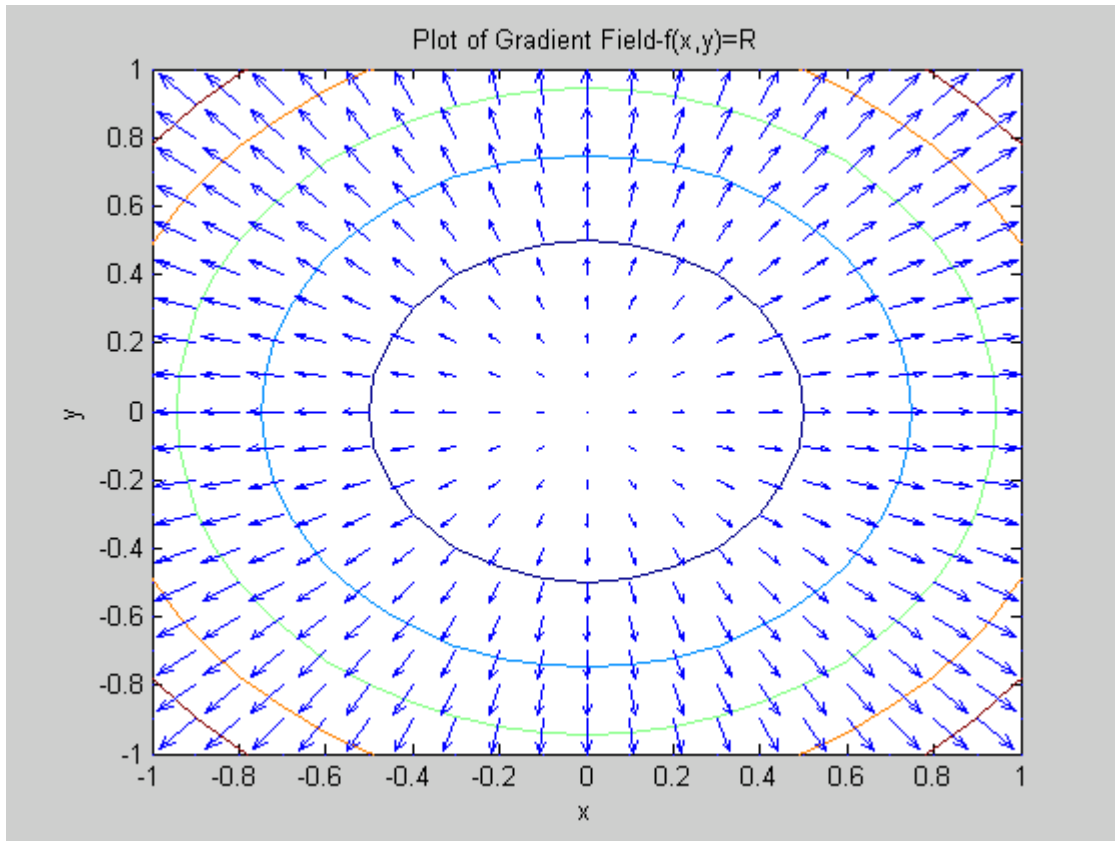
$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = \frac{\partial}{\partial x_i} \tag{28}$$

This shows how to replace ∇ when using suffix notation, so we can now re-visit the main applications of ∇ and see how they look with the new notation.

4.1 The gradient operator

Say we have a scalar variable such as $T(\underline{x})$. The *gradient* of this *scalar field* is a vector gthat gives the direction in which T changes most quickly, as well as the rate of change:

$$\nabla T = \left(\frac{\partial T}{\partial x_1}, \frac{\partial T}{\partial x_2}, \frac{\partial T}{\partial x_3} \right) = \frac{\partial T}{\partial x_i}$$



(29)

You will also sometimes see the notation **grad** T . The suffix notation version gives a compact form that makes it clear what you actually need to do when using real numbers.

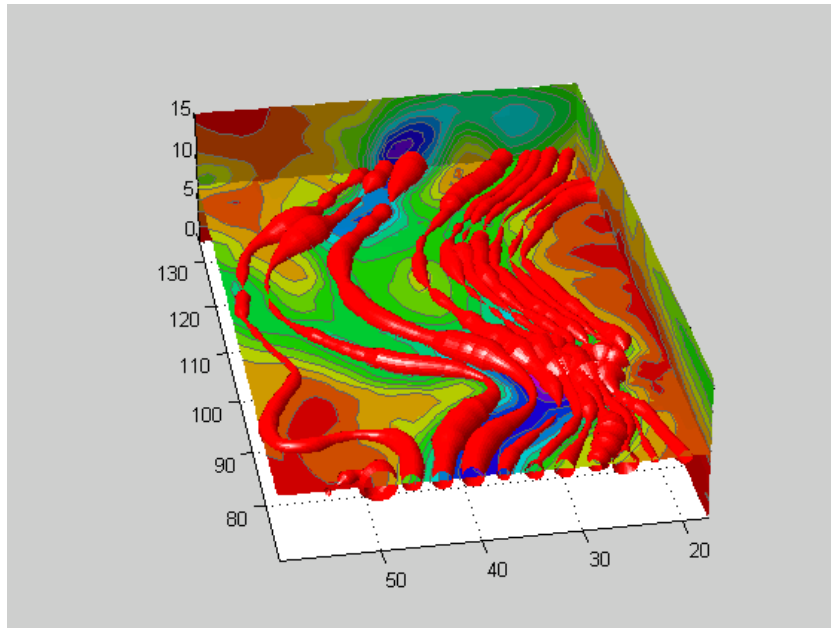
4.2 The divergence operator

The *divergence* of a vector field $u_i(\underline{x})$ is a measure of the rate of flow of material into or out of an element of volume around the point \underline{x} (easiest to imagine if u_i is the flow velocity of a compressible fluid). It is defined as:

$$\nabla \bullet \underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i}$$

(30)

You will also sometimes see the notation **div** \underline{u} .



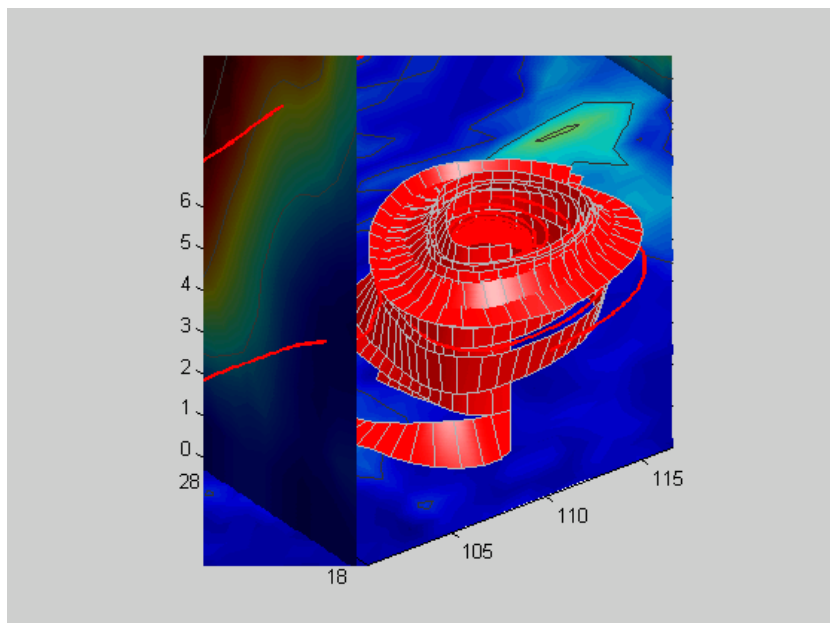
4.3 The curl operator

The *curl* of a vector field $u_i(\underline{x})$ is a measure of the maximum rate of rotation of material around the point \underline{x} , also giving the direction of the axis of the maximum rotation (easiest to image if u_i is the flow of a fluid that has vortices). It is defined as:

$$\nabla \times \underline{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \quad (31)$$

You will also sometimes see the notation *curl* \underline{u} .

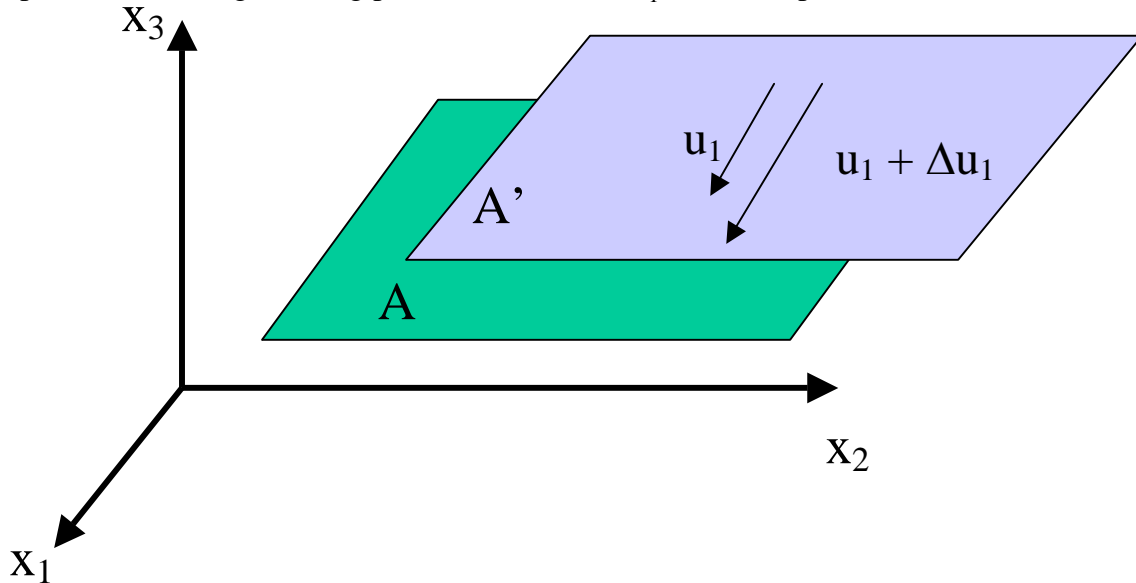
EXERCISE 10: Notice that we have identified three “flavours” of first derivative operator: the gradient, the divergence and the curl. By examining the free and fixed suffices in the suffix notation version of each, decide what kind of quantity (scalar or vector) each operates on, and what kind of quantity is produced/



5. Analysis of strain

When a continuous body is deformed, a point initially at \underline{x} moves by an amount $\underline{u}(\underline{x})$ (e.g, u_1 in the x_1 direction). If \underline{u} is constant, the entire body is in uniform translation. If an adjacent point (Δx_2 away) moves by a small additional increment Δu_1 then the body is being strained. We therefore consider how all those additional increments add up as we move incrementally across the body, i.e., we consider *gradients* in displacement

Displacement of neighbouring points. Difference Δu_1 is the component of strain and rotation



5.1 Displacement gradient tensor

There are three independent components of the displacement vector at any point (u_1, u_2, u_3), and three independent directions in which they can change (x_1, x_2, x_3). Hence there are nine independent values of displacement gradient, which can be assembled into a *displacement gradient tensor* $D_{ij}(x)$:

$$\frac{\Delta u_i}{\Delta x_j} = \frac{u_i(x + \Delta x_j) - u_i(x)}{\Delta x_j} = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} = D_{ij}(x)$$

Or writing the displacement gradient tensor out in full: (32)

$$D_{ij}(x) = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

5.2 Rotation and strain

We want to deal with rotations and strain separately, so we can separate out the rigid body rotations of plates from how they are deforming. So we separate out the displacement gradient tensor D_{ij} into strains and rotations:

$$\begin{aligned}
 D_{ij} = \frac{\partial u_i}{\partial x_j} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) \\
 &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \\
 &= \quad \quad \quad \varepsilon_{ij} \quad + \quad \quad \quad \omega_{ij}
 \end{aligned}
 \tag{34}$$

The first term ε_{ij} is the *strain tensor*, and describes the internal deformation of the material. This tensor is *symmetric*, i.e. $\partial u_i / \partial x_j = \partial u_j / \partial x_i$. It has 6 independent components in 3-D. The second, antisymmetric term ω_{ij} is called the *rotation tensor*, and has 3 independent components in 3-D.

5.3 Strain kinematics

The strain-rate and rotation-rate tensors are defined by:

$$\begin{aligned}
 \dot{\varepsilon}_{ij} &= \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \\
 \dot{\omega}_{ij} &= \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)
 \end{aligned}
 \tag{35, 36}$$

These are the formal definitions of strain and rotation and should be learnt

ε_{ij} has 9 components in 3-D, because each of i and j can take the values 1, 2 and 3 independently.

However, ε_{ij} is symmetric, i.e., $\varepsilon_{ij} = \varepsilon_{ji}$

$$\text{e.g., } \dot{\varepsilon}_{12} = \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) = \frac{1}{2} \left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) = \dot{\varepsilon}_{21}
 \tag{37}$$

Hence, only 6 components are independent:

$$\dot{\varepsilon}_{11} \quad \dot{\varepsilon}_{22} \quad \dot{\varepsilon}_{33} \quad \dot{\varepsilon}_{12} = \dot{\varepsilon}_{21} \quad \dot{\varepsilon}_{13} = \dot{\varepsilon}_{31} \quad \dot{\varepsilon}_{23} = \dot{\varepsilon}_{32}$$

ω_{ij} also has 9 components in 3-D.

The three diagonal components are always zero.

$$\text{e.g., } \dot{\omega}_{11} = \frac{1}{2} \left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_1}{\partial x_1} \right) = 0 \quad (38)$$

The remaining components are antisymmetric, i.e., $\omega_{ij} = -\omega_{ji}$

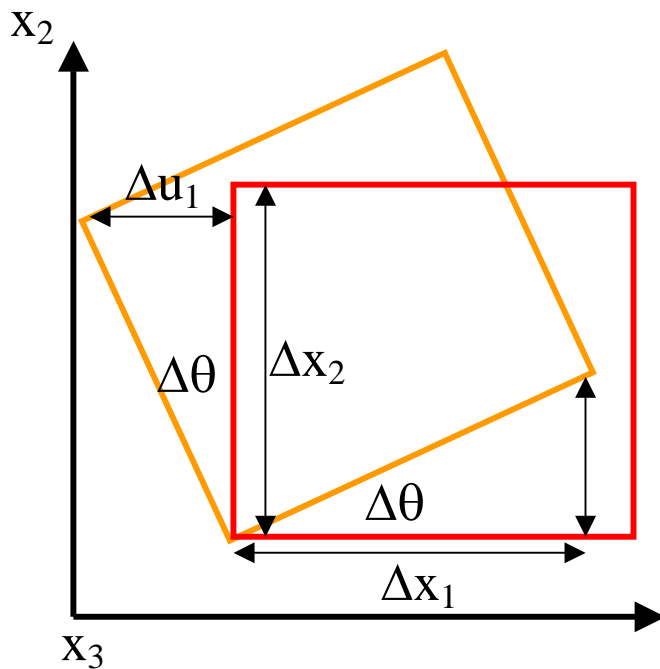
$$\text{e.g., } \dot{\omega}_{12} = \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) = -\frac{1}{2} \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) = -\dot{\omega}_{21} \quad (39)$$

Hence, only 3 components are independent:

$$\dot{\omega}_{12} = -\dot{\omega}_{21} \quad \dot{\omega}_{13} = -\dot{\omega}_{31} \quad \dot{\omega}_{23} = -\dot{\omega}_{32} \quad (40)$$

5.4 Analysis of Rotation

Consider the 2-D displacement gradient tensor describing a positive rotation of an angle $\Delta\theta$ about the x_3 -direction:



$$\frac{\partial u_1}{\partial x_2} = -\frac{\partial u_2}{\partial x_1}$$

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = 0$$

Hence,

$$D_{ij} = \begin{pmatrix} 0 & -\Delta\theta \\ \Delta\theta & 0 \end{pmatrix}$$

(41,42,43)

This is *antisymmetric*, i.e., $\partial u_i / \partial x_j = -\partial u_j / \partial x_i$.

EXERCISE 11: The motion between two plates is accommodated in a fault zone of width L . Surveys across this zone have been fitted by the following velocity field:

$$v_1 = -0.1 \frac{V x_1}{L}$$

$$v_2 = \frac{V x_1}{L}$$

(44, 45)

where x_1 and x_2 are horizontal coordinates measuring distance across and along the zone, and V is the strike-slip component of relative plate motion. Sketch the variation of velocity with position across the fault zone. Calculate the two-dimensional strain-rate and rotation-rate tensors for the material in the fault zone. Assuming that the material is incompressible, calculate the vertical component of strain rate (ϵ_{33}).

The San Andreas Fault is a major strike-slip plate boundary but is also marked by a series of small mountain ranges. In certain places, motion between the plates is accommodated across a broad zone rather than by discrete faults

EXERCISE 12: To apply this model to the San Andreas, assume that $L = 30$ km and $V = 30$ mm/yr. What is the maximum uplift that could be produced after 1 Myr in the fault zone if the plates are 20 km thick? Why would you expect the observed uplift to be less than this?