6. FUNCTIONS OF A COMPLEX VARIABLE

6.1 INTRODUCTION

For $z = x + iy \in \mathbb{C}$, the complex function $f(z) \in \mathbb{C}$ has real and imaginary parts:

$$f(z) = u(x,y) + i\,v(x,y)$$

e.g. $f(z) = z^2 = (x+iy)^2$

$$= x^2 - y^2 + 2ixy$$

$$= u(x,y) + i\,v(x,y)$$

where $u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

Just as a complex number $z = x + iy$ is equivalent to a pair of real numbers x, y, so a function of z is equivalent to a pair of real functions $u(x,y)$ and $v(x,y)$ of the real variables x and y.

e.g. $f(z) = e^z = e^{x+iy}$

$$= e^x e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

$$\Rightarrow \quad u(x,y) = e^x \cos y$$

$$v(x,y) = e^x \sin y$$

e.g. $f(z) = e^{iz} = e^{i(x+iy)} = e^{-y+ix}$

$$= e^{-y}(\cos x + i \sin x)$$

$$\Rightarrow \quad u(x,y) = e^{-y} \cos x$$

$$v(x,y) = e^{-y} \sin x$$

e.g. $f(z) = \cosh z = \frac{1}{2}\left(e^z + e^{-z}\right)$

$$= \frac{1}{2}\left(e^{x+iy} + e^{-x-iy}\right)$$

$$= \frac{1}{2}\Big(e^x(\cos y + i \sin y) + e^{-x}(\cos(-y) + i \sin(-y))\Big)$$

$$= \frac{1}{2} e^x \cos y + \frac{1}{2} e^{-x} \cos y + \frac{i}{2} e^x \sin y - \frac{i}{2} e^{-x} \sin y$$

$$= \cosh x \cos y + i \sinh x \sin y$$

$\Rightarrow \quad u(x,y) = \cosh x \cos y$

$\quad\quad v(x,y) = \sinh x \sin y$

e.g. $f(z) = \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

$$= \frac{1}{2} e^{i(x+iy)} + \frac{1}{2} e^{-i(x+iy)}$$

$$= \frac{1}{2} e^{-y+ix} + \frac{1}{2} e^{y-ix}$$

$$= \frac{1}{2} e^{-y}\left(\cos x + i \sin x\right)$$

$$+ \frac{1}{2} e^{y}\left(\cos(-x) + i \sin(-x)\right)$$

$$= \left(\frac{1}{2} e^{-y} + \frac{1}{2} e^{y}\right)\cos x$$

$$+ \left(\frac{1}{2} e^{-y} - \frac{1}{2} e^{y}\right) i \sin x$$

$\Rightarrow \quad u(x,y) = \cosh y \cos x$

$\quad\quad v(x,y) = -\sinh y \sin x$

Functions are usually single-valued, i.e. $f(z)$ has just one (complex) value for each value of z.

But $\quad z = |z| e^{i\theta}$ where $|z| = \sqrt{x^2 + y^2}$

and $\theta = \tan^{-1} \frac{y}{x}$ $\quad$ ($|z|$ is modulus of z
θ is argument of z)

$$\Rightarrow \ln z = \ln\left(|z| e^{i\theta}\right)$$

$$= \ln|z| + i\left(\theta + 2n\pi\right)$$

$$= \frac{1}{2}\ln\left(x^2+y^2\right) + i\left(\theta + 2n\pi\right)$$

Therefore, for each z, $\ln z$ has an infinite set of values. If θ is allowed a range of only 2π then $\ln z$ has one value for each z and is single-valued.

6.2 ANALYTIC FUNCTIONS

For a real function, $f(x)$ we had

$$f'(x) = \frac{df}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

for the derivative.

For a complex function, $f(z)$ we have

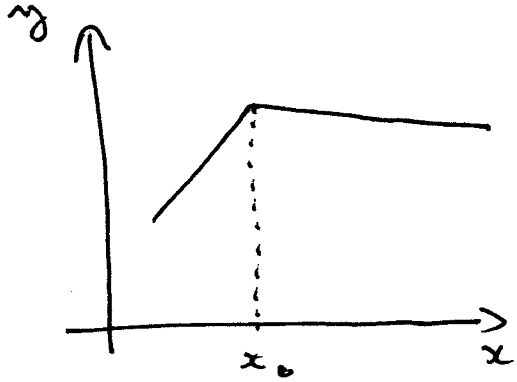
$$f'(z) = \frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

where $\delta z = \delta x + i\,\delta y$

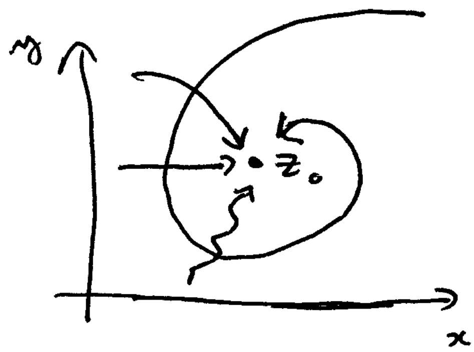
a function $f(z)$ is analytic (or regular or holomorphic or monogenic) in a region of the complex plane if it has a unique derivative at every point of the region.

The statement "$f(z)$ is analytic at a point $z = a$" means that $f(z)$ has a derivative at every point inside some small circle about $z = a$.

Note that isolated points and curves are not regions; a region must be two-dimensional.



For a function $f(x)$ it is possible for the limit $\delta f/\delta x$ to have two values at a point x_0 depending on whether we approach from the left or right.



For a function $f(z)$ there are an infinite number of ways we can approach a point z_0. When we say that $f(z)$ has a derivative at $z = z_0$ we mean that $f'(z)$ has the same value no matter how we approach z_0.

e.g. To find $f'(z)$ where $f(z) = z^2$ we have

$$\frac{d}{dz}(z^2) = \lim_{\delta z \to 0} \frac{(z+\delta z)^2 - z^2}{\delta z} = \lim_{\delta z \to 0} \frac{z^2 + 2z\delta z + (\delta z)^2 - z^2}{\delta z}$$

$$= \lim_{\delta z \to 0} (2z + \delta z) = 2z.$$

This result is independent of how δz tends to zero. Therefore $f(z) = z^2$ is an analytic function.

Similarly, $\frac{d}{dz} z^n = n z^{n-1}$ where n is a positive integer.

Now consider $\frac{d}{dz}(|z|^2)$. We note that $|x|^2 = x^2$ and its derivative is $2x$. If $|z|^2$ has a derivative then

$$\frac{d}{dz}(|z|^2) = \lim_{\delta z \to 0} \frac{|z+\delta z|^2 - |z|^2}{\delta z}$$

Then $|z+\delta z|^2 - |z|^2$ is always real (because absolute values are always real). But consider $\delta z = \delta x + i\delta y$. As we approach z_0 (see fig.), i.e. as $\delta z \to 0$, δz has different values depending on the approach. E.g. if we approach along a horizontal line then $\delta y = 0$ and $\delta z = \delta x$; along a vertical line $\delta x = 0$ and $\delta z = i\delta y$ and along other directions δz is some complex number so is neither real nor purely imaginary. Therefore, even though $|z+\delta z|^2 - |z|^2$ is always real, the denominator δz is in general complex. Hence $\lim_{\delta z \to 0} \delta f/\delta z$ has different values for different directions of approach to z_0 and so $|z|^2$ is not analytic.

However, we can use the Cauchy-Riemann conditions to determine whether or not a function has a derivative.

If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a region, then in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} , \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} .$$

These are the Cauchy-Riemann conditions.

<u>PROOF</u>: $f = f(z)$, $z = x + iy$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{df}{dz} \cdot \frac{\partial z}{\partial x} = \frac{df}{dz} \cdot 1$$

$$\frac{\partial f}{\partial y} = \frac{df}{dz} \cdot \frac{\partial z}{\partial y} = \frac{df}{dz} \cdot i$$

But $f = u(x,y) + i\,v(x,y)$ and so

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Combining these we have

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{df}{dz} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Since we assumed that df/dz exists and is unique (i.e. analytic) these two expressions must be equal.

$$\Rightarrow \quad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Comparing real and imaginary parts:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

It can be shown that if $u(x,y)$ and $v(x,y)$ and their partial derivatives w.r.t. x and y are continuous and satisfy the Cauchy-Riemann conditions in a region, then $f(z)$ is analytic at all points inside the region.

e.g. Find df/dz assuming we approach z_0 along a straight line of slope m and show that df/dz does not depend on m if u and v satisfy the Cauchy-Riemann conditions.

The equation of the straight line is

$$y = m(x - x_0) + y_0 .$$

Along this line we have

$$\frac{dy}{dx} = m.$$

$$\Rightarrow \frac{df}{dz} = \frac{du + i\,dv}{dx + i\,dy}$$

$$= \frac{\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + i\left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right)}{dx + i\,dy}$$

$$= \frac{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} m + i\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} m\right)}{1 + im}$$

Using the Cauchy-Riemann conditions we get

$$\frac{df}{dz} = \frac{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} m + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} m\right)}{1 + im}$$

$$= \frac{\frac{\partial u}{\partial x}\left(1 + im\right) + i\frac{\partial v}{\partial x}\left(1 + im\right)}{1 + im}$$

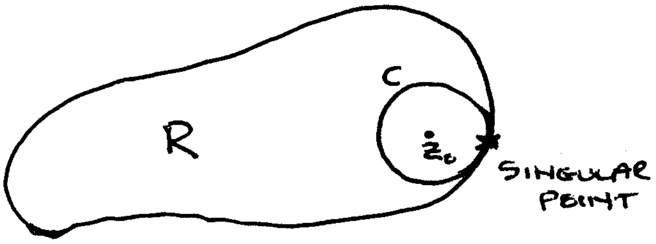
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Hence df/dz has the same value for approach along any curve.

A <u>regular point</u> of $f(z)$ is a point at which $f(z)$ is analytic.
A <u>singular point</u> or <u>singularity</u> of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an <u>isolated</u> singular point if $f(z)$ is analytic everywhere else

inside some small circle about the singular point.

If $f(z)$ is analytic in a region R then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges inside the circle C about z_0 that extends to the nearest singular point.



So if $f(z)$ has a first derivative w.r.t. z then it has derivatives of all orders and they are all analytic functions.

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)$$

$$\text{where} \quad f^{(n)}(z_0) = \left(\frac{d}{dz}\right)^n f(z)\bigg|_{z_0} .$$

Consider the function $f(x) = \dfrac{1}{1+x^2}$. There is nothing unusual about its behavior at $x = \pm 1$. However, if we expand it in a power series,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \qquad (1)$$

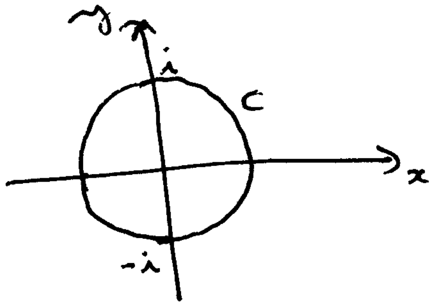
we see that the series only converges for $|x| < 1$.

We can see why this happens by considering

$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots \qquad (2)$$

When $z = \pm i$, $f(z)$ and its derivatives become infinite

$\Rightarrow$ $f(z)$ is not analytic in any region containing $z = \pm i$.



The point z_0 is the origin and the circle C (bounding the disk of convergence of the series) passes through the nearest singular points $\pm i$.

Since a power series in z always converges inside its disk of convergence and diverges outside, we see that (1) (which is (2) for $y = 0$) converges for $|x| < 1$ and diverges for $|x| > 1$. Hence our study of $f(z)$ gives us insights about the corresponding $f(x)$. Many formulas involving functions are more easily derived and understood by considering them as functions of z.

A function $\phi(x, y)$ which satisfies Laplace's equation in two dimensions, i.e.

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

is called a harmonic function. It can be shown that if $f(z) = u + iv$ is analytic in a region then u and v satisfy Laplace's equation in the region (that is, u and v are harmonic functions).

A corollary is that any function u or v satisfying Laplace's equation in a simply connected region is the real or imaginary part of an analytic function.

e.g. Consider the function $u(x,y) = x^2 - y^2$.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\Rightarrow$ u satisfies Laplace's equation.

Now we will find the function $v(x,y)$ such that $u + iv$ is an analytic function of z. By the Cauchy-Riemann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x$$

Integrating partially w.r.t. y we have

$$v(x,y) = 2xy + g(x)$$

where $g(x)$ is a function of x to be determined

$$\Rightarrow \frac{\partial v}{\partial x} = 2y + g'(x) = -\frac{\partial u}{\partial y} \quad \text{(by the Cauchy-Riemann equations)}$$

$$= 2y$$

$$\Rightarrow g'(x) = 0 \quad \text{or} \quad g = \text{const.}$$

$$\Rightarrow f(z) = u + iv = x^2 - y^2 + 2ixy + \text{const.}$$

$$= z^2 + \text{const.}$$

The pair of functions u, v are called <u>conjugate harmonic functions</u>.

6.3 <u>CONTOUR INTEGRALS</u>

Let C be a simple closed curve with a continuously turning tangent except possibly at a finite number of points. <u>Cauchy's Theorem</u> states that if $f(z)$ is analytic on and inside C then

$$\oint_{\text{around } C} f(z)\, dz = 0$$

(This is a contour integral.)

<u>PROOF</u> We shall prove Cauchy's theorem assuming that $f'(z)$ is continuous

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

Green's theorem in the plane states that if $P(x,y)$ and $Q(x,y)$ and their partial derivatives are continuous in a simply connected region R, then

$$\oint_C (P\,dx + Q\,dy) = \iint_{\substack{\text{area}\\ \text{inside } C}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where C is a simple closed curve lying entirely in R. The curve C is traversed in a direction so that the area enclosed is always to the left; the area integral is over the area inside C. Applying the Green's theorem result we get

$$\oint_C (u\,dx - v\,dy) = \iint_{\substack{\text{area}\\ \text{inside } C}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Since $f'(z)$ is assumed to be continuous, then u and v and their derivatives are continuous. Therefore, since the Cauchy-Riemann equations give

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

and so

$$\oint_C (u\,dx - v\,dy) = 0$$

Similarly,

$$\oint_C (v\,dx + u\,dy) = \iint_{\substack{\text{area}\\ \text{inside } C}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

and the Cauchy-Riemann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

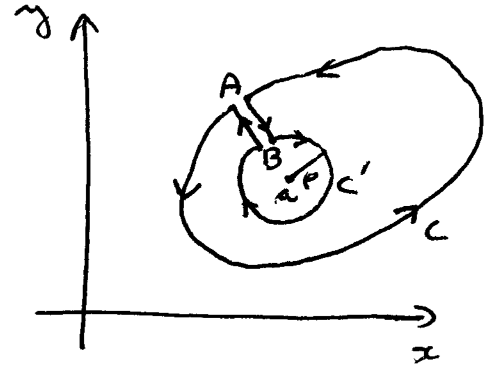
Hence $$\oint_C (v\,dx + u\,dy) = 0$$

$$\Rightarrow \quad \oint_C f(z)\,dz = 0 \quad \text{as required.}$$

<u>Cauchy's integral formula</u>: If $f(z)$ is analytic on and inside a simple closed curve C, the value of $f(z)$ at a point $z = a$ inside C is given by the following contour integral along C:

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\,dz$$

<u>PROOF</u>: Let a be a fixed point inside the simple closed curve C and consider the function



$$\phi(z) = \frac{f(z)}{z-a}$$

where $f(z)$ is analytic on and inside C. Let C' be a small circle (inside C) with centre at a and radius ρ. Make a cut between C and C' along AB. We now integrate along the path from A, around C, to B, around C', and back to A. Notice that the area between C and C' is always to the left of the integration path and is enclosed by it. In this area between C and C', $\phi(z)$ is analytic; we have cut out a small disk about the point $z = a$ at which $\phi(z)$ is not analytic. Cauchy's theorem then applies to the integral along the combined path consisting of C counterclockwise, C' clockwise, and the two cuts. The two integrals, in opposite directions along the cuts, cancel each other when the cuts coincide. Hence

$$\oint_{\substack{C \text{ COUNTER-}\\ \text{CLOCKWISE}}} \phi(z)\,dz + \oint_{C' \text{ CLOCKWISE}} \phi(z)\,dz = 0$$

or

$$\oint_C \phi(z)\,dz = \oint_{C'} \phi(z)\,dz$$ where both are counterclockwise.

Along the circle C', $z = a + \rho e^{i\theta}$, $dz = \rho i e^{i\theta} d\theta$

and so

$$\oint_C \phi(z)\,dz = \oint_{C'} \phi(z)\,dz = \oint_{C'} \frac{f(z)}{z-a}\,dz$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}} \rho i e^{i\theta}\,d\theta = \int_0^{2\pi} f(z)\, i\,d\theta$$

Now let $\rho \to 0$ (i.e. $z \to a$). Because $f(z)$ is continuous at $z = a$ (it is analytic inside C), $\lim_{z \to a} f(z) = f(a)$. Hence

$$\oint_C \phi(z)\,dz = \oint_C \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} f(z)\, i\,d\theta$$

$$= \int_0^{2\pi} f(a)\, i\,d\theta = 2\pi i f(a)$$

or $$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz \qquad (a \text{ inside } C)$$

e.g. $$\oint_{\text{CIRCLE}} z\,dz = \int_0^{2\pi} (re^{i\theta})(rie^{i\theta})\,d\theta$$

$$= ir^2 \int_0^{2\pi} e^{2i\theta}\,d\theta = ir^2\left(\frac{1}{2i}\right)\left[e^{2i\theta}\right]_0^{2\pi}$$

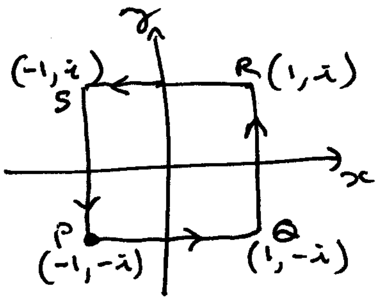
$$= ir^2\left(\frac{1}{2i}\right)(1-1) = 0$$

More generally

$$\oint_{\text{CIRCLE}} z^n \, dz = \frac{r^{n+1}}{n+1}\left[e^{in\theta}\right]_0^{2\pi} = 0$$

(n integer)

e.g. Calculate $\oint_C z \, dz$ around the square joining $(-1,-i)$, $(1,-i)$, $(1,i)$, $(-1,i)$.



We split C into four parts: PQ, QR, RS and SP

(a) PQ: $z = -i + x$, $-1 \le x \le 1$; $dz = dx$

$$\Rightarrow \int_{PQ} z \, dz = \int_{-1}^{1} (-i + x)\, dx = \left[-ix + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= -i - i + \frac{1}{2} - \frac{1}{2} = -2i$$

(b) QR: $z = 1 + iy$, $-1 \le y \le 1$, $dz = i\,dy$

$$\Rightarrow \int_{QR} z \, dz = i\int_{-1}^{1} (1 + iy)\, dy = i\left[y + i\frac{y^2}{2}\right]_{-1}^{1}$$

$$= i\left[1 - (-1) + \frac{i}{2} - \frac{i}{2}\right] = 2i$$

(c) RS: $z = i + x$, $+1 \ge x \ge -1$, $dz = dx$

$$\Rightarrow \int_{RS} z \, dz = \int_{1}^{-1} (i + x)\, dx = \left[ix + \frac{x^2}{2}\right]_{1}^{-1}$$

$$= -i - i + \tfrac{1}{2} - \tfrac{1}{2} = -2i$$

(d) SP : $z = -1 + iy \quad 1 \geq y \geq -1$, $dz = i\,dy$

$$\Rightarrow \int_{SP} z\,dz = i\int_{1}^{-1} (-1 + iy)\,dy = i\left[-y + i\frac{y^2}{2}\right]_1^{-1}$$

$$= i\left(1 + 1 + \frac{i}{2} - \frac{i}{2}\right) = 2i$$

Hence $\displaystyle\oint_{\text{SQUARE}} z\,dz = -2i + 2i - 2i + 2i = 0$

Cauchy's theorem states that if $f(z)$ is analytic, $\oint_C f(z) = 0$ if C is a closed curve. We have verified this for $f(z) = z$.

6.4 <u>LAURENT SERIES</u>

Let C_1 and C_2 be two circles with centre at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

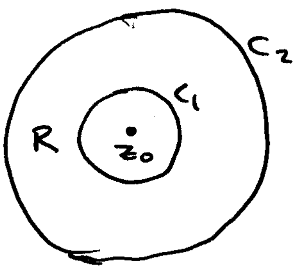
$$\cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

convergent in R. This is <u>Laurent's Theorem</u> and such a series is called a <u>Laurent series</u>. The "b" series is called the <u>principal part</u> of the Laurent series.

e.g. Consider the Laurent series

$$f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots + \left(\frac{z}{2}\right)^n + \dots$$
$$+ \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \dots + \frac{(-1)^n}{z^n} + \dots\right)$$

We can use the ratio test to show that the series of +ve powers converges for $|z/2| < 1$, i.e. $|z| < 2$. The series of -ve powers converges for $|1/z| < 1$, i.e. $|z| > 1$. Therefore the Laurent series converges for $1 < |z| < 2$. This corresponds to a ring between two circles of radii 1 and 2.



The "a" series converges <u>inside</u> some circle C_2.

The "b" series converges <u>outside</u> some circle C_1.

The Laurent series converges (if it converges at all) between the two circles.

The coefficients a_n and b_n can be calculated for a function $f(z)$ using

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}, \quad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$$

where C is any simple closed curve surrounding z_0 and lying in R.

e.g. Find the Laurent series for the function

$$f(z) = \frac{12}{z(2-z)(1+z)}$$

This function has three singular points:

$$z = 0, \quad z = 2 \text{ and } z = -1$$

Thus there are two circles C_1 and C_2 about $z_0 = 0$ and three Laurent series about $z_0 = 0$, one valid in each of the 3 regions

$R_1 \quad (0 < |z| < 1)$

$R_2 \quad (1 < |z| < 2)$

$R_3 \quad (|z| > 2)$

First we express $f(z)$ using partial fractions

$$f(z) = \frac{4}{z}\left(\frac{1}{1+z} + \frac{1}{2-z}\right)$$

<u>CHECK</u> :

$$\frac{12}{z(2-z)(1+z)} = \frac{1}{z}\left(\frac{A}{2-z} + \frac{B}{1+z}\right)$$

$$= \frac{1}{z}\left(\frac{A(1+z) + B(2-z)}{(2-z)(1+z)}\right) = \frac{1}{z}\left(\frac{(A+2B) + (A-B)z}{(2-z)(1+z)}\right)$$

$$\Rightarrow \left.\begin{array}{l} A + 2B = 12 \\ A - B = 0 \end{array}\right\} \Rightarrow 3B = 12, \; B = 4, \; A = 4$$

Now $(1+z)^{-1} = 1 - z + z^2 - z^3 + \dots$

$$(2-z)^{-1} = \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$

$$= \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \cdots$$

$$\Rightarrow \frac{4}{z}\left(\frac{1}{(1+z)} + \frac{1}{(2-z)}\right)$$

$$= \frac{4}{z}\Big(1 - z + z^2 - z^3 + \cdots + \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \cdots\Big)$$

$$= \frac{6}{z} - 3 + \frac{9z}{2} - \frac{15z^2}{4} + \frac{33z^3}{8} + \cdots$$

This is valid in R_1 $(0 < |z| < 1)$

For R_3 $(|z| > 2)$ we have

$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{1+1/z}, \quad \frac{1}{2-z} = -\frac{1}{z}\frac{1}{1-2/z}$$

and expanding each in powers of $1/z$ we have:

$$\frac{1}{1+z} = \frac{1}{z}(1+1/z)^{-1} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots$$

$$\frac{1}{2-z} = -\frac{1}{z}(1 - 2/z)^{-1} = -\frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \cdots\right)$$

$$= -\frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3} - \frac{8}{z^4} - \cdots$$

$$\Rightarrow \frac{4}{z}\left(\frac{1}{(1+z)} + \frac{1}{(2-z)}\right)$$

$$= \frac{4}{z}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\right.$$

$$\left. - \frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3} - \frac{8}{z^4} - \cdots\right)$$

$$= \frac{4}{z}\left(-\frac{3}{z^2} - \frac{3}{z^3} - \frac{9}{z^4} + \cdots\right)$$

$$= -\frac{12}{z^3}\left(1 + \frac{1}{z} + \frac{3}{z^2} + \frac{5}{z^3} + \frac{11}{z^4} + \cdots\right)$$

This is valid in R_3 $(|z| > 2)$.

In R_2 $(1 < |z| < 2)$ we expand $1/(2-z)$ in powers of z and $1/(1+z)$ in powers of $1/z$.

$$\frac{1}{1+z} = \frac{1}{z}\left(1 + 1/z\right)^{-1} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots$$

$$\frac{1}{2-z} = \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} = \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$

$$= \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \cdots$$

$$\Rightarrow \frac{4}{z}\left(\frac{1}{(1+z)} + \frac{1}{(2-z)}\right)$$

$$= \frac{4}{z}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots \right.$$

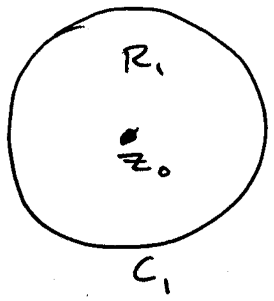
$$\left. + \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \cdots \right)$$

$$= 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots$$

$$+ \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \cdots\right)$$

This is valid in R_2 $(1 < |z| < 2)$ and is the series used in the example above.

The 3 series all represent $f(z)$ in $f(z) = \frac{12}{z(2-z)(1+z)}$ but in three different regions.



Let z_0 be either a regular point or an isolated singular point and assume that there are no other singular points inside C_1. Let $f(z)$ be expanded in the Laurent series about z_0 which converges inside C_1 (except possibly at z_0). We say that we have expanded $f(z)$ in the Laurent series which converges near z_0. We have the following definitions:

If all the b's are zero, $f(z)$ is analytic at $z = z_0$ and we call z_0 a <u>regular point</u>.

If $b_n \neq 0$ but all b's after b_n are zero, $f(z)$ is said to have a pole of order n at $z = z_0$. If $n = 1$ we say that $f(z)$ has a simple pole.

If there are an infinite number of b's different from zero, $f(z)$ has an essential singularity at $z = z_0$.

The coefficient b_1 of $1/(z - z_0)$ is called the residue of $f(z)$ at $z = z_0$.

e.g. $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

is analytic at $z = 0$; the residue at $z = 0$ is 0.

e.g. $$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!\,z} + \frac{1}{3!} + \cdots$$

has a pole of order 3 at $z = 0$; the residue of e^z/z^3 at $z = 0$ is $1/2!$.

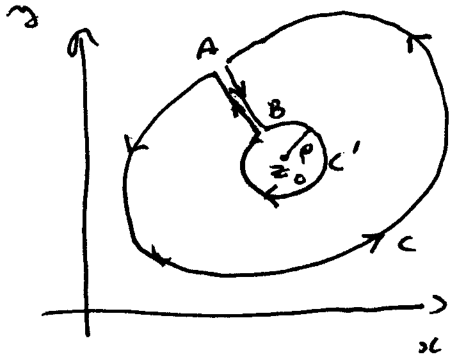
e.g. $$e^{1/z} = 1 + 1/z + \frac{1}{2!\,z^2} + \cdots$$

has an essential singularity at $z = 0$; the residue of $e^{1/z}$ at $z = 0$ is 1.

We can often see that a function has a pole and find the order of the pole without finding the Laurent series.

6.5 THE RESIDUE THEOREM

Let z_0 be an isolated singular point of $f(z)$. We want to find $\oint_C f(z)\,dz$ around a simple closed curve C surrounding z_0 but enclosing no other singularities. Expand $f(z)$ in a Laurent series about $z = z_0$ that converges near $z = z_0$.



By Cauchy's theorem the integral of the "a" series is zero since this part is analytic. To evaluate the integrals of the "b" series we replace the integrals around C by integrals around the circle C' with centre at z_0 and radius ρ.

Along C', $\quad z = z_0 + \rho e^{i\theta}$

and for the integral of the b_1 term we have

$$\oint_C \frac{b_1\,dz}{(z-z_0)} = b_1 \int_0^{2\pi} \frac{\rho\, i e^{i\theta}}{\rho e^{i\theta}}\,d\theta = 2\pi i\, b_1$$

But, for all the other values of n, the integral

$$\oint_C \frac{dz}{(z-z_0)^n} = 0$$

CHECK: $\displaystyle \oint_C \frac{b_2}{(z-z_0)^2}\,dz = b_2 \int_0^{2\pi} \frac{\rho\, i e^{i\theta}\,d\theta}{\rho^2 e^{2i\theta}} = \frac{b_2 i}{\rho}\int_0^{2\pi} e^{-i\theta}\,d\theta$

$$= \frac{b_2 i}{\rho}\left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi} = -\frac{b_2}{\rho}\left[e^{-2\pi i} - e^0\right] = 0$$

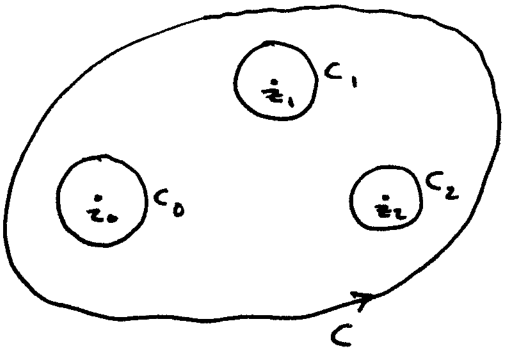
$$\Rightarrow \oint_C f(z)\,dz = 2\pi i\, b_1$$

Since b_1 is called the residue of $f(z)$ at $z = z_0$ we have

$$\oint_C f(z)\,dz = 2\pi i \times \text{residue of } f(z) \text{ at the singular point inside } C.$$

Note that the b_1 term is the only term in the Laurent series which has survived the integration process (hence the term "residue").

If there are several singularities inside C, say at $z_0, z_1, z_2, \ldots$ we draw small circles about each so that $f(z)$ is analytic in the region between C and the circles.



Then we find (as before) that the integral around C counterclockwise, plus the integrals around $C_0, C_1, C_2, \ldots$ is zero (since the integrals along the cuts cancel). Therefore the integral along C is the sum of the integrals around the circles.

But the integral around each circle is $2\pi i$ times the residue of $f(z)$ at the singular point inside. This gives the Residue Theorem:

$$\oint_C f(z)\,dz = 2\pi i \times \text{sum of the residues of } f(z) \text{ inside } C$$

where the integral around C is in the counterclockwise direction.

6.6 METHODS OF FINDING RESIDUES

6.6.1 LAURENT SERIES

If it is easy to write the Laurent series for $f(z)$ about $z = z_0$ that is valid near z_0 then the residue is just the coefficient b_1 of the term $1/(z - z_0)$.

e.g. Given $f(z) = e^z/(z-1)$, find the residue $R(1)$ of $f(z)$ at $z = 1$.

We want to expand e^z in powers of $z-1$:

$$\frac{e^z}{z-1} = \frac{e \cdot e^{z-1}}{z-1} = \frac{e}{z-1}\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \dots\right]$$

$$= \frac{e}{z-1} + e + \dots$$

The residue is the coefficient of $1/(z-1)$, i.e. $R(1) = e$.

6.6.2 SIMPLE POLE

If $f(z)$ has a simple pole at $z = z_0$, we find the residue by multiplying $f(z)$ by $(z - z_0)$ and evaluating the result at $z = z_0$.

e.g. Find $R(-\frac{1}{2})$ and $R(5)$ for

$$f(z) = \frac{z}{(2z+1)(5-z)}$$

Multiply $f(z)$ by $(z + \frac{1}{2})$ and evaluate the result at $z = -\frac{1}{2}$.

$$(z + \tfrac{1}{2}) f(z) = (z + \tfrac{1}{2}) \frac{z}{(2z+1)(5-z)} = \frac{z}{2(5-z)}$$

$$R(-\tfrac{1}{2}) = \frac{-\frac{1}{2}}{2(5 + \frac{1}{2})} = -\frac{1}{22}$$

Similarly

$$(z - 5) f(z) = (z - 5) \frac{z}{(2z+1)(5-z)} = \frac{-z}{2z+1}$$

$$R(5) = -\frac{5}{11}.$$

e.g. Find $R(0)$ for $f(z) = (\cos z)/z$.

Since $z f(z) = \cos z$ we have

$$R(0) = (\cos z)_{z=0} = \cos 0 = 1$$

In some problems we may have to evaluate an indeterminate form, so in general we write

$$R(z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

when z_0 is a simple pole.

e.g. Find the residue of $\cot z$ at $z = 0$.

Using the above equation we have

$$R(0) = \lim_{z \to 0} \frac{z \cos z}{\sin z} = \cos 0 . \lim_{z \to 0} \frac{z}{\sin z} = 1.1 = 1$$

If $f(z)$ can be written as $g(z)/h(z)$ where $g(z)$ is analytic and not zero at z_0 and $h(z_0) = 0$, then

$$R(z_0) = \lim_{z \to z_0} \frac{(z - z_0) g(z)}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{z - z_0}{h(z)}$$

$$= g(z_0) \lim_{z \to z_0} \frac{1}{h'(z)} = \frac{g(z_0)}{h'(z_0)}$$

by L'Hopital's rule and the definition of $h'(z)$.

Hence

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \quad \text{if} \quad \begin{cases} f(z) = g(z)/h(z) \ \underline{\text{and}} \\ g(z_0) = \text{finite const.} \neq 0 \ \underline{\text{and}} \\ h(z_0) = 0, \ h'(z_0) \neq 0 \end{cases}$$

e.g. Find the residue of $\dfrac{\sin z}{1-z^4}$ at $z = i$.

We have

$$R(i) = \left.\frac{\sin z}{-4z^3}\right|_{z=i} = \frac{\sin i}{-4i^3} = \frac{e^{-1}-e}{(2i)(4i)}$$

$$= \frac{1}{8}\left(e - e^{-1}\right) = \frac{1}{4}\sinh 1$$

6.6.3 MULTIPLE POLES

When $f(z)$ has a pole of order n, we can use the following method of finding residues:

Multiply $f(z)$ by $(z-z_0)^m$ where m is an integer greater than or equal to the order n of the pole, differentiate $m-1$ times, divide by $(m-1)!$ and evaluate the resulting expression at $z = z_0$.

e.g. Find the residue of $f(z) = \dfrac{z \sin z}{(z-\pi)^3}$ at $z = \pi$.

We take $m = 3$ to eliminate the denominator before differentiating.

$$R(\pi) = \left.\frac{1}{2!}\frac{d^2}{dz^2}\left(z \sin z\right)\right|_{z=\pi}$$

$$= \frac{1}{2}\Big[-z\sin z + 2\cos z\Big]_{z=\pi} = -1$$

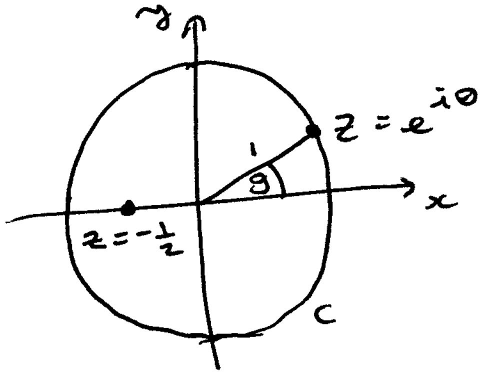
6.7 EVALUATION OF DEFINITE INTEGRALS BY USE OF THE RESIDUE THEOREM

The residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \times \text{sum of the residuals of } f(z) \text{ inside } C.$$

can be used to evaluate several different types of definite integrals.

e.g. Find $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$



Let $z = e^{i\theta}$. As θ goes from 0 to 2π, z traverses the unit circle $|z| = 1$ in the counterclockwise direction and we have a contour integral.

$$dz = i e^{i\theta} d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}dz$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}$$

$$I = \oint_C \frac{(1/iz)\,dz}{5 + 2(z + 1/z)} = \frac{1}{i}\oint_C \frac{dz}{5z + 2z^2 + 2}$$

$$= \frac{1}{i}\oint \frac{dz}{(2z+1)(z+2)} \quad \text{where } C \text{ is the unit circle.}$$

The integrand has poles at $z = -\frac{1}{2}$ and $z = -2$ but only $z = -\frac{1}{2}$ is inside the contour C.

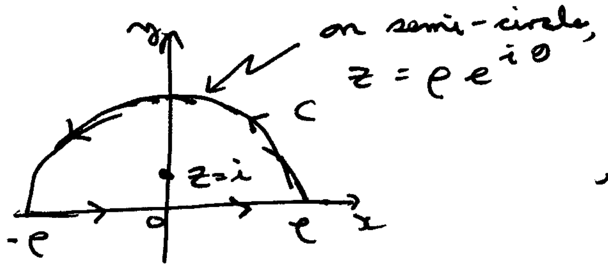
The residue of $\dfrac{1}{(2z+1)(z+2)}$ at $z = -\frac{1}{2}$ is

$$R\left(-\tfrac{1}{2}\right) = \lim_{z \to -1/2} \left(z + 1/2\right) \cdot \frac{1}{(2z+1)(z+2)} = \left.\frac{1}{2(z+2)}\right|_{z=-\frac{1}{2}} = \frac{1}{3}$$

Then, by the residue theorem,

$$I = \frac{1}{i}\, 2\pi i\, R\left(-\tfrac{1}{2}\right) = 2\pi \cdot \frac{1}{3} = \frac{2\pi}{3}.$$

e.g. Evaluate $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.



Consider $\displaystyle\oint_C \frac{dz}{1+z^2}$

where C is the closed boundary of the semi-circle

For any $\rho > 1$ the semi-circle encloses the singular point $z = i$ and no others. The residue of the integral at $z = i$ is

$$R(i) = \lim_{z \to i} (z - i) \frac{1}{(z-i)(z+i)} = \frac{1}{2i}$$

$$\Rightarrow \oint_C \frac{dz}{1+z^2} = 2\pi i \cdot \frac{1}{2i} = \pi$$

Now write the same integral in two parts:

$$\oint_C \frac{dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{dx}{1+x^2} + \int_0^{\pi} \frac{\rho i e^{i\theta}}{1+\rho^2 e^{2i\theta}} d\theta = \pi$$

No matter how large ρ becomes there are no other singularities ~~besides~~ $z = i$ in the upper half-plane.

Let $\rho \to \infty$; the second integral tends to 0

since $$\lim_{\rho \to \infty} \int_0^{\pi} \frac{\rho i e^{i\theta}}{1+\rho^2 e^{2i\theta}} d\theta = \lim_{\rho \to \infty} \int_0^{\pi} \frac{i e^{i\theta}}{1/\rho + \rho e^{2i\theta}} d\theta$$

$$= 0$$

$$\Rightarrow \underline{I} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi .$$

The method can be used to evaluate any integral of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$$

where $P(x)$, $Q(x)$ are polynomials with the degree of Q at least two greater than the degree of P, and if $Q(x)$ has no real zeros (i.e. zeros along the x axis).

e.g. Evaluate $\underline{I} = \displaystyle\int_0^{\infty} \frac{\cos x}{1+x^2} dx$

Consider the contour integral

$$\oint_C \frac{e^{iz}}{1+z^2} dz$$

where C is the same semi-circular contour as above. The singular point is again $z = i$.

$$R(i) = \lim_{z \to i} (z - i) \frac{e^{iz}}{(z-i)(z+i)} = \frac{e^{-1}}{2i}$$

$$\Rightarrow \quad \oint_C \frac{e^{iz} dz}{1+z^2} = 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

As before, we write

$$\oint_C \frac{e^{iz} dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{e^{ix} dx}{1+x^2} + \int_{\substack{\text{along} \\ \text{upper half} \\ \text{of } z = \rho e^{i\theta}}} \frac{e^{iz}}{1+z^2} dz$$

We want to show that the second integral on the RHS $\to 0$ as $\rho \to \infty$.

$$|e^{iz}| = |e^{ix-y}| = |e^{ix}||e^{-y}| = e^{-y} \leq 1$$

since $y \geq 0$ on the contour we are considering.

Since $|e^{iz}| \leq 1$ this factor does not change the proof in the previous example and so

$$\lim_{\rho \to \infty} \int \frac{e^{iz}}{1+z^2} dz = 0$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx = \frac{\pi}{e}$$

Taking the real part of both sides gives

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}$$

But $$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = 2\int_{0}^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{2e}$$