4. PARTIAL DIFFERENTIAL EQUATIONS

Previously we studied methods of solving (in the main) linear first and second order ordinary differential equations (ODE) – that is equations with $y(x)$ only dependent on a single variable x.

Partial differential equations (PDEs) are equations involving functions of several variables and their partial derivatives. As such, PDEs play a central role in mathematical physics.

e.g. $$\frac{\partial \psi}{\partial x}(x,t) = \frac{1}{c}\frac{\partial \psi}{\partial t}(x,t)$$ linear PDE, 1st order

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$$ Laplace equation in 3-D; 2nd order PDE

$$\frac{\partial^2 \psi}{\partial x^2}(x,t) = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}(x,t)$$ Wave equation in 1-D. (2nd order, linear PDE)

$$-\frac{\hbar^2}{2m}\nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}$$ Schrödinger equation (2nd order PDE)

$$\nabla^2 \psi = \frac{1}{\alpha}\frac{\partial \psi}{\partial t}$$ Diffusion equation (2nd order PDE)

There are several techniques one can use in an attempt to solve PDEs. We shall consider only 1st and 2nd order PDEs in this course.

4.1 FIRST ORDER PDEs

Let's look at a simple example:

$$\frac{\partial \psi}{\partial x}(x,t) - \frac{1}{a}\frac{\partial \psi}{\partial t}(x,t) = 0$$

$\psi = \psi(x,t)$ depends on two independent variables x, t and a is a constant.

Consider a change of variables

$$x^- = x - at \quad , \quad x^+ = x + at$$

Then
$$\frac{\partial}{\partial x} = \frac{\partial x^-}{\partial x}\cdot\frac{\partial}{\partial x^-} + \frac{\partial x^+}{\partial x}\cdot\frac{\partial}{\partial x^+} = \frac{\partial}{\partial x^-} + \frac{\partial}{\partial x^+}$$

$$\frac{\partial}{\partial t} = \frac{\partial x^-}{\partial t}\cdot\frac{\partial}{\partial x^-} + \frac{\partial x^+}{\partial t}\cdot\frac{\partial}{\partial x^+} = -a\frac{\partial}{\partial x^-} + a\frac{\partial}{\partial x^+}$$

$$\Rightarrow \quad \frac{\partial}{\partial x} - \frac{1}{a}\frac{\partial}{\partial t} = 2\frac{\partial}{\partial x^-}$$

Our equation can therefore be written as

$$\frac{\partial}{\partial x^-}\psi(x^-, x^+) = 0$$

Now an equation like $\frac{d f(x)}{dx} = 0$ has a solution $f(x) = $ constant. But when $\psi(x^-, x^+)$ depends on 2 variables,

$$\frac{\partial}{\partial x^-}\psi(x^-, x^+) = 0 \quad \Rightarrow \quad \psi(x^-, x^+) = f(x^+)$$

(since $\frac{\partial}{\partial x^-} f(x^+) = 0$ as x^- and x^+ are independent)

But $\quad x^+ = x + at$

Hence the general solution of $\frac{\partial \psi}{\partial x} - \frac{1}{a}\frac{\partial \psi}{\partial t} = 0$ is

$$\psi = \psi(x + at)$$

i.e. only depends on x and t through the combination $x + at$.

4.2 GENERAL FIRST ORDER PDEs

Consider the PDE of the form

$$\frac{\partial \psi}{\partial x}(x,y) + P(x,y)\frac{\partial \psi}{\partial y}(x,y) = Q(x,y)$$

This is a linear PDE since ψ appears at most as linear.

Let us look at some simple cases:

$$Q = 0 \; ; \quad P(x,y) = f(x)\, g(y) \quad \text{(factorizable)}$$

We look for a factorized solution with separation of variables:

$$\psi(x,y) = u(x)\, v(y)$$

$$\frac{\partial \psi}{\partial x} = \left(\frac{\partial u}{\partial x}\right) v(y) + u(x) \cancel{\frac{\partial v(y)}{\partial x}}^{\to 0} = \frac{\partial u}{\partial x} \cdot v(y)$$

Similarly

$$\frac{\partial \psi}{\partial y} = u(x) \frac{\partial v}{\partial y}$$

Our differential equation is

$$\frac{\partial \psi}{\partial x}(x,y) + f(x) g(y) \frac{\partial \psi}{\partial y}(x,y) = 0$$

If we substitute our expressions for $\partial \psi / \partial x$ and $\partial \psi / \partial y$ we have

$$\frac{\partial u(x)}{\partial x} \cdot v(y) + f(x) g(y) u(x) \frac{\partial v(y)}{\partial y} = 0$$

Dividing by $f\,u\,v$ gives

$$\left(\frac{\partial u}{\partial x}\right) \frac{1}{fu} + \frac{g(y)}{v(y)} \frac{\partial v(y)}{\partial y} = 0$$

Now, since u and f are <u>only</u> functions of x and g and v are <u>only</u> functions of y, the above equation can only be satisfied if

$$\frac{\partial u}{\partial x} \cdot \frac{1}{f(x) u(x)} = \text{constant} = c$$

$$\frac{g(y)}{v(y)} \cdot \frac{\partial v(y)}{\partial y} = -c$$

$$\Rightarrow \quad \frac{du}{dx} = c f(x) u(x)$$

$$\frac{dv}{dy} = -c \frac{v(y)}{g(y)}$$

$$\Rightarrow \int \frac{du}{u} = c\int f(x)\,dx \quad \Rightarrow u(x) = c_1 e^{c\int f(x)dx}$$

$$\int \frac{dv}{v} = -c\int \frac{1}{g(y)}\,dy \quad \Rightarrow v(y) = c_2 e^{-c\int \frac{dy}{g(y)}}$$

Therefore

$$\psi(x,y) = u(x)v(y)$$
$$= c_3 e^{c\int f(x)\,dx} e^{-c\int \frac{dy}{g(y)}} \qquad (c_3 = \text{const.})$$

e.g. Solve $\dfrac{\partial \psi}{\partial x} + xy\dfrac{\partial \psi}{\partial y} = 0$

We take $f(x) = x$ and $g(y) = y$.

$$\psi(x,y) = (\text{const.})\, e^{c\int x\,dx} e^{-c\int \frac{dy}{y}}$$

$$= \text{const.}\; e^{cx^2/2} \cdot e^{-c\ln y}$$

$$= \text{const.}\; \frac{e^{cx^2/2}}{y^c}$$

4.3 Examples of First Order PDEs in Physics

4.3.1 Magnetic Fields / Magnetic Potential

Since (apparently) there are no magnetic monopoles in ~~an~~ nature, the physical magnetic field $\underset{\sim}{B}$ is divergence-free, i.e.

$$\nabla \cdot \underset{\sim}{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z = 0$$

This equation means that we can always write

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} \quad \text{where } \underset{\sim}{A} \text{ is the magnetic potential.}$$

EXAMPLE 1

Let's take $\underset{\sim}{B}$ to be constant and pointing along the z-axis:

$$\underset{\sim}{B} = B\,\hat{\underset{\sim}{z}} \; ; \qquad \text{where } B \text{ is a constant.}$$

So, recalling that in component form,

$$\nabla \times \underset{\sim}{A} = \begin{vmatrix} \hat{\underset{\sim}{x}} & \hat{\underset{\sim}{y}} & \hat{\underset{\sim}{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\partial_y A_z - \partial_z A_y\right)\hat{\underset{\sim}{x}} + \left(-\partial_x A_z + \partial_z A_x\right)\hat{\underset{\sim}{y}} + \left(\partial_x A_y - \partial_y A_x\right)\hat{\underset{\sim}{z}}$$

This gives us three equations to solve:

$$\partial_y A_z - \partial_z A_y = 0$$

$$-\partial_x A_z + \partial_z A_x = 0$$

$$\partial_x A_y - \partial_y A_x = B$$

Let's assume that $A_z = 0$. Then:

$$\partial_z A_y = 0 \qquad (1)$$

$$\partial_z A_x = 0 \qquad (2)$$

$$\partial_x A_y - \partial_y A_x = B \qquad (3)$$

$$\left.\begin{aligned} (1) &\Rightarrow A_y = A_y(x,y) \\ (2) &\Rightarrow A_x = A_x(x,y) \end{aligned}\right\} \text{ i.e. independent of } z.$$

Put these back into (3):

$$\partial_x A_y(x,y) - \partial_y A_x(x,y) = B \quad \text{(a constant)}$$

We note that

$$A_y = \alpha x B \qquad (\partial_x A_y = \alpha B)$$

$$A_x = (\alpha - 1) y B \qquad (\partial_y A_x = (\alpha-1)B)$$

solves the differential equation.

$$\Rightarrow \quad \underset{\sim}{A} = (\alpha - 1) y B \,\hat{\underset{\sim}{x}} + \alpha x B \,\hat{\underset{\sim}{y}}$$

solves $\quad \nabla \times \underset{\sim}{A} = B \hat{\underset{\sim}{z}}$

<u>CHECK</u>: $\nabla \times \hat{\underset{\sim}{A}} = \begin{vmatrix} \hat{\underset{\sim}{x}} & \hat{\underset{\sim}{y}} & \hat{\underset{\sim}{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (\alpha-1) y B & \alpha x B & 0 \end{vmatrix} = (0, 0, B)$

4.3.2 MECHANICS

In mechanics the force is minus the gradient of the potential:

$$\underset{\sim}{F} = -\nabla V$$

EXAMPLE 2

Given the force

$$\underset{\sim}{F} = (x^2+y^2+z^2)(x\,\hat{\underline{x}} + y\,\hat{\underline{y}} + z\,\hat{\underline{z}})$$

calculate $V(x,y,z)$.

In components:

$$-\frac{\partial V}{\partial x} = F_x = (x^2+y^2+z^2)x \qquad (1)$$

$$-\frac{\partial V}{\partial y} = F_y = (x^2+y^2+z^2)y \qquad (2)$$

$$-\frac{\partial V}{\partial z} = F_z = (x^2+y^2+z^2)z \qquad (3)$$

Solve (1) keeping y, z fixed:

$$\frac{\partial V}{\partial x} = -(x^2+y^2+z^2)x \Big|_{y,\, z \text{ const.}}$$

$$\Rightarrow \quad V(x,y,z) = -\frac{x^4}{4} - \frac{x^2}{2}(y^2+z^2) + f(y,z)$$

(indep. of x)

Substitute into (2):

$$-\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}\left(-\frac{x^4}{4} - \frac{x^2}{2}(y^2+z^2) + f(y,z)\right) = (x^2+y^2+z^2)y$$

$$\Rightarrow \quad x^2y - \frac{\partial f}{\partial y} = x^2y + y^3 + z^2y$$

$$\Rightarrow \quad -\frac{\partial f}{\partial y} = y^3 + z^2 y$$

$$\Rightarrow \quad f(y,z) = -\frac{y^4}{4} - \frac{z^2 y^2}{2} + g(z)$$

To determine $g(z)$, substitute back into (3):

$$-\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left(-\frac{x^4}{4} - \frac{x^2}{2}(y^2+z^2) - \frac{y^4}{4} - \frac{z^2 y^2}{2} + g(z)\right)$$

$$= (x^2+y^2+z^2)z$$

$$\Rightarrow \quad +x^2 z + y^2 z - \frac{dg}{dz} = x^2 z + y^2 z + z^3$$

$$\Rightarrow \quad \frac{dg}{dz} = -z^3 \ ; \qquad g(z) = -\frac{z^4}{4}$$

So finally,

$$V(x,y,z) = -\frac{x^4}{4} - \frac{y^4}{4} - \frac{z^4}{4} - \frac{x^2}{2}(y^2+z^2) - \frac{z^2 y^2}{2}$$

$$= -\frac{1}{4}\left(x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2\right)$$

$$\Rightarrow \quad V(x,y,z) = -\frac{1}{4}(x^2+y^2+z^2)^2 = -\frac{1}{4}r^4$$

<u>CHECK</u>: $\quad V = -\frac{1}{4}r^4$

$$\Rightarrow \quad \nabla V = \frac{\partial V}{\partial r}\hat{\underset{\sim}{r}} = -r^3\hat{\underset{\sim}{r}} = -r^2\underset{\sim}{r} = -\underset{\sim}{F}$$

and so $\underset{\sim}{F} = -\nabla V$

(Note that the calculation is easier in spherical polar coordinates)

4.4 Solving Second Order PDEs

We will examine several examples from physics.

4.4.1 Wave Equation in 1+1 Dimensions

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi(x,t)}{\partial t^2}$$

This is a linear, 2nd order PDE (1-space dim., 1-time dims.)

We will try different methods.

First consider separation of variables, i.e. we will look for solutions of the form

$$\psi(x,t) = f(x)\,g(t)$$

Substituting in wave equation

$$\left(\frac{d^2 f(x)}{dx^2}\right) g(t) = \frac{1}{c^2} f(x)\left(\frac{d^2 g(t)}{dt^2}\right)$$

or

$$\underbrace{\frac{1}{f(x)}\left(\frac{d^2 f(x)}{dx^2}\right)}_{\text{(only depends on } x)} = \underbrace{\frac{1}{c^2}\left(\frac{d^2 g(t)}{dt^2}\right)\frac{1}{g(t)}}_{\text{(only depends on } t)}$$

Since x and t are independent variables, the only solution is

$$\frac{1}{f(x)}\left(\frac{d^2 f}{dx^2}\right) = \underset{\text{(constant)}}{K} = \frac{1}{c^2}\frac{1}{g(t)}\left(\frac{d^2 g}{dt^2}\right)$$

Thus we obtain two equations for f and g:

$$\frac{d^2 f(x)}{dx^2} = K f(x) \; ; \quad \frac{d^2 g(t)}{dt^2} = K c^2 g(t) .$$

These equations determine $f(x)$ and $g(t)$.

E.g. If $\underline{K < 0}$ (i.e. $K = -k^2$) then the solutions are

$$f(x) = A \sin kx + B \cos kx$$

$$g(t) = C \sin \omega t + D \cos \omega t \qquad (\omega = kc)$$

Then $\psi(x,t) = f(x) g(t)$

$$= (A \sin kx + B \cos kx)(C \sin \omega t + D \cos \omega t)$$

The constants A, B, C, D are determined from boundary conditions.

E.g. If $\psi(x,0) = \sin kx$

$$\Rightarrow (A \sin kx + B \cos kx)(D) = \sin kx$$

$$\Rightarrow B = 0 , \quad AD = 1$$

If $\frac{\partial \psi}{\partial t}(x,0) = 0$

$$\Rightarrow (A \sin kx + B \cos kx)(C\omega \cos \omega t - D\omega \sin \omega t)\Big|_{t=0}$$

$$= (A \sin kx + B \cos kx)(C\omega) = 0$$

$$= C = 0$$

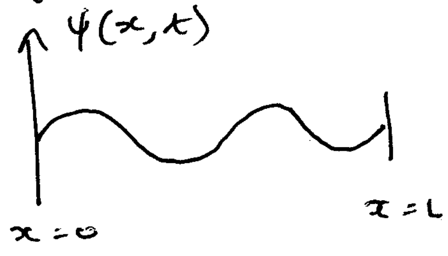
$$\Rightarrow \psi(x,t) = \sin kx \cos \omega t$$

If $K > 0$ (i.e. $K = +k^2$) then the solutions are

$$f(x) = Ae^{kx} + Be^{-kx}$$

$$g(t) = Ce^{\omega t} + De^{-\omega t}$$

Consider the boundary conditions for the vibrations of a stretched string:



The physical boundary conditions give

$$\psi(x=0, t) = \psi(x=L, t) = 0$$

These are only consistent with our previous solutions if we assume $K < 0 = -k^2$.

$$\psi(x=0, t) = 0 \Rightarrow f(x=0) = 0$$
$$\Rightarrow A \sin(0) + B\cos(0) = 0 \Rightarrow B = 0$$

$$\psi(x=L, t) = 0 \Rightarrow f(x=L) = 0$$
$$\Rightarrow A\sin(kL) = 0 \Rightarrow kL = n\pi \quad n = 1, 2, \ldots$$

$$\Rightarrow B = 0 \quad \text{and} \quad k = \frac{n\pi}{L}$$

i.e. $$f(x) = A \sin\left(\frac{n\pi x}{L}\right)$$

If we also impose another boundary condition, e.g. $\partial\psi/\partial t\big|_{t=0} = 0$, then $g(t) = \cos\left(\frac{n\pi}{L} ct\right)$

and so

$$\psi(x,t) = A \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}ct\right) \quad n = 1, 2, \ldots$$

The most general solution of a wave equation is a linear combination of particular solutions:

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}ct\right)$$

We can also see the general structure of solutions to the 1+1 dimensional wave equation in another way:

Define $\quad x^+ = x + ct \qquad$ (c = velocity of wave)

$\qquad\qquad x^- = x - ct$

We have:

$$\frac{\partial}{\partial x} = \frac{\partial x^-}{\partial x}\cdot\frac{\partial}{\partial x^-} + \frac{\partial x^+}{\partial x}\cdot\frac{\partial}{\partial x^+} = \frac{\partial}{\partial x^-} + \frac{\partial}{\partial x^+}$$

$$\frac{\partial}{\partial t} = \frac{\partial x^-}{\partial t}\frac{\partial}{\partial x^-} + \frac{\partial x^+}{\partial t}\cdot\frac{\partial}{\partial x^+} = -c\frac{\partial}{\partial x^-} + c\frac{\partial}{\partial x^+}$$

$$\Rightarrow \quad \frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 4\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-}\psi(x^+, x^-)$$

So our new wave equation when written in terms of new variables x^+, x^- instead of x, t becomes

$$\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-}\psi(x^+, x^-) = 0$$

$\Rightarrow$ general solution $\psi(x^+, x^-) = F(x^+) + G(x^-)$

where F only depends on x^+, so $\frac{\partial}{\partial x^-}F = 0$

and G only depends on x^-, so $\frac{\partial}{\partial x^+}G = 0$.

The physical meaning of these functions is clear.

$F(x^+) = F(x+ct)$:  right moving waves

$G(x^-) = G(x-ct)$:  left moving waves

4.4.2 HEAT CONDUCTION EQUATION

The heat conduction equation in 1 space + 1 time dimension is

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

where k is a constant related to thermal conductivity and u is the temperature.

The method we shall use to solve this is the separation of variables.

Let's consider that

$$u(x,t) = f(x)\,g(t)$$

Substitute in our PDE :

$$\frac{\partial^2}{\partial x^2}\big(f(x)\,g(t)\big) = \frac{1}{k}\frac{\partial}{\partial t}\big(f(x)\,g(t)\big)$$

i.e. $$\left(\frac{\partial^2 f}{\partial x^2}\right) g(t) = \frac{1}{k}\left(\frac{\partial g}{\partial t}\right) f(x)$$

$$\Rightarrow \quad \underbrace{\frac{1}{f}\frac{\partial^2 f}{\partial x^2}}_{(x\text{-dep. only})} = \underbrace{\frac{1}{k}\frac{1}{g}\frac{\partial g}{\partial t}}_{(t\text{-dep. only})}$$

Since x and t are indep. variables, the only consistent solution is that

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = \frac{1}{f}\frac{d^2 f}{dx^2} = \text{const.} = c \qquad (1)$$

$$\frac{1}{k}\frac{1}{g}\frac{\partial g}{\partial t} = \frac{1}{k}\frac{1}{g}\frac{dg}{dt} = c \qquad (2)$$

Solving (2) : $\quad g(t) = g_0 e^{ckt}$ assuming $k > 0$

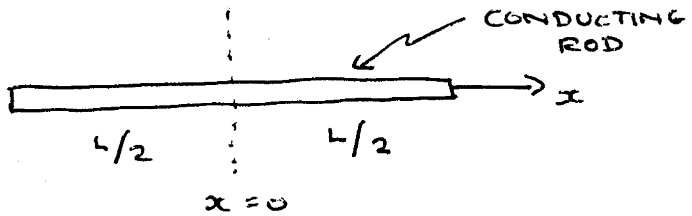
Physical solutions will have $c < 0$ (i.e decaying) and so we write $c = -\omega^2$

i.e. $\quad g(t) = g_0 e^{-\omega^2 kt}$

(1) $\Rightarrow$ $$\frac{d^2 f}{dx^2} = -\omega^2 f$$

The solutions will depend on the <u>boundary conditions</u>

e.g.



Choose $f(x) = 0$ at $x = \pm L/2$ (with $u(x,t) = 0$ at these points, i.e. we are keeping the temperature zero at the rod ends).

The solutions of (1) are

$$f_n(x) = \cos\left(\frac{n\pi}{L}x\right)$$

since at $x = \pm L/2$, $\cos\left(\frac{n\pi}{L}x\right) = 0 \quad (n = 1, 3, 5, ..)$

$$\frac{\partial^2}{\partial x^2}\cos\left(\frac{n\pi x}{L}\right) = -\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi}{L}x\right) = -\omega^2\cos\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow \quad \omega_n = \frac{n\pi}{L}$$

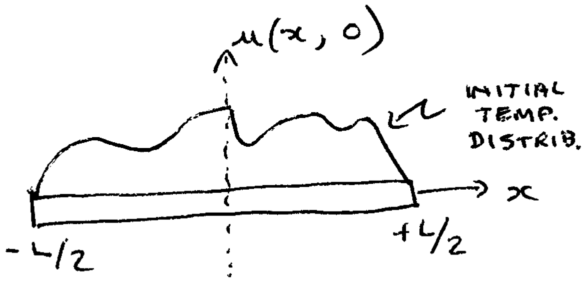
and hence

$$u_n(x,t) = \cos\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt}$$

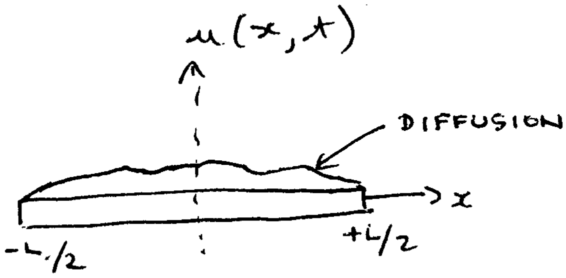
The most general solution is any linear combination of $u_n(x,t)$ above with const. coefficients.

$$u(x,t) = \sum_{n\ \mathrm{odd}} a_n \cos\left(\frac{n\pi}{L}x\right)e^{-\frac{n^2\pi^2}{L^2}kt}$$

with $\quad u(x,t=0) = \sum_{n\ \mathrm{odd}} a_n \cos\left(\frac{n\pi}{L}x\right)$



$t=0$



$t>0$

It's clear from our solution that the higher frequency modes (i.e large n) decay faster in time than lower frequency modes. Physically, higher frequency modes correspond to inhomogeniaties in the temperature distribution $f(x)$ (regions where df/dx is much larger than zero) and these tend to diffuse faster.

4.4.3 Schrodinger Equation with $V=0$

Schrodinger's equation with $V=0$ can be written

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}$$

Looking for separable solutions gives

$$\psi = e^{-iEt/\hbar}\left[A\sin\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right) + B\cos\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right)\right]$$

For the boundary conditions $\psi(0,t) = \psi(L,t) = 0$,

we require

$$\psi = \sum_{n=0}^{\infty} A_n \sin\left(\sqrt{\frac{2mE_n}{\hbar^2}}\,x\right) e^{-iE_n t/\hbar}$$

where $E_n = \dfrac{n^2\hbar^2\pi^2}{2mL^2}$ and A_n determined by

$\psi(x)$ at $t=0$.

4.5 Green's Function in Two and Three Dimensions

In order to solve inhomogeneous PDEs we must extend concept of Green's function (used to solve ODEs) to functions of more than one variable.

4.5.1 DELTA FUNCTIONS IN 2D AND 3D

We can extend $\delta(x)$ to two dimensions.

$$\delta^{(2)}(\underset{\sim}{x}) \text{ with } \underset{\sim}{x} = (x, y) \text{ is } \begin{cases} 0 & \text{at } \underset{\sim}{x} \neq \underset{\sim}{0} \\ \infty & \text{at } \underset{\sim}{x} = \underset{\sim}{0}. \end{cases}$$

We can regard this as a product $\delta(x)\delta(y)$.

Then $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta^{(2)}(\underset{\sim}{x})\, dx\, dy$

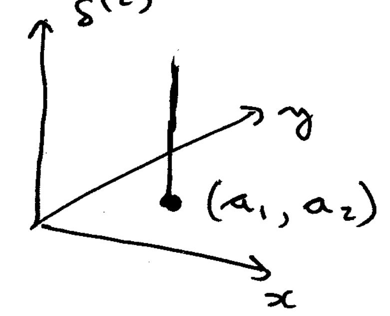
$$= \left(\int_{-\infty}^{\infty} \delta(x)\, dx\right)\left(\int_{-\infty}^{\infty} \delta(y)\, dy\right) = 1$$

$$\Rightarrow \iint \delta(\underset{\sim}{x} - \underset{\sim}{a})\, dx\, dy = 1$$

provided the region of integration contains $\underset{\sim}{a}$.

Also $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\underset{\sim}{x})\, \delta^{(2)}(\underset{\sim}{x} - \underset{\sim}{a})\, dx\, dy = f(\underset{\sim}{a})$

$(\underset{\sim}{a} = (a_1, a_2))$



Similarly for 3D:

$$\delta^{(3)}(\underset{\sim}{x}) = \delta(x)\,\delta(y)\,\delta(z)$$

and $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\underset{\sim}{x})\, \delta^{(3)}(\underset{\sim}{x} - \underset{\sim}{a})\, dx\, dy\, dz = f(\underset{\sim}{a})$

where $\underset{\sim}{x} = (x, y, z)$

$\underset{\sim}{a} = (a_1, a_2, a_3)$

4.5.2 GREEN'S FUNCTION FOR POISSON'S EQUATION

Consider Poisson's equation in 3D :

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \eta(x, y, z)$$

where $\eta(x, y, z)$ is arbitrary

First solve for $\eta = \delta^{(3)}(\underset{\sim}{x})$.

In electrostatics,

$$\nabla . \underset{\sim}{E} = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \underset{\sim}{E} = -\nabla V$$

giving $$\nabla^2 V = \nabla . (\nabla V) = -\frac{\rho}{\varepsilon_0}$$

where $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Here $\underset{\sim}{E}$ is the electric field

V is the electric potential

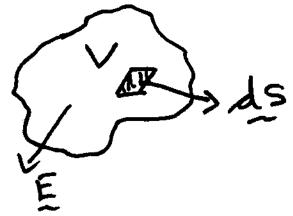
ρ is the charge density

ε_0 is the vacuum permittivity (or electric const.)

Gauss's Law gives

$$\iint_S \underset{\sim}{E} . d\underset{\sim}{S} = \iiint_V (\nabla . \underset{\sim}{E})\, dV = \frac{1}{\varepsilon_0} \iiint_V \rho\, dV = \frac{Q}{\varepsilon_0}$$

for the surface S enclosing the volume V and charge Q.



For a point charge at the origin, $\rho(\underset{\sim}{x}) = Q\,\delta^{(3)}(\underset{\sim}{x})$

$$\Rightarrow \iiint_V \rho(\underset{\sim}{x})\,dx\,dy\,dz = Q$$

Also, $\underset{\sim}{E} = E(r)\,\underset{\sim}{\hat{r}}$ is radial, so taking S as a sphere of radius r gives

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \Rightarrow \quad V = \frac{Q}{4\pi\varepsilon_0 r}$$

Since $V = \dfrac{Q}{4\pi\varepsilon_0 r}$ solves $\nabla^2 V = -\dfrac{Q}{\varepsilon_0}\,\delta^{(3)}(\underset{\sim}{x})$,

a solution of

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = \delta(x)\,\delta(y)\,\delta(z) \text{ is } G(r) = -\frac{1}{4\pi r}$$

This is Green's function for the Laplacian operator $\hat{L} = \nabla^2$.

Explicitly, applying Gauss's theorem to a sphere (where $\underset{\sim}{dS} = |\underset{\sim}{dS}|\,\underset{\sim}{\hat{r}}$) gives

$$\iint_S (\nabla G)\cdot \underset{\sim}{dS} = \iint_S \frac{dG}{dr}\, r^2 \sin\theta\, d\theta\, d\phi = 4\pi r^2 \frac{dG}{dr}$$

and

$$\iiint_V (\nabla^2 G)\, dV = \iiint_V \delta^{(3)}(\underline{x})\, dV = 1$$

$$\Rightarrow \quad \frac{dG}{dr} = \frac{1}{4\pi r^2} \quad \Rightarrow \quad G(r) = -\frac{1}{4\pi r}$$

Equivalently

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\, \delta(r) \qquad \text{(see HOMEWORK 9.1)}$$

More generally, for sources at $\underline{x}'$ rather than the origin,

$$\nabla^2 G(\underline{x}, \underline{x}') = \delta^{(3)}(\underline{x} - \underline{x}')$$

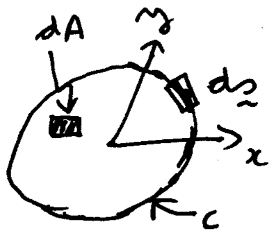
$$\Rightarrow \quad G(\underline{x}, \underline{x}') = G(\underline{x} - \underline{x}') = -\frac{1}{4\pi |\underline{x} - \underline{x}'|}.$$

4.5.3 2D Poisson's Equation

We must solve

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x)\,\delta(y)$$

Integrating over a disc of radius r and using 2D divergence theorem (i.e. $\iint_A (\nabla \cdot \underline{V})\, dA = \int_C \underline{V} \cdot d\underline{s}$



for closed curve C enclosing area A)

we have

$$\int_C \nabla G \cdot d\underset{\sim}{s} = \int_C \frac{dG}{dr}\, r\, d\theta = 2\pi r \frac{dG}{dr} = \iint_A \nabla^2 G\, dA = 1$$

$$\Rightarrow \frac{dG}{dr} = \frac{1}{2\pi r} \quad \Rightarrow \quad G(r) = \frac{1}{2\pi} \ln r$$

For the electrostatic problem:

charge at origin

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{Q}{\varepsilon_0}\,\delta(x)\,\delta(y).$$

which gives

$$V(r) = \frac{-Q}{2\pi\varepsilon_0} \ln r$$

$$\Rightarrow \underset{\sim}{E} = \frac{Q}{2\pi\varepsilon_0 r}\,\hat{\underset{\sim}{r}}$$

4.5.4 More General Inhomogeneous PDE

Generally $\hat{L}\,\psi(x,y,z,t) = J(x,y,z,t)$ where $\hat{L}$ is a linear, 2nd order, differential operator.

If $J = 0$ then we have a homogeneous 2nd order PDE.

e.g. $$\hat{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

for the wave equation in 3D

e.g. $$\hat{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{k}\frac{\partial}{\partial t}$$

for the heat conduction equation in 3D.

For a source at the origin, Green's function for the operator $\hat{L}$ satisfies

$$\hat{L}\, G(x, y, z, t) = \delta(x)\, \delta(y)\, \delta(z)\, \delta(t).$$

More generally

$$\hat{L}\, G(\underline{x}, t, \underline{x}', t') = \delta^{(3)}(\underline{x} - \underline{x}')\, \delta(t - t')$$

and so the solution of $\hat{L}\psi = J$ becomes

$$\psi(\underline{x}, t) = \iiiint G(\underline{x}, t, \underline{x}', t')\, J(\underline{x}', t')\, dx'\, dy'\, dz'\, dt'$$

since

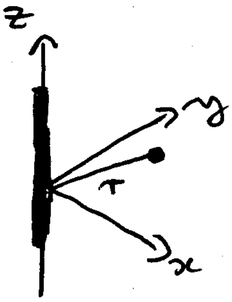
$$\hat{L}\psi = \iiiint \hat{L}\, G(\underline{x}, t, \underline{x}', t')\, J(\underline{x}', t')\, dx'\, dy'\, dz'\, dt'$$

$$= \iiiint \delta(x - x')\, \delta(y - y')\, \delta(z - z')\, \delta(t - t')\, J(x', y', z', t')\, dx'\, dy'\, dz'\, dt'$$

$$= J(x, y, z, t)$$

Hence we can solve the inhomogeneous problem.

For example, solve Poisson's equation for an electric potential $V(x, y)$ due to an infinite line charge along the z-axis (HOMEWORK 9.2)



Then

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\lambda}{\varepsilon_0}\delta(x)\delta(y)$$

where λ is the charge per unit length.

This is a 3D problem but V depends only on $r = \sqrt{x^2+y^2}$ by symmetry.

$$V(x,y,z) = -\frac{1}{\varepsilon_0}\iiint G(x,y,z,x',y',z')\,\rho(x',y',z')\,dx'\,dy'\,dz'$$

$$= \iiint \left(\frac{-1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)\left(-\frac{\lambda}{\varepsilon_0}\delta(x')\delta(y')\right)dx'\,dy'\,dz'$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dz'}{\sqrt{x^2+y^2+(z-z')^2}} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{x^2+y^2} + \text{const.}$$

The constant is infinite for infinite line but the field $\underset{\sim}{E} = -\nabla V$ is finite!

The above analysis is good approximation for finite string $(-L < x < L)$ providing $L \gg \sqrt{x^2+y^2+z^2}$.