SPA5218 MATHEMATICAL TECHNIQUES 3

EXERCISE SHEET 6 : SOLUTIONS

Q1.
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

$$y = e^{\lambda x} \; ; \quad \Rightarrow \quad \frac{dy}{dx} = \lambda e^{\lambda x} \, , \quad \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$$

The DE becomes
$$(\lambda^2 + P\lambda + Q)e^{\lambda x} = 0$$

which is solved if $\lambda^2 + P\lambda + Q = 0$

If $P^2 \neq 4Q$ the general solution to $\lambda^2+P\lambda+Q=0$ is given by the distinct solutions
$$\lambda_+ = -\frac{P}{2} + \frac{1}{2}\sqrt{P^2-4Q}$$

$$\lambda_- = -\frac{P}{2} - \frac{1}{2}\sqrt{P^2-4Q}$$

and so the general solution of the differential equation is
$$y = Ae^{\lambda_+ x} + Be^{\lambda_- x}$$

If $P^2 = 4Q$ the equation $\lambda^2 + P\lambda + Q = 0$ has a double root at $\lambda = -P/2$.

Let us show that $y = Ae^{\lambda x} + Bxe^{\lambda x}$ is a solution :

$$\frac{dy}{dx} = \lambda A e^{\lambda x} + B e^{\lambda x} + B\lambda x e^{\lambda x}$$

$$= \lambda y + B e^{\lambda x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \lambda \frac{dy}{dx} + B\lambda e^{\lambda x}$$

$$= \lambda^2 y + 2B\lambda e^{\lambda x}$$

So the DE becomes (LHS)

$$\lambda^2 y + 2\lambda B e^{\lambda x} + P(\lambda y + B e^{\lambda x}) + Q y$$

$$= \underbrace{(\lambda^2 + P\lambda + Q)}_{\substack{\text{This part is zero since} \\ \lambda \text{ is a root of } \lambda^2 + P\lambda + Q = 0}} y + \underbrace{(2\lambda + P)}_{\substack{\text{This is also zero} \\ \text{since } \lambda = -P/2}} B e^{\lambda x}$$

$= 0$ and so $y = A e^{\lambda x} + B x e^{\lambda x}$ satisfies the differential equation.

Q2. $\dfrac{dy}{dx} = y$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

But $\dfrac{dy}{dx} = y$ and so

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$$

and so by comparing powers of x we have

$$(n+1)a_{n+1} = a_n$$

$$\Rightarrow \quad a_{n+1} = \frac{a_n}{n+1}$$

Since we are given $a_0 = 1$, we have

$$a_1 = \frac{a_0}{1+0} = \frac{1}{1+0} = 1$$

$$a_2 = \frac{a_1}{1+1} = \frac{1}{2}$$

$$a_3 = \frac{a_2}{2+1} = \frac{1/2}{2+1} = \frac{1}{6}$$

$$a_4 = \frac{a_3}{3+1} = \frac{1/6}{3+1} = \frac{1}{24}$$

Q3.

(a)

$$a_{m+2} = \frac{2(m-\alpha)}{(m+1)(m+2)} a_m$$

$\underline{m=0}$: $a_2 = \dfrac{2(-\alpha)}{1.2} a_0 = -\alpha a_0$

$\underline{m=2}$: $a_4 = \dfrac{2(2-\alpha)}{3.4} a_2 = \dfrac{-\alpha(2-\alpha)}{6} a_0$

$\underline{m=4}$: $a_6 = \dfrac{2(4-\alpha)}{5.6} a_4 = -\dfrac{\alpha(-\alpha+2)(-\alpha+4)}{90} a_0$

(b) Relation is

$$a_{2j} = \frac{2^{j}(-\alpha)(-\alpha+2)\dots(-\alpha+2j-2)}{(2j)!}a_0$$

Assume relation is true for $j = J$

i.e. $$a_{2J} = \frac{2^{J}(-\alpha)(-\alpha+2)\dots(-\alpha+2J-2)}{(2J)!}a_0$$

We want to prove that this implies the relation is also true for $j = J+1$

$$a_{2(J+1)} = \frac{2^{(J+1)}(-\alpha)(-\alpha+2)\dots(-\alpha+2(J+1)-2)}{(2(J+1))!}a_0$$

But $a_{2(J+1)} = a_{2J+2}$

$$= \frac{2(2J-\alpha)}{(2J+1)(2J+2)}a_{2J}$$

$$= \frac{2(2(J+1)-2-\alpha)}{(2J+1)(2J+2)}a_{2J}$$

$$= \frac{2(-\alpha+2(J+1)-2)}{(2J+1)(2J+2)}\left[\frac{2^{J}(-\alpha)(-\alpha+2)\dots(-\alpha+2J-2)}{(2J)!}a_0\right]$$

But $2 \cdot 2^{J} = 2^{J+1}$ and $(2J+1)(2J+2)((2J)!)$
$= 2(J+1)!$

and so

$$a_{2(J+1)} = \frac{2^{J+1}(-\alpha)(-\alpha+2)\dots(-\alpha+2(J+1)-2)}{(2(J+1))!}\,a_0$$

Finally, when $2j = 2$ we have

$$a_2 = \frac{2^1(-\alpha)}{2!}\,a_0 = -\alpha\, a_0 \checkmark \quad \text{(from part(a))}$$

Therefore the relation is true for all j.

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HOMEWORK SHEET 6 : SOLUTIONS

Q1. $$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - xy = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \quad (1)$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n \quad (1)$$

Substituting in the differential equation gives

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}x^n - (n+1)a_{n+1}x^{n+2} - a_n x^{n+1}\right] = 0 \quad (1)$$

We can re-write the x^{n+2} term as

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} = \sum_{m=2}(m-1)a_{m-1}x^m$$

We can re-write the x^{n+1} term as

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{m=1}^{\infty} a_{m-1}x^m$$

(2)

i.e. $$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n - \sum_{n=1}^{\infty}a_{n-1}x^n = 0$$

$\underline{x^0}$: $2a_2 = 0$

$\underline{x^1}$: $3.2a_3 - a_0 = 0$

$\underline{x^2}$: $4.3a_4 - a_1 - a_1 = 0$

$\underline{x^3}$: $5.4a_5 - 2a_2 - a_2 = 0$

$\underline{x^4}$: $6.5a_6 - 3a_3 - a_3 = 0$

$\underline{x^n}$: $(n+2)(n+1)a_{n+2} - (n-1)a_{n-1} - a_{n-1} = 0$

This gives

$$a_2 = 0$$ (1)

$$a_3 = \frac{1}{3!}a_0$$ (1)

$$a_4 = \frac{1}{6}a_1$$ (1)

$$a_5 = \frac{3a_2}{20} = 0$$ (1)

$$a_6 = \frac{4}{30}a_3 = \frac{2}{15}a_3 = \frac{4^2}{6!}a_0$$ (1)

In general

$$a_{n+2} = \frac{n}{(n+2)(n+1)}a_{n-1}$$ (2)

or $$a_{n+3} = \frac{n+1}{(n+3)(n+2)} a_n$$

We can write this as

$$a_{n+3} = \frac{(n+1)^2}{(n+3)(n+2)(n+1)} a_n$$ (multiplying top & bottom by $n+1$)

Since $a_2 = 0$ we have

$$a_5 = a_8 = a_{11} = \cdots = 0$$ 1

With $n = 0, 3, 6, 9, \ldots$ we have

$$a_3 = \frac{1}{3!} a_0$$ 1

$$a_6 = \frac{4^2}{6!} a_0$$ 1

$$a_9 = \frac{7^2}{9.8.7} a_6 = \frac{7^2.4^2}{9!} a_0$$ 1

Similarly we find $a_4, a_7, \ldots$ in terms of a_1

$$a_4 = \frac{2^2}{4!} a_1$$ 1

$$a_7 = \frac{5^2}{7.6.5} a_4 = \frac{5^2.2^2}{7!} a_1$$ 1

$$a_{10} = \frac{8^2}{10.9.8} a_7 = \frac{(8.5.2)^2}{10!} a_1$$ 1

Thus the solution of the differential equation is

$$y = a_0\left[1 + \frac{1}{3!}x^3 + \frac{4^2}{6!}x^6 + \frac{(7.4)^2}{9!}x^9 + \cdots\right]$$

$$+ a_1\left[x + \frac{2^2}{4!}x^4 + \frac{(5.2)^2}{7!}x^7 + \frac{(8.5.2)^2}{10!}x^{10} + \cdots\right]$$

1

1 + 1+1+2 +1+1+1+1+1+2 +1+1+1+1+1+1+1+1+1 = 20

Q2. $(x^2+1)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \qquad 1$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n \qquad 1$$

Substituting in the differential equation gives

$$\sum_{n=0}^{\infty}\Big[(n+1)(n+2)a_{n+2}x^{n+2} + (n+1)(n+2)a_{n+2}x^n$$

$$- 2(n+1)a_{n+1}x^{n+1} + 2a_n x^n\Big] = 0 \qquad 1$$

We can re-write the x^{n+2} term as

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^{n+2} = \sum_{m=2}^{\infty}(m-1)m\,a_m x^m$$

We can re-write the x^{n+1} term as

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} = \sum_{m=1}^{\infty} m a_m x^m$$

i.e. $$\sum_{n=2}^{\infty}(n-1)(n)a_n x^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$$

$$-2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0 \qquad 2$$

$\underline{x^0}$: $2a_2 + 2a_0 = 0$ 1

$\underline{x^1}$: $2.3a_3 - 2a_1 + 2a_1 = 0$ 1

$\underline{x^2}$: $2a_2 + 3.4a_4 - 4a_2 + 2a_2 = 0$ 1

$\underline{x^3}$: $2.3a_3 + 4.5a_5 - 6a_3 + 2a_3 = 0$ 1

$\underline{x^n}$: $n(n-1)a_n + (n+1)(n+2)a_{n+2} - 2na_n + 2a_n = 0$ 1

This gives

$2(a_0 + a_2) = 0 \Rightarrow a_2 = -a_0$ 1

$6a_3 = 0 \Rightarrow a_3 = 0$

$12a_4 = 0 \Rightarrow a_4 = 0$

$20a_5 + 2a_3 = 0 \Rightarrow a_5 = 0$

$$(n+1)(n+2)a_{n+2} + (n-1)(n-2)a_n = 0$$

Since $a_3 = a_4 = 0$ and $a_5 = 0$ this means

$$a_n = 0 \text{ for all } n \geq 3$$ (2)

The coefficients ~~of~~ a_0 and a_1 are arbitrary and

$a_2 = -a_0$.

Hence the general solution is

$$y_0 = a_0(1-x^2) + a_1 x$$ (2)

$1+1+1+2+1+1+1+1+1+1+2+2 = 15$

Q3. $$x^2 \frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0$$

Let $y(x) = \sum_{j=0}^{\infty} a_j x^{j+c}$

$$\frac{dy}{dx} = \sum_{j=0}^{\infty} a_j (j+c) x^{j+c-1}$$ (1)

$$\frac{d^2y}{dx^2} = \sum_{j=0}^{\infty} a_j (j+c)(j+c-1) x^{j+c-2}$$ (1)

Substituting these in the differential equation gives

$$\sum_{j=0}^{\infty} a_j \Big[(j+c)(j+c-1)x^{j+c-1} + (j+c)x^{j+c-1} - (j+c)x^{j+c} + \lambda x^{j+c} \Big] = 0$$ (1)

The lowest power of x is x^{c-1} which occurs when $j = 0$

Equating the coefficients of x^{c-1} to zero gives

$$a_0\left[c(c-1)+c\right] = a_0 c^2 = 0 \qquad (2)$$

$\Rightarrow c = 0$ is a solution. The general power of x^j for $c=0$ has a coefficient

$$a_{j+1}(j)(j+1) + a_{j+1}(j+1) - j a_j + \lambda a_j \quad (*)$$

(2)

(This comes from writing

$$\sum_{j=0}^{\infty}\Big[j(j-1)a_j x^{j-1} + j a_j x^{j-1}$$

$$- j a_j x^j + \lambda a_j x^j\Big]$$

$$= \sum_{j=0}^{\infty}\Big[j(j+1)a_{j+1}x^j + (j+1)a_j x^j$$

$$- j a_j x^j + \lambda a_j x^j\Big] \Bigg)$$

From $(*)$ we have

$$a_{j+1}\big(j(j+1) + (j+1)\big) = (j-\lambda)a_j$$

$$\Rightarrow (j+1)^2 a_{j+1} = (j-\lambda)a_j$$

$$\Rightarrow \quad a_{j+1} = \frac{j-\lambda}{(j+1)^2}a_j \qquad (2)$$

$$\Rightarrow a_1 = (-\lambda)a_0 \qquad (1)$$

$$a_2 = \frac{1-\lambda}{2^2}a_1 = \frac{(-\lambda)(1-\lambda)}{2^2}a_0$$

$$a_3 = \frac{2-\lambda}{3^2} a_2 = \frac{(-\lambda)(1-\lambda)(2-\lambda)}{2^2\,3^2} a_0$$

(1)

$$a_4 = \frac{3-\lambda}{4^2} a_3 = \frac{(-\lambda)(1-\lambda)(2-\lambda)(3-\lambda)}{2^2\,3^2\,4^2} a_0$$

The series for $y(x)$ with $c=0$ is therefore

$$y(x) = a_0\left\{1 - \lambda x - \frac{\lambda(1-\lambda)}{4}x^2 - \frac{\lambda(1-\lambda)(2-\lambda)}{4.9}x^3 - \frac{\lambda(1-\lambda)(2-\lambda)(3-\lambda)}{4.9.16}x^4 - \dots\right\}$$

Therefore the series will always terminate if λ is integer. Taking $a_0 = 1$ we have

(1)

$\underline{\lambda = 0}$ $\quad y(x) = L_0(x) = 1$

$\underline{\lambda = 1}$ $\quad y(x) = L_1(x) = 1 - x$ (1)

$\underline{\lambda = 2}$ $\quad y(x) = L_2(x) = 1 - 2x - \dfrac{2(1-2)}{4}x^2$

$$= 1 - 2x + \frac{1}{2}x^2$$ (1)

$\underline{\lambda = 3}$ $\quad y(x) = L_3(x) = 1 - 3x - \dfrac{3(1-3)}{4}x^2 - \dfrac{3(1-3)(2-3)}{4.9}x^3$

$$= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$ (1)

$1+1+1+2+2+2+1+1+1+1+1+1 = 15$