SPA5218 MATHEMATICAL TECHNIQUES 3

EXERCISE SHEET 7 : SOLUTIONS

Q1.

(a) Substituting $x = e^{mt}$ the auxiliary equation is

$$m^2 + \gamma m + \omega_0^2 = 0$$

with solutions

$$m = -\frac{\gamma}{2} \pm i\left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2} \qquad (\gamma < 2\omega_0)$$

The general solution is therefore of the form

$$x(t) = e^{-\frac{\gamma}{2}t}\left(A_1 \cos\omega t + A_2 \sin\omega t\right)$$

where $$\omega = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}$$

Now apply the boundary conditions $x = A$, $dx/dt = 0$ at $t = 0$

$$A = A_1 \quad (\text{from } x(0))$$

$$\frac{dx}{dt} = -\frac{\gamma}{2} e^{-\frac{\gamma}{2}t}\left(A_1 \cos\omega t + A_2 \sin\omega t\right) + e^{-\frac{\gamma}{2}t}\left(-A_1 \omega \sin\omega t + A_2 \omega \cos\omega t\right)$$

$$\left(\frac{dx}{dt}\right)_{t=0} = -\frac{\gamma}{2} A_1 + A_2 \omega = -\frac{\gamma}{2} A + A_2 \omega = 0$$

$$\Rightarrow \quad A_2 = \frac{\gamma}{2}\frac{A}{\omega}$$

i.e. $$x(t) = A e^{-\frac{\gamma}{2}t}\left(\cos\omega t + \frac{\gamma}{2\omega}\sin\omega t\right)$$

(b) For $\gamma = 2\omega_0$ the auxiliary equation reduces to

$$\left(m + \frac{\gamma}{2}\right)^2 = 0$$

with the degenerate roots

$$m_1 = m_2 = m = -\frac{\gamma}{2}$$

The general solution is therefore

$$x(t) = (A_1 + A_2 t)e^{-\frac{\gamma}{2}t}$$

$x = A$, $dx/dt = 0$ at $t=0$ gives

$$A = A_1$$

$$\frac{dx}{dt} = A_2 e^{-\frac{\gamma}{2}t} + (A_1 + A_2 t)\left(-\frac{\gamma}{2}\right)e^{-\frac{\gamma}{2}t}$$

$$\left(\frac{dx}{dt}\right)_{t=0} = A_2 - \frac{A_1\gamma}{2} = 0 \quad \Rightarrow \quad A_2 = \frac{\gamma}{2}A_1$$

i.e. $$x(t) = A\left(1 + \frac{\gamma}{2}t\right)e^{-\frac{\gamma}{2}t}$$

(c) For $\gamma > 2\omega_0$ the solutions of the auxiliary equation are

$$m = -\alpha_1 \, , \, -\alpha_2$$

where
$$\alpha_1 = \frac{\gamma}{2} + \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$$

$$\alpha_2 = \frac{\gamma}{2} - \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$$

The general solution is therefore

$$x(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$$

$x = A$, $dx/dt = 0$ at $t = 0$ gives

$$A = A_1 + A_2$$

$$\frac{dx}{dt} = -\alpha_1 A_1 e^{-\alpha_1 t} - \alpha_2 A_2 e^{-\alpha_2 t}$$

$$\left(\frac{dx}{dt}\right)_{t=0} = -\alpha_1 A_1 - \alpha_2 A_2 = 0$$

$$\Rightarrow A_2 = -\frac{\alpha_1}{\alpha_2} A_1 \; ; \quad A = A_1\left(1 - \frac{\alpha_1}{\alpha_2}\right)$$

$$\Rightarrow A_1 = A\left(\frac{\alpha_2}{\alpha_2 - \alpha_1}\right) \; ; \quad A_2 = A\left(\frac{-\alpha_1}{\alpha_2 - \alpha_1}\right) \; ;$$

i.e. $$x(t) = \frac{A}{\alpha_2 - \alpha_1}\left(\alpha_2 e^{-\alpha_1 t} - \alpha_1 e^{-\alpha_2 t}\right)$$

Q2.

(i) $$\int_{-\infty}^{\infty} F(x)\,\delta(x)\,dx = F(0)$$

$$\int_{-\infty}^{\infty} F(x)\,\delta(-x)\,dx = F(0)$$

$$\Rightarrow \delta(x) = \delta(-x)$$

(ii) $$\int_{-\infty}^{\infty} F(x)\,\delta(ax)\,dx = \int_{-\infty}^{\infty} F\left(\frac{u}{a}\right)\delta(u)\,\frac{du}{a} = \frac{F(0)}{a} \text{ for } a>0$$

$$\int_{-\infty}^{\infty} F(x)\delta(ax)\,dx = \int_{+\infty}^{-\infty} F\left(\frac{u}{a}\right)\delta(u)\,\frac{du}{a} = \int_{-\infty}^{\infty} F\left(\frac{u}{a}\right)\delta(u)\,\frac{du}{(-a)}$$

$$= \frac{F(0)}{|a|} \quad \text{for } a<0$$

and so $$\int_{-\infty}^{\infty} F(x)\left(\frac{\delta(x)}{|a|}\right)dx = \frac{F(0)}{|a|}$$

and hence $$\delta(ax) = \frac{1}{|a|}\,\delta(x).$$

(iii) $$\int_{-\infty}^{\infty} F(x)\left(\frac{d}{dx}\,\delta(x-x_0)\right)dx$$

$$= \Big[F(x)\,\delta(x-x_0)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dF}{dx}\cdot\delta(x-x_0)\,dx$$

($=0$ since $\delta(x-x_0) = 0$ for $x \neq x_0$)

$$= -\int_{-\infty}^{\infty} \frac{dF}{dx} \cdot \delta(x - x_0)\, dx = -\left[\frac{d}{dx} F(x)\right]_{x=x_0} = -F'(x_0)$$

(iv) $$\int_{-\infty}^{\infty} F(x)\, x\, \delta(x)\, dx = \left[F(x)\, x\right]_{x=0} = 0$$

$$\Rightarrow \quad x\,\delta(x) = 0$$

(v) $$\int_{-\infty}^{\infty} F(x)\, x\, \delta'(x)\, dx = \int_{-\infty}^{\infty} F(x)\, x \frac{d}{dx}\delta(x)\, dx$$

$$= \left[F(x)\, x\, \delta(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx}\left(x F(x)\right)\delta(x)\, dx$$

$$= -\int_{-\infty}^{\infty} \left(F(x) - x \frac{d}{dx} F(x)\right)\delta(x)\, dx$$

$$= -\left[F(x) - x \frac{d}{dx} F(x)\right]_{x=0} = -F(0) - 0 \cdot \frac{d}{dx}F(0)$$

$$= -F(0)$$

and $$\int_{-\infty}^{\infty} F(x)\left(-\delta(x)\right) dx = -F(0)$$

so $$x\,\delta'(x) = -\delta(x)$$

(vi) $$\int_{-\infty}^{\infty} F(x)\, \delta\left((x-a)(x-b)\right) dx$$

$$= \int_{a-\varepsilon}^{a+\varepsilon} F(x)\, \delta\left((x-a)(x-b)\right) dx$$

$$+ \int_{b-\varepsilon}^{b+\varepsilon} F(x)\, \delta\left((x-a)(x-b)\right) dx$$

Integrals outside these ranges vanish since $\delta((x-a)(x-b)) = 0$ for $x \neq a$, $x \neq b$. If we consider one of the remaining integrals:

$$\int_{a-\varepsilon}^{a+\varepsilon} F(x)\,\delta((x-a)(x-b))\,dx = \int_{a-\varepsilon}^{a+\varepsilon} F(x)\,\delta((x-a)(a-b))\,dx$$

$$= \int_{a-\varepsilon}^{a+\varepsilon} F(x)\,\frac{\delta(x-a)}{|a-b|}\,dx = \frac{F(a)}{|a-b|}$$

The other term is similar, hence

$$\int_{-\infty}^{\infty} F(x)\,\delta((x-a)(x-b))\,dx = \frac{F(a)+F(b)}{|a-b|}$$

We also have

$$\int_{-\infty}^{\infty} F(x)\,\frac{(\delta(x-a)+\delta(x-b))}{|a-b|}\,dx = \frac{F(a)+F(b)}{|a-b|}$$

$$\Rightarrow \quad \delta((x-a)(x-b)) = \frac{1}{|a-b|}\left(\delta(x-a)+\delta(x-b)\right)$$

Comparing this with

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \qquad *$$

we have $f(x) = (x-a)(x-b)$ which has zeroes $x_1 = a$ and $x_2 = b$ such that

$$f'(a) = a-b = -f'(b)$$

and so * becomes the same as the expression just derived.

Q3. $$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = x \ln x$$

Divide through by x^2:

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = \frac{1}{x}\ln x$$

i.e. $P(x) = -\frac{3}{x}$, $Q(x) = \frac{3}{x^2}$, $F(x) = \frac{1}{x}\ln x$

The homogeneous equation is

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 0$$

Trying $y = x^n$ gives $\frac{dy}{dx} = nx^{n-1}$, $\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$

and $$n(n-1)x^{n-2} - \frac{3}{x}.nx^{n-1} + \frac{3}{x^2}x^n$$

$$= \left(n(n-1) - 3n + 3\right)x^{n-2}$$

$$= 0 \text{ provided } n(n-1) - 3n + 3 = 0$$

$$\Rightarrow n^2 - 4n + 3 = 0$$

$$\Rightarrow (n-3)(n-1) = 0 \Rightarrow y_1 = x, \; y_2 = x^3$$

$$y_1' = 1, \; y_2' = 3x^2$$

Then $w(x) = y_1 y_2' - y_1' y_2$

$$= x \cdot 3x^2 - 1 \cdot x^3 = 2x^3$$

$$y_2(x) \int^x \frac{y_1(x') F(x')}{w(x')} dx' = x^3 \int^x \frac{\ln x'}{2x'^3} dx'$$

$$= x^3 \left[-\frac{\ln x}{4x^2} + \int^x \frac{dx'}{4x'^3} \right]$$ (integrating by parts)

$$= x^3 \left[-\frac{\ln x}{4x^2} - \frac{1}{8x^2} \right]$$

Also

$$y_1(x) \int^x \frac{y_2(x') F(x')}{w(x')} dx' = x \int^x \frac{\ln x'}{2x'} dx'$$

$$= \frac{1}{4} x (\ln x)^2$$

$$\Rightarrow y(x) = -\frac{1}{4} x (\ln x)^2 - \frac{1}{4} x \ln x - \frac{1}{8} x$$

SPA5218 MATHEMATICAL TECHNIQUES 3

HOMEWORK SHEET 7 : SOLUTIONS

Q1. $$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{2x} + e^{x}$$

Homogeneous equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

The auxiliary equation is

$$(m^2 + m - 6) = (m-2)(m+3) = 0 \qquad (2)$$

with solutions $m = 2$ and $m = -3$. The complementary function is of the form

$$y_c(x) = Ae^{2x} + Be^{-3x} \qquad (1)$$

where A and B are constants. The particular solution is found by using the trial solution

$$y_p(x) = Cxe^{2x} + De^{x} \qquad (1)$$

(note the extra factor x in front of e^{2x} because e^{2x} occurs in y_c)

Substitute into the differential eqn.

$$\frac{dy_p(x)}{dx} = 2Cxe^{2x} + Ce^{2x} + De^{x} \qquad (1)$$

$$\frac{d^2y_p(x)}{dx^2} = 4Cxe^{2x} + 2Ce^{2x} + 2Ce^{2x} + De^{x} \qquad (1)$$

i.e $\frac{d^2y_p}{dx^2} + \frac{dy_p}{dx} - 6y_p$

$= 4Cxe^{2x} + 2Ce^{2x} + 2Ce^{2x} + De^{x}$

$+ 2Cxe^{2x} + Ce^{2x} + De^{x}$

$- 6Cxe^{2x} - 6De^{x}$

$= 5Ce^{2x} - 4De^{x} = e^{2x} + e^{x}$

$\Rightarrow C = \frac{1}{5}, D = -\frac{1}{4}$

i.e. $y_p(x) = \frac{1}{5}xe^{2x} - \frac{1}{4}e^{x}$ (2)

The complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$= Ae^{2x} + Be^{-3x} + \frac{1}{5}xe^{2x} - \frac{1}{4}e^{x}$$

$$= \left(A + \frac{1}{5}x\right)e^{2x} + Be^{-3x} - \frac{1}{4}e^{x}$$ (2)

$2+1+1+1+1+2+2 = 10$

Q2. Prove that

$$\int_{-\infty}^{\infty} f(x)\frac{d}{dx}\delta(x)\,dx = -\int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\frac{d}{dx}\delta(x)\,dx = \Big[f(x)\delta(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx$$ (2)

(using integration by parts)

But $\left[f(x)\, \delta(x) \right]_{-\infty}^{\infty} = 0$ (2)

because $\delta(x) = 0$ except at $x = 0$.

Hence $\displaystyle\int_{-\infty}^{\infty} f(x) \frac{d}{dx} \delta(x)\, dx = -\int_{-\infty}^{\infty} f'(x)\, \delta(x)\, dx$

Now let $f(x) = x\, g(x)$

i.e. $\displaystyle\int_{-\infty}^{\infty} x\, g(x)\, \delta'(x)\, dx = -\int_{-\infty}^{\infty} \delta(x) \frac{d}{dx}\left(x\, g(x) \right) dx$

$$= -\int_{-\infty}^{\infty} \delta(x) \left[g(x) + x\, g'(x) \right] dx \qquad (3)$$

$$= -\int_{-\infty}^{\infty} g(x)\, \delta(x)\, dx \quad \left(\text{since } \int_{-\infty}^{\infty} x\, g'(x)\, \delta(x)\, dx = 0 \right) \qquad (2)$$

Therefore, from the definition of the δ function,

$$x\, \delta'(x) = -\delta(x) \qquad (1)$$

$2+2+3+2+1 = 10$

Q3. $\displaystyle\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = f_0 \cos \omega t$

The homogeneous equation is

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

(This was solved in EXERCISE 7.1)

The auxiliary equation is

$$m^2 + \gamma m + \omega_0^2 = 0$$

with solutions

$$m = -\frac{\gamma}{2} \pm i \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}$$

for $\underline{\gamma < 2\omega_0}$. Hence the solution is of the form

$$x(t) = e^{-\frac{\gamma}{2}t}\left(A_1 \cos \omega t + A_2 \sin \omega t\right) \qquad (3)$$

where $\qquad \omega = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}$

As $t \to \infty$, $x(t) \to 0$

For the case $\underline{\gamma = 2\omega_0}$ the solutions of the auxiliary equation are $m_1 = m_2 = m = -\frac{\gamma}{2}$. Hence the solution is of the form

$$x(t) = (A_1 + A_2 t)e^{-\frac{\gamma}{2}t} \qquad (3)$$

As $t \to \infty$, $x(t) \to 0$

For the case $\underline{\gamma > 2\omega_0}$ the solutions of the auxiliary equation are $m_1 = -\alpha_1$, $m_2 = -\alpha_2$ where

$$\alpha_1 = \frac{\gamma}{2} + \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}, \quad \alpha_2 = \frac{\gamma}{2} - \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$$

and so the solution is of the form

$$x(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} \qquad (3)$$

As $t \to \infty$, $x(t) \to 0$.

Hence all these solutions to the homogeneous equation $\to 0$ as $t \to \infty$. Therefore we must investigate the particular solution. The trial function is

$$x(t) = A \cos \omega t + B \sin \omega t \qquad (1)$$

Substitution in the ODE gives

$$-\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$
$$- \gamma \omega A \sin \omega t + \gamma \omega B \cos \omega t$$
$$+ \omega_0^2 A \cos \omega t + \omega_0^2 B \sin \omega t$$
$$= f_0 \cos \omega t \qquad (2)$$

Equating terms in $\sin \omega t$:

$$-\omega^2 B - \gamma \omega A + \omega_0^2 B = 0 = B(\omega_0^2 - \omega^2) - A\gamma\omega \qquad (1)$$

Equating terms in $\cos \omega t$:

$$-\omega^2 A + \gamma \omega B + \omega_0^2 A = f_0 = A(\omega_0^2 - \omega^2) + B\gamma\omega \qquad (1)$$

The solution is

$$A = \frac{f_0(\omega_0^2 - \omega^2)}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]} \; ; \; B = \frac{f_0 \gamma \omega}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]} \; ; \qquad (2)$$

Now, writing

$$x = A\cos\omega t + B\sin\omega t = C\cos(\omega t + \alpha)$$

$$= C\cos\alpha\cos\omega t - C\sin\alpha\sin\omega t \qquad (1)$$

we have

$$\tan\alpha = -\frac{B}{A} = \frac{\gamma\omega}{\omega^2 - \omega_0^2} \qquad (1)$$

and
$$C^2 = A^2 + B^2 = \frac{f_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]} \qquad (1)$$

so that we have

$$x(t) = \frac{f_0}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right]^{1/2}}\cos(\omega t + \alpha) \qquad (1)$$

where
$$\alpha = \tan^{-1}\left(\frac{\gamma\omega}{\omega^2 - \omega_0^2}\right)$$

[Note: the amplitude of the oscillation peaks when the forcing frequency ω equals the natural frequency ω_0. This is the phenomenon of <u>resonance</u>.]

3 + 3 + 3 + 1 + 2 + 1 + 1 + 2 + 1 + 1 + 1 + 1
= 20

Q4. $$\frac{d^2y}{dx^2} + y = \sin 2x = F(x)$$

where $y(0) = y(\pi/2) = 0$

The homogeneous equation is $\frac{d^2y}{dx^2} = -y$

$\Rightarrow y(x)$ is $\cos x$ or $\sin x$

$y_1(0) = 0 \quad \Rightarrow y_1(x) = \sin x \; ; \; y_1' = \cos x$ (2)

$y_2(\pi/2) = 0 \quad \Rightarrow y_2(x) = \cos x \; ; \; y_2' = -\sin x$

$$W(x) = y_1 y_2' - y_1' y_2 = (\sin x)(-\sin x) - (\cos x)(\cos x)$$

$$= -\sin^2 x - \cos^2 x = -1$$ (1)

So $$y_2(x) \int_0^x \frac{y_1(x') F(x')\, dx'}{W(x')} = \cos x \int_0^x \frac{\sin x' . \sin 2x'}{(-1)} dx'$$

$$= -\cos x \int_0^x \sin x' . \sin 2x' \, dx'$$

$$= -2 \cos x \int_0^x \sin x' . \sin x' \cos x' \, dx'$$

$$= -2 \cos x \int_0^x \sin^2 x' \, d(\sin x') = -2 \cos x \left[\frac{1}{3} \sin^3 x' \right]_0^x$$

$$= -\frac{2}{3} \cos x . \sin^3 x$$ (2)

$$y_1(x) \int_x^{\pi/2} \frac{y_2(x') F(x')\, dx'}{W(x')} = \sin x \int_x^{\pi/2} \frac{(\cos x') . \sin 2x'}{(-1)} dx'$$

$$= -\sin x \int_x^{\pi/2} \cos x' \cdot \sin 2x' \, dx'$$

$$= -2\sin x \int_x^{\pi/2} \sin x' \cos^2 x' \, dx'$$

$$= -2\sin x \int_x^{\pi/2} \cos^2 x' \, d(-\cos x') = 2\sin x \left[\frac{1}{3}\cos^3 x'\right]_x^{\pi/2}$$

2

$$= -\frac{2}{3}\sin x \cdot \cos^3 x$$

$$y(x) = y_1(x) \int_x^{\pi/2} \frac{y_2(x') F(x')}{W(x')} dx' + y_2(x) \int_0^x \frac{y_1(x') F(x')}{W(x')} dx'$$

$$= -\frac{2}{3}\sin x \cdot \cos^3 x - \frac{2}{3}\cos x \cdot \sin^3 x$$

$$= -\frac{2}{3}\sin x \cos x \left(\cos^2 x + \sin^2 x\right)$$

$$= -\frac{1}{3} \cdot \sin 2x$$

3

$2 + 1 + 2 + 2 + 3 = 10$