BSc/MSci EXAMINATION

PHY-218 Mathematical Techniques 3

Time Allowed: 2 hours 30 minutes

Date: 6^{th} May, 2011

Time: 2.30 p.m - 5.00 p.m

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. Course work comprises 40 % of the final mark.**

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. S. Ramgoolam, Dr. A. Brandhuber

This (reverse cover) page is left blank.

SECTION A. Attempt answers to all questions.

A1 Give the definition of linear independence for a set of M vectors $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_M\}$. Use the definition to determine whether the following pairs of vectors are linearly independent.

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad , \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad , \qquad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

[5]

A2 Calculate the Matrix product AB for

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

[5]

A3 For a matrix A, the inverse A^{-1} obeys $AA^{-1} = A^{-1}A = \mathbf{1}$. The transpose of a matrix A is denoted by A^T. Use the fact that $(AB)^T = B^T A^T$ to show that $(A^T)^{-1} = (A^{-1})^T$. **[5]**

A4 For the inner product between wavefunctions defined by

$$(\Psi_1, \Psi_2) = \int_{-\infty}^{\infty} \Psi_1^*(x)\Psi_2(x)dx$$

show that $P = i\frac{d}{dx}$ is a hermitian operator, i.e, it obeys $(\Psi_1, P\Psi_2) = (P\Psi_1, \Psi_2)$. You may assume that the wavefunctions vanish at $\pm\infty$. **[5]**

A5 Prove that

$$\mathcal{L} = \frac{d^2}{dx^2} + P(x)\frac{d}{dx}$$

is a linear operator. **[5]**

A6 $\mathcal{L}$ is taken to be defined as in the previous question. The inhomogeneous equation

$$\mathcal{L}y(x) = F(x)$$

is solved by the particular solution $y_p(x)$. $y_c(x)$ is a complementary function solving the corresponding homogeneous equation. Show that $y_p(x) + y_c(x)$ is also a solution of the inhomogeneous equation. Show also that for a fixed $y_p(x)$, this form gives the most general solution of the inhomogenous equation. **[5]**

A7 Solve the differential equation

$$x^2\frac{dy}{dx} + xy^2 = 5y^2$$

[5]

A8 By extremizing

$$I = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt$$

under variations of of the paths $x(t)$ subject to the condition that $\delta x(t_1) = \delta x(t_2) = 0$, derive the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

[5]

A9 For the Hermite differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\alpha y = 0$$

where α is a constant, consider the series solution ansatz $y = \sum_{n=0}^{\infty} a_n x^{k+n}$ and derive the indicial equation $k(k-1) = 0$. **[5]**

A10 Derive the contour integrals

$$\begin{aligned} \oint \frac{dz}{z} &= 2\pi i \\ \oint \frac{dz}{z^2} &= 0 \end{aligned}$$

where the contour is a unit circle in the complex plane, centred at the origin. **[5]**

SECTION B. Answer only TWO of the four questions in this section.

B1

(a) For two vectors in $\mathbb{C}^N$,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \end{pmatrix} \quad , \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \cdots \\ w_N \end{pmatrix}$$

the standard inner product is

$$(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{N} v_i^* w_i$$

Show that the following properties of inner products hold :

$$\begin{aligned} &(\mathbf{v}, \lambda\mathbf{w}) = \lambda(\mathbf{v}, \mathbf{w}) \text{ for any } \lambda \in \mathbb{C} \\ &(\lambda\mathbf{v}, \mathbf{w}) = \lambda^*(\mathbf{v}, \mathbf{w}) \end{aligned}$$

[4+4]

(b) A linear operator H is said to be hermitian if

$$(\mathbf{v}, H\mathbf{w}) = (H\mathbf{v}, \mathbf{w})$$

for any pair of vectors $\mathbf{v}, \mathbf{w}$.

An eigenvector $\mathbf{v}$ of H, with eigenvalue λ obeys the equation

$$H\mathbf{v} = \lambda\mathbf{v}$$

Prove that eigenvalues of hermitian operators are real, and that two eigenvectors with distinct eigenvalues are orthogonal. **[5+7]**

(c) A unitary operator U obeys

$$(\mathbf{v}, U\mathbf{w}) = (U^{-1}\mathbf{v}, \mathbf{w})$$

for any pair of vectors $\mathbf{v}, \mathbf{w}$. The matrix elements of the operator U with respect to an orthonormal basis $\{\mathbf{e}_i\}$ are defined by

$$U\mathbf{e}_i = \sum_j U_{ji}\mathbf{e}_j$$

Show that $U^{-1}_{ij} = U^*_{ji}$. **[5]**

B2

The Dirac delta function obeys the property

$$\int_{-\infty}^{\infty} F(x)\delta(x-x')dx = F(x')$$

Prove the following properties of the Dirac delta function :

(i)

$$\delta(x) = \delta(-x)$$

[3]

(ii)

$$x\delta(x) = 0$$

[3]

(iii)

$$x\delta'(x) = -\delta(x)$$

[4]

(iv)

$$\delta(ax) = \frac{1}{|a|}\delta(x) \text{ for real } a \neq 0$$

[5]

(v) Calculate the integral

$$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty}(z+1)\sin(x+z)\cos(y)\delta(x-\frac{\pi}{2})\delta(y-\pi)\delta(z)dxdydz$$

[5]

(vi) By considering the regions on the x-axis near the roots of the polynomial $(x+2)(x^2-1)$, calculate

$$\int_{-\infty}^{\infty} dx\ \delta((x+2)(x^2-1))$$

[5]

B3

The equation governing heat flow in one space dimension is

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t}$$

where $\alpha > 0$ is a constant related to the thermal conductivity.

(i) Use separation of variables to obtain a general form for the solutions. **[5]**

(ii) A metal bar is placed along the x-axis so that its ends are located at $x = 0$ and $x = L$. Both ends are kept at zero temperature at all times

$$u(0,t) = u(L,t) = 0$$

Given these (Dirichlet) boundary conditions, find the most general solution for $u(x,t)$, expressing it as an infinite sum. **[5]**

(iii) For integers $m, n > 0$, prove the result

$$\int_0^L dx \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) = \frac{L}{2}\delta_{n,m}$$

[7]

(iv) Given the temperature distribution at time $t = 0$ is given by $u(x,0) = \delta(x - \frac{L}{4}) + \delta(x - \frac{3L}{4})$, what will be the temperature of the bar at a later time t ? **[8]**

B4

(i) Given a differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$$

for $y(x)$ defined in the interval $a \leq x \leq b$, the Green's Function is required to satisfy

$$[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)]G(x,x') = \delta(x - x')$$

Show that

$$y(x) = \int_a^b G(x,x')F(x')dx'$$

is a particular integral. **[5]**

(ii) For the above differential equation, with boundary conditions $y(a) = y(b) = 0$, the Green's function is known to be :

$$G(x,x') = \frac{y_2(x')y_1(x)}{W(x')} \qquad \text{for} \quad x < x'$$
$$G(x,x') = \frac{y_1(x')y_2(x)}{W(x')} \qquad \text{for } x > x'$$

where $W(x') = y_1(x')y_2'(x') - y_2(x')y_1'(x')$. and $y_1(x), y_2(x)$ are solutions of the homogeneous equation.

Show that

$$y(x) = y_2(x)\int_a^x \frac{y_1(x')F(x')}{W(x')}dx' + y_1(x)\int_x^b \frac{y_2(x')F(x')}{W(x')}dx'$$

solves the differential equation. **[5]**

Show, using the conditions $y_1(a) = y_2(b) = 0$, that $y(x)$ satisfies the boundary conditions. **[5]**

(iii) For the equation

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = (x^2+1)^2$$

in the interval $0 \leq x \leq 1$, obtain the solution obeying the boundary conditions $y(0) = y(1) = 0$. You may use the fact that x and $(1 - x^2)$ are solutions of the homogeneous equation. **[10]**