SPA5218 Mathematical Techniques 3
Exercise Sheet 7

There is a lot of material here and we will use two tutorials to cover it.

Solve the starred problems in the exercise class. These are the ones you will be able to solve given the amount of material we have covered so far. Attempt the rest after we have covered Green's functions on Friday. We will cover some of the exercises on the Dirac delta-function in class.

*1. A damped harmonic oscillator is described by an equation of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0$$

where ω_0 is a frequency and $\gamma\rangle 0$ is a real damping parameter. If $x(t=0) = A$ and $\mathrm{d}x/\mathrm{d}t(t=0) = 0$, solve the equation for (a) 'light damping', $\gamma\langle 2\omega_0$, (b) 'critical damping', $\gamma = 2\omega_0$, and (c) 'heavy damping', $\gamma\rangle 2\omega_0$.

*2. The Dirac delta function, $\delta(x)$, has the property that

$$\int_{-\infty}^{\infty} F(x)\,\delta(x-x')\,\mathrm{d}x = F(x')\,.$$

Show that

$$\text{(i)}\quad \delta(x) = \delta(-x)$$

$$\text{(ii)}\quad \delta(ax) = \frac{1}{|a|}\delta(x) \text{ for real } a \neq 0\ ,$$

$$\text{(iii)}\quad \int_{-\infty}^{\infty} F(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x_0)\right)\mathrm{d}x = -F'(x_0),$$

$$\text{(iv)}\quad x\delta(x) = 0,$$

$$\text{(v)}\quad x\delta'(x) = -\delta(x)\,.$$

$$\text{(vi)}\quad \delta\left((x-a)(x-b)\right) = \frac{1}{|a-b|}\left[\delta(x-a) + \delta(x-b)\right] \text{ for } a \neq b\,.$$

Show also that the last result is a special case of

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$$

where x_i are simple zeroes of the function f, i.e.

$$f(x_i) = 0\,, \quad f'(x_i) \neq 0\,.$$

3. A particular solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = F(x)$$

is

$$y(x) = y_2(x)\int^x \frac{y_1(x')F(x')}{W(x')}\mathrm{d}x' - y_1(x)\int^x \frac{y_2(x')F(x')}{W(x')}\mathrm{d}x'$$

where the function $W(x)$ is defined by

$$W(x) = y_1(x)\frac{\mathrm{d}y_2(x)}{\mathrm{d}x} - y_2(x)\frac{\mathrm{d}y_1(x)}{\mathrm{d}x}$$

and $y_1(x)$ and $y_2(x)$ are the solutions of the homogeneous equation ($F = 0$). Consider the differential equation,

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = x\ln x\,.$$

By trying a solution of the form $y = x^n$, show that the homogeneous equation has solutions $y_1 = x$ and $y_2 = x^3$. Then use the method of Green's function given above to find a particular solution of the differential equation.

*4. Find the complete solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = e^{2x} + e^x\,.$$

Hint: Finding the complementary function should be easy. Find the particular solution by assuming it is proportional to the source function $e^{2x} + e^x$, but take care to make this guess orthogonal to the complementary function. See RHB §15.1.2.

*5. Prove that

$$\int_{-\infty}^{\infty} f(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x)\mathrm{d}x = -\int_{-\infty}^{\infty} f'(x)\,\delta(x)\mathrm{d}x\,,$$

where $\delta(x)$ is the Dirac delta function.

By letting $f(x) = x\,g(x)$ in this equation, show that

$$\int_{-\infty}^{\infty} x\,g(x)\delta'(x)\,\mathrm{d}x = -\int_{-\infty}^{\infty} g(x)\delta(x)\,\mathrm{d}x$$

and hence that

$$x\,\delta'(x) = -\delta(x)\,.$$

6. The differential equation describing a forced, harmonic oscillator is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = f_0\,\cos\omega t$$

where γ is a damping coefficient, ω_0 and ω are the natural and forcing frequencies, respectively, and f_0 is a constant that determines the magnitude of the forcing term. Show that as $t \to \infty$ the solution can be written in the form $x(t) = C\cos(\omega t + \alpha)$ and find expressions for C and α in terms of γ, ω, ω_0 and f_0.

7. Use the Green function method to find the solution of the differential equation,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin 2x$$

with $y(0) = y(\pi/2) = 0$.