3. INHOMOGENEOUS DIFFERENTIAL EQUATIONS

Consider $\mathcal{L}y(x) = f(x)$ where $\mathcal{L}$ is a linear differential operator.

i.e. $\mathcal{L}(y_1 + y_2) = \mathcal{L}y_1 + \mathcal{L}y_2$

Generally the solution of the differential equation is

$$y = y_c + y_p$$

where y_c is the solution of the homogeneous equation (with $f = 0$) and y_p is the particular integral. y_c is the complementary function and contains the integration constants. Here we mainly assume that

$$\mathcal{L} = \frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)$$

but we can also consider more general situations.

3.1 ODEs WITH CONSTANT COEFFICIENTS

Homogeneous case :

For
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

where P and Q are constants, we put

$$y \propto e^{\lambda x}$$

and obtain the auxiliary equation

$$\lambda^2 + P\lambda + Q = 0$$

[CHECK: If $y \propto e^{\lambda x}$ then $y = ke^{\lambda x}$ where k is a constant. Substituting in the DE gives

$$\frac{dy}{dx} = k\lambda e^{\lambda x}, \quad \frac{d^2y}{dx^2} = k\lambda^2 e^{\lambda x}$$

i.e. $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy$$

$$= k\lambda^2 e^{\lambda x} + Pk\lambda e^{\lambda x} + Qke^{\lambda x} = 0$$

$\Rightarrow$ $k = 0$ (trivial solution which we will ignore)

or $$\lambda^2 + P\lambda + Q = 0$$

Note that there is no ~~k~~ finite value of λ for which $e^{\lambda x} = 0$.]

So, solving the quadratic to find λ gives

$$\lambda = \alpha \pm \beta$$

(i.e. two roots because the auxiliary eqn. is a quadratic) where

$$\alpha = -\tfrac{1}{2}P \quad , \quad \beta = \tfrac{1}{2}\sqrt{P^2 - 4Q}$$

This just comes from

$$\lambda = \frac{-P \pm \sqrt{P^2 - 4Q}}{2} = -\frac{P}{2} \pm \frac{1}{2}\sqrt{P^2 - 4Q}$$

If $P^2 - 4Q < 0$ then β is purely imaginary and the solution is of the form

$$y = e^{\alpha x}\left(A \sin \bar{\beta} x + B \cos \bar{\beta} x\right)$$

$\bar{\beta} = \mathrm{Im}(\beta)$

[recall that $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$]

If $P^2 - 4Q > 0$ then β is real and the solution is of the form

$$y = A e^{(\alpha + \beta)x} + B e^{(\alpha - \beta)x} .$$

If $P^2 = 4Q$ then the auxiliary equation has a double root

$$\lambda = \alpha$$

and the solution has the form

$$y = Ae^{\lambda x} + Bxe^{\lambda x}$$

where $\lambda = \alpha = -\frac{1}{2}P$. (see EXERCISE 6.1).

In the case of a nth order ODE,

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 = 0$$

the auxiliary equation is

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

If the n roots, λ_i, are all real and distinct then the solution has the form

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \ldots + c_n e^{\lambda_n x}.$$

As we have seen above, if $n = 2$ then for a pair of complex conjugate roots we have

$$y = e^{(\mathrm{Re}\,\lambda)x}\left(A\sin(\mathrm{Im}\,\lambda)x + B\cos(\mathrm{Im}\,\lambda)x\right)$$

For a repeated real root (k times) we have

$$y = (c_1 + c_2 x + \ldots + c_k x^{k-1})e^{\lambda_1 x}$$

For two repeated roots we have

$$y = (c_1 + c_2 x + \ldots + c_k x^{k-1}) e^{\lambda_1 x}$$

$$+ \ (c_{k+1} + c_{k+2} x + \ldots + c_{k+l} x^{l-1}) e^{\lambda_2 x}$$

$$+ \ c_{k+l+1} e^{\lambda_{k+l+1} x} + \ldots + c_n e^{\lambda_n x}.$$

Note that we always have n constants.

<u>Inhomogeneous case</u> :

We seek solutions with the same form as $f(x)$.

E.g. If $f(x) = a e^{rx}$ then

$$y_p = b e^{rx}$$

If $f(x) = a_1 \sin rx + a_2 \cos rx$ then

$$y_p = b_1 \sin rx + b_2 \cos rx$$

If $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ then

$$y_p = b_0 + b_1 x + \ldots + b_n x^n$$

unless this is already contained in the complementary function, in which case we multiply by the smallest power of x such that it is not contained in CF.

3.2 MORE GENERAL 2ND ORDER ODEs

We want to solve

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x).$$

We can still use

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x)$ can be obtained, e.g., from a series solution. But how do we find $y_p(x)$?

We start off by assuming $F(x)$ is a Dirac delta function.

3.2.1 DIRAC DELTA FUNCTION, $\delta(t)$

This is defined by

$$\delta(t) = 0 \qquad (t \neq 0)$$

$$\delta(t) = \infty \qquad (t = 0)$$

$$\int \delta(t-a) f(t)\,dt = f(a) \quad \text{if integ. range includes } a$$

$$= 0 \quad \text{otherwise.}$$

Therefore, if the integration range includes a,

$$\int \delta(t-a)\,dt = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

It can be shown that:

$$\delta(t) = \delta(-t)$$

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

$$t\,\delta(t) = 0$$

[If $b > 0$, $\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta(bt)\,dt = \int_{-\infty}^{\infty} f\left(\frac{t'}{b}\right)\delta(t')\,\frac{dt'}{b}$

$$= \frac{1}{b} f(0) = \frac{1}{b}\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt$$

where we substituted $t' = bt$. But $f(t)$ is arbitrary and so $\delta(bt) = \delta(t)/b = \delta(t)/|b|$ for $b > 0$.

If $b = -c < 0$, $\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta(bt)\,dt = \int_{\infty}^{-\infty} f\left(\frac{t'}{-c}\right)\delta(t')\,\frac{dt'}{(-c)}$

$$= \int_{-\infty}^{\infty} \frac{1}{c}\, f\left(\frac{t'}{-c}\right)\delta(t')\,dt'$$

$$= \frac{1}{c} f(0) = \frac{1}{|b|} f(0) = \frac{1}{|b|}\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt$$

where we substituted $t' = bt = -ct$. But $f(t)$ is arbitrary and so $\delta(bt) = \delta(t)/|b|$ for all b.]

We also have:

$$\delta\big((t-a)(t-b)\big) = \frac{1}{|a-b|}\left[\delta(t-a)+\delta(t-b)\right]$$

$$\delta\big(h(t)\big) = \sum_i \frac{\delta(t-t_i)}{|h'(t_i)|} \quad \text{where} \quad h(t_i)=0 \quad \text{and} \quad h' = \frac{dh}{dt}.$$

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = \Big[f(t)\,\delta(t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt = -f'(0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t-t')\,dt = -f'(t')$$

$$\int_{-\infty}^{\infty} f(t)\,\frac{d^2}{dt^2}\,\delta(t-t')\,dt = -\int \frac{df}{dt}\cdot\frac{d\delta}{dt}(t-t')\,dt$$

$$= \int \frac{d^2 f}{dt^2}\,\delta(t-t')\,dt = +f''(t')$$

Consider the impulse J:

$$J = \int F\,dt = \int m\,\frac{dv}{dt}\,dt = \int m\,dv = \Delta P$$

(the change in momentum)

In the limit as $F \to \infty$ and $\Delta t \to 0$ we have

$$F(t) = J\,\delta(t-t_0)$$

i.e. we can use δ function to represent a large force applied over a short time interval

We can also use δ functions to describe localised positions as well as events.

Consider the charge distribution

$$\rho(\underset{\sim}{r}) = q\,\delta^{(3)}(\underset{\sim}{r} - \underset{\sim}{r}_0)$$
$$= q\,\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)$$

where the charge q is located at a point $\underset{\sim}{r}_0 = (x_0, y_0, z_0)$. The total charge contained within a volume V is

$$\int_V \rho(\underset{\sim}{r})\,dV = q \quad \text{if } \underset{\sim}{r}_0 \in V$$
$$= 0 \quad \text{otherwise.}$$

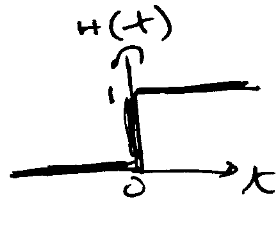
Similarly we could have a mass distribution,

$$M = \int_V \rho(\underset{\sim}{r})\,dV \Rightarrow \rho(\underset{\sim}{r}) = M\,\delta^{(3)}(\underset{\sim}{r} - \underset{\sim}{r}_0)$$

for point mass at $\underset{\sim}{r}_0$.

The δ function is also related to the Heaviside (or step) function,

$$H(t) = \begin{cases} 1, & (t > 0) \\ 0, & (t < 0) \end{cases}$$



Then $H'(t) = \delta(t)$ because

$$\int_{-\infty}^{\infty} f(t) H'(t)\,dt = \left[f(t) H(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t) H(t)\,dt$$
$$= f(\infty) - \int_0^{\infty} f'(t)\,dt = f(0)$$

3.2.2 DELTA FUNCTION AND FOURIER TRANSFORM

Recall that we can write

$$f(x) = \sum_{-\infty}^{\infty} c_r \exp\left(\frac{2\pi i r x}{L}\right)$$

where the coefficients c_r are given by

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x) \exp\left(-\frac{2\pi i r x}{L}\right) dx$$

We can also write

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega \qquad \text{and}$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

We can use the Fourier inversion theorem to write

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\int_{-\infty}^{\infty} du\, f(u) e^{-i\omega u}$$

$$= \int_{-\infty}^{\infty} du\, f(u) \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)} d\omega \right\}$$

$$= \int_{-\infty}^{\infty} du\, f(u)\, \delta(t-u)$$

and we have

$$\delta(t-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\,d\omega$$

i.e. the sum of waves at all frequencies with the same amplitude and in phase at $t=0$.

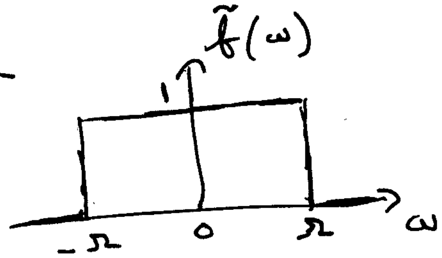
Equivalently,

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega$$

and so the Fourier transform of $\delta(t)$ is $\frac{1}{\sqrt{2\pi}}$.

For the rectangular distribution

$$\tilde{f}_\Omega(\omega) = 1 \quad \text{for } |\omega| < \Omega$$



we have

$$f_\Omega(t) = \frac{1}{\sqrt{2\pi}}\int_{-\Omega}^{\Omega} \tilde{f}_\Omega(\omega)\, e^{i\omega t}\,d\omega$$

(taking the inverse FT)

$$= \frac{1}{\sqrt{2\pi}}\int_{-\Omega}^{\Omega} e^{i\omega t}\,d\omega$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i\omega t}}{it}\right]_{-\Omega}^{\Omega} = \frac{1}{\sqrt{2\pi}}\left[-i\frac{e^{i\omega t}}{t}\right]_{-\Omega}^{\Omega}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{-i\frac{e^{i\Omega t}}{t} + i\frac{e^{-i\Omega t}}{t}\right\}$$

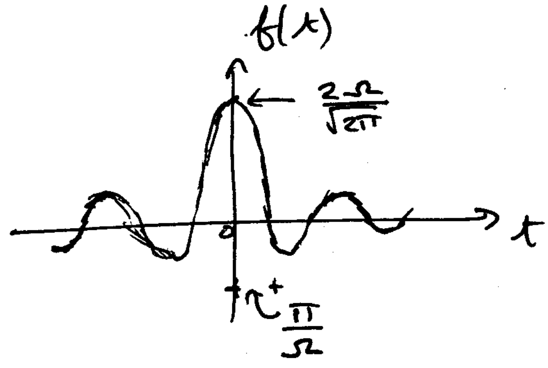
But $e^{i\Omega t} = \cos\Omega t + i\sin\Omega t$

$$\Rightarrow f_\Omega(t) = \frac{1}{\sqrt{2\pi}} \left\{ -\frac{i}{t}\left(\cos\Omega t + i \sin\Omega t\right) + \frac{i}{t}\left(\cos(-\Omega t) + i\sin(-\Omega t)\right) \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{t} \cdot 2 \sin \Omega t \right\}$$

$$= \frac{2\Omega}{\sqrt{2\pi}} \frac{\sin \Omega t}{\Omega t}$$

$$= \sqrt{\frac{2}{\pi}}\, \Omega \quad \text{at } t = 0$$



Therefore $f_\Omega(t) \to \sqrt{2\pi}\, \delta(t)$ as $\Omega \to \infty$.

Hence the δ function can also be represented by

$$\delta(t) = \lim_{\Omega \to \infty} \left(\frac{\sin \Omega t}{\pi t} \right).$$

3.2.3 Green's Function

Consider $G(x, x')$ such that

$$\mathcal{L}_x G(x, x') = \delta(x - x')$$

Then the function

$$y_p = \int G(x, x') F(x')\, dx'$$

solves the differential equation $\mathcal{L}_x y = F(x)$

since

$$\mathcal{L}_x y_p = \int \mathcal{L}_x G(x, x') F(x') \, dx'$$

$$= \int \delta(x - x') F(x') \, dx'$$

$$= F(x)$$

Schematically:

$$\mathcal{L} G = \delta \quad \Rightarrow \quad G = \mathcal{L}^{-1} \delta$$

$$y = \int G F \rightarrow \mathcal{L}^{-1} F \quad \text{since} \quad \mathcal{L} y = \mathcal{L}(\mathcal{L}^{-1} F) = F$$

The δ function $\delta(x - x')$ is like the identity operator in function space since

$$\int dx' \, \delta(x - x') F(x') = F(x)$$

3.2.4 GREEN'S FUNCTION FOR 1D PROBLEM

Consider the differential equation

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = F(x)$$

with boundary conditions $y(a) = y(b) = 0$.

Suppose that the solutions of the homogeneous equation are $y_1(x)$ and $y_2(x)$ such that $y_1(a) = 0$ and $y_2(b) = 0$.

The function $G(x,x')$ obeys

$$\mathcal{L}_x G(x,x') = \delta(x-x')$$

and we require $G(a,x')=0$ and $G(b,x')=0$.

We also need continuity of $G(x,x')$ at $x=x'$ but discontinuity of $\frac{dG}{dx}(x,x')$ at $x=x'$ since integrating over a small interval containing $x=x'$ gives

$$\int_{x'-\varepsilon}^{x'+\varepsilon} \frac{d}{dx}\left(\frac{dG}{dx}\right) dx + \int_{x'-\varepsilon}^{x'+\varepsilon} P(x)\frac{dG}{dx}\, dx + \int_{x'-\varepsilon}^{x'+\varepsilon} Q(x)\, G\, dx$$

$$= \int_{x'-\varepsilon}^{x'+\varepsilon} \delta(x-x')\, dx = 1$$

By continuity $\int_{x'-\varepsilon}^{x'+\varepsilon} Q\, G\, dx \to 0$ and

$$\int_{x'-\varepsilon}^{x'+\varepsilon} P\frac{dG}{dx}\, dx = \Big[PG\Big]_{x'-\varepsilon}^{x'+\varepsilon} - \int_{x'-\varepsilon}^{x'+\varepsilon} G\frac{dP}{dx}\, dx \to 0 \text{ as } \varepsilon \to 0$$

Then

$$\lim_{\varepsilon\to 0}\left[\frac{dG}{dx}\right]_{x'-\varepsilon}^{x'+\varepsilon} = \left(\text{discontinuity in } \frac{dG}{dx}\right) = 1$$

For $x \lessgtr x'$: $\mathcal{L}_x G = 0 \Rightarrow G(x,x') = \begin{cases} A(x')\, y_1(x), & x<x' \\ B(x')\, y_2(x), & x>x' \end{cases}$

in order to ensure $G(a,x') = G(b,x') = 0$.

Continuity of G at x'

$$\Rightarrow A(x')y_1(x') = B(x')y_2(x')$$

$$\Rightarrow \frac{A(x')}{B(x')} = \frac{y_2(x')}{y_1(x')}$$

Discontinuity of $\frac{dG}{dx}$ at x'

$$\Rightarrow B(x')y_2'(x') - A(x')y_1'(x') = 1$$

(where prime on y means d/dx).

Then $$\left(\frac{Ay_1}{y_2}\right)y_2' - Ay_1' = 1$$

$$\Rightarrow A\left(y_1 y_2' - y_2 y_1'\right) = y_2$$

$$\Rightarrow A = \frac{y_2}{W(y_1, y_2)}, \quad B = \frac{y_1}{W(y_1, y_2)}$$

where $$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

and all quantities are evaluated at $x = x'$.

Then

$$G(x, x') = \begin{cases} y_2(x')y_1(x)/W(y_1(x'), y_2(x')), & x < x' \\ y_1(x')y_2(x)/W(y_1(x'), y_2(x')), & x > x' \end{cases}$$

The solution to the inhomogeneous equation is

$$y(x) = \int_a^b G(x, x') F(x') \, dx'$$

$$= y_1(x) \int_x^b \frac{y_2(x') F(x')}{W(x')} \, dx' \quad \swarrow x' > x$$

$$+ \; y_2(x) \int_a^x \frac{y_1(x') F(x')}{W(x')} \, dx' \quad \swarrow x' < x \; .$$

e.g. Solve the differential equation

$$\frac{d^2 y}{dx^2} + y = \sec x$$

in $[0, \pi/2]$ with $y(0) = y(\pi/2) = 0$.

The homogeneous equation is

$$\frac{d^2 y}{dx^2} = -y$$

$\Rightarrow$ y is $\cos x$ or $\sin x$.

$y_1(0) = 0 \Rightarrow y_1(x) = \sin x$

$y_2(\pi/2) = 0 \Rightarrow y_2(x) = \cos x$.

Then $W(x') = y_1 y_2' - y_1' y_2$

$$= \sin x' (-\sin x') - \cos x' \cos x'$$

$$= -(\sin^2 x' + \cos^2 x') = -1$$

So
$$y(x) = y_1(x) \int_x^{\pi/2} \frac{y_2(x') \sec x'}{W(x')} dx'$$
$$+ y_2(x) \int_0^x \frac{y_1(x') \sec x'}{W(x')} dx'$$
$$= \sin x \int_x^{\pi/2} \frac{\cos x'}{(-1)\cos x'} dx' + \cos x \int_0^x \frac{\sin x'}{(-1)\cos x'} dx'$$
$$= -\sin x \int_x^{\pi/2} dx' - \cos x \int_0^x \tan x' \, dx'$$
$$= -\sin x \left[x' \right]_x^{\pi/2} + \cos x \left[\ln(\cos x') \right]_0^x$$
$$= -\sin x \left(\pi/2 - x \right) + \cos x \ln(\cos x)$$

A string vibrating between two fixed points has the differential equation,
$$-y'' + \ddot{y} = F(x) e^{i\omega t}$$
(where $\dot{}$ denotes d/dt). Try a solution of the form
$$y(x,t) = u(x) e^{i\omega t}$$
$$\Rightarrow y'' = u'' e^{i\omega t}, \quad \ddot{y} = -\omega^2 u e^{i\omega t}$$
giving the equation
$$u'' + \omega^2 u = -F(x)$$
This has the form assumed above.