SPA5218 Mathematical Techniques 3
Exercise Sheet 6

1. For the homogeneous equation with constant coefficients,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

show that the trial solution $y = e^{\lambda x}$ leads to the *auxiliary equation*

$$\lambda^2 + P\lambda + Q = 0\,.$$

Hence find the most general solution for the case $P^2 \neq 4Q$, giving an expression for $y(x)$ containing the parameters P and Q.

In the case where $P^2 = 4Q$, verify that the most general solution has the form

$$y = Ae^{\lambda x} + Bxe^{\lambda x}\,.$$

Write λ in terms of P in this case.

2. The exponential function $y(x) = e^x$ may be defined as the function whose value at the origin is 1, i.e. $y(0) = 1$, and whose derivative at a point x is equal to its value at that point, i.e.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\,.$$

The series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

can be derived by considering the series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

with $a_0 = 1$. Taking this solution, calculate the derivative of the series and substitute in the equation $\mathrm{d}y/\mathrm{d}x = y$. Hence derive the recursion relation

$$a_{n+1} = \frac{a_n}{(n+1)}$$

and use this to obtain expressions for a_1, a_2, a_3 and a_4.

3. The solution of the Hermite differential equation by the series method involves the recursion relation

$$a_{m+2} = \frac{2(m-\alpha)}{(m+1)(m+2)} a_m$$

(a) Use the recursion relation to calculate a_2, a_4 and a_6 in terms of a_0. (b) For a general even number $2j \geq 2$, the expression for the coefficient is

$$a_{2j} = \frac{2^j(-\alpha)(-\alpha+2)\cdots(-\alpha+2j-2)}{(2j)!} a_0 .$$

Prove this result by the method of induction, i.e. first assume that the equation is true for some $j = J$ and then use the recursion relation to show that it is also true for $j = J+1$. To complete the proof show that the above result is true in the case $2j = 2$.

4. Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x^2 \frac{\mathrm{d}y}{\mathrm{d}x} - xy = 0$$

as a power series of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ about the point $x = 0$. You should derive explicit expressions for the coefficients a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 and a_{10} in terms of the coefficients a_0 and a_1.

5. Find the general solution of the differential equation

$$(x^2+1)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

as a power series of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ about the point $x = 0$.

6. Laguerre's differential equation is

$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda y = 0 ,$$

where λ is a constant. By writing

$$y(x) = \sum_{j=0}^{\infty} a_j \, x^{j+c} ,$$

show that $c = 0$ is a solution of the indicial equation. Taking $c = 0$, derive the recursion relation for the coefficients a_j and use the relation to find explicit expressions for the coefficients a_1, a_2, a_3 and a_4 in terms of a_0 and λ. Show that when λ is a positive integer the series terminates. Taking $a_0 = 1$, find the explicit form of the resulting solutions $y(x) = L_\lambda(x)$, called Laguerre polynomials, for the fixed values $\lambda = 0$, 1, 2 and 3.