2. Solving First and Second Order Linear Differential Equations

2.1 Introduction

In physics we very often deal with quantities which are smoothly varying functions of space and time. It is not surprising then that physical laws are often expressed as equations involving various derivatives of a quantity with respect to position and/or time. Such equations are referred to as differential equations and are to be contrasted with purely "algebraic" equations which do not involve derivatives.

e.g. (i) $$m\frac{d^2x(t)}{dt^2} = F \qquad (F = ma)$$

(ii) $$\frac{dy(x)}{dx} + 2y(x) + 3 = 0$$

The order of a given differential equation is the order of the highest derivative present. In the above (i) is a second-order differential equation whilst (ii) is a first-order differential equation.

Differential equations can be either ordinary or partial.

<u>Ordinary differential equations</u> (ODEs) are those where the unknown function depends on a single variable

e.g. $$\frac{dy(t)}{dt} + 2t = 4$$

or $$\frac{d^2x}{dt^2} = 3t$$

<u>Partial differential equations</u> (PDEs) are differential equations where the unknown function depends on more than one variable:

e.g. <u>Laplace's equation</u>:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

where $V = V(x, y, z)$

or <u>Schrödinger's Equation</u> (time independent)

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + V(x, y, z)\psi = E\psi$$

or <u>Wave equation</u>

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0$$

or <u>Diffusion Equation</u>:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = \frac{1}{a^2}\frac{\partial P}{\partial t}$$

Finally a differential equation (ODE or PDE) can be classified as either <u>linear</u> or <u>non-linear</u>.

An example of a general, linear, 1st order ODE:

$$\frac{dy(x)}{dx} + P(x)\,y(x) = Q(x)$$

Here $P(x)$, $Q(x)$ are functions of x only. Note that the highest power of y (or $\frac{dy}{dx}$) is 1 – this is why it is a <u>linear</u> ODE.

An example of a non-linear differential equation:

$$\frac{dy}{dx} = \sin\left(y(x)\right)$$

$\sin(y(x))$ is a non-linear function of $y(x)$.

There are no general rules for solving non-linear differential equations (as opposed to linear differential equations), so we will not consider them in any great detail.

2.2 Solving First-Order ODEs

These can be generally written as

$$\frac{dy}{dx} = P(x, y)$$

(1st order as it only involves $\frac{dy}{dx}$, not $\frac{d^2y}{dx^2}$, etc.)

for some function P depending on x <u>and</u> $y(x)$.

Let's look at some simplifying cases for $P(x,y)$:

(i) <u>Factorizable form</u> : $P(x, y) = f(x)\, g(y)$

Here f is any function of x only, and g is only a function of y but <u>not</u> explicitly x.

$$\frac{dy}{dx} = f(x)\, g(y)$$

$$\Rightarrow \quad \frac{dy}{g(y)} = f(x)\, dx$$

$$\Rightarrow \quad \int \frac{dy}{g(y)} = \int f(x)\, dx$$

Thus <u>if</u> we can calculate the indefinite integrals

$$\int \frac{dy}{g(y)} = G(y) \quad ; \quad \int f(x)\, dx = F(x)$$

for <u>known</u> functions $G(y)$ and $F(x)$,

then
$$G(y) = F(x) + c$$
$$y = G^{-1}\left(F(x) + c\right)$$

Note that this method can cope even with non-linear 1st order ODEs as long as we can compute the 2 integrals yielding $G(y)$ and $F(x)$.

e.g. $P(x,y) = xy^2$; $\dfrac{dy}{dx} = xy^2$

$$G(y) = \int \frac{dy}{y^2} \quad ; \quad F(x) = \int x\,dx .$$

$$\Rightarrow \quad G(y) = -\frac{1}{y} \quad ; \quad F(x) = \frac{1}{2}x^2 + c$$ constant of integration

$$-\frac{1}{y(x)} = \frac{1}{2}x^2 + c$$

$$\Rightarrow \quad y(x) = \frac{-1}{\frac{1}{2}x^2 + c}$$

(ii) Linear, 1st order ODEs : $P(x,y) = -f(x)y + g(x)$ for convenience

$$\frac{dy}{dx} + f(x)y = g(x)$$ for any functions $f(x)$, $g(x)$

The trick to solving this equation for general $f(x)$, $g(x)$ is to introduce some function $\alpha(x)$, and multiply our linear ODE with it:

$$\alpha(x)\frac{dy}{dx} + \alpha(x) f(x) y(x) = \alpha(x) g(x)$$

Now we require that the LHS of the above equation is a total derivative

$$\alpha(x)\frac{dy}{dx} + \alpha(x) f(x) y(x) \equiv \frac{d}{dx}\Big(\alpha(x) y(x)\Big)$$

Therefore, since

$$\frac{d}{dx}\Big(\alpha(x) y(x)\Big) = \left(\frac{d\alpha}{dx}\right) y + \alpha \frac{dy}{dx}$$

we require $\alpha(x)$ to satisfy its own differential equation:

$$\cancel{y}\left(\frac{d\alpha}{dx}\right) = \alpha(x) f(x) \cancel{y(x)}$$

$$\text{i.e.} \quad \frac{d\alpha}{dx} = \alpha(x) f(x)$$

But we can solve this (implicitly)

$$\int \frac{d\alpha}{\alpha} = \int f(x)\, dx$$

$$\Rightarrow \quad \ln \alpha = \int f(x)\, dx + c$$

$$\Rightarrow \quad \alpha(x) = C\, e^{\int f(x)\, dx}$$

where C is a constant of integration.

Therefore, having defined $\alpha(x)$ in this way, our linear, 1st order ODE becomes

$$\frac{d}{dx}\left(\alpha(x)\, y(x)\right) = \alpha(x)\, g(x)$$

$$\Rightarrow \quad \int \frac{d}{dx}\left(y(x)\,\alpha(x)\right) = \int \alpha(x)\, g(x)\, dx$$

$$\Rightarrow \quad y(x)\,\alpha(x) = \int \alpha(x)\, g(x)\, dx + \underbrace{c_2}_{\text{constant of integ.}}$$

and so finally,

$$y(x) = \frac{1}{\alpha(x)}\left[\int \alpha(x)\, g(x)\, dx + c_2\right]$$

Substituting our expression for $\alpha(x)$ found earlier gives

$$y(x) = e^{-\int f(x)\, dx} \times \left\{ \int e^{\int f(x)\, dx} g(x)\, dx + C \right\}$$

where we have combined all integration constants into C.

Note that all integrals are indefinite.

The form of $y(x)$ is :

$$y(x) = y_p(x) + y_0(x)$$

where $y_0(x)$ is a solution of our ODE with $g(x) = 0$.

This means that $y_0(x)$ satisfies

$$\frac{d y_0(x)}{dx} + f(x) y_0(x) = 0$$

$$\Rightarrow \quad y_0(x) = \text{const.} \times e^{-\int f(x)\,dx}$$

and $y_p(x)$ is a particular solution depending on the choice of $g(x)$:

$$y_p(x) = e^{-\int f(x)\,dx} \times \left\{ \int e^{\int f(x)\,dx} g(x)\,dx \right\}$$

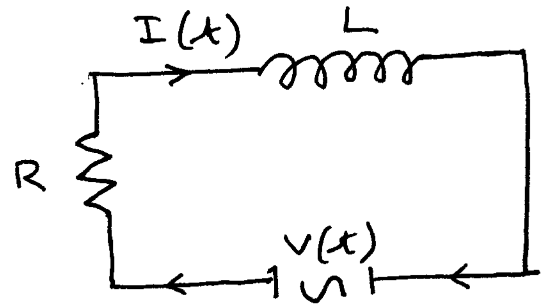
Sometimes $y_0(x)$ is called the <u>complementary function</u>.

The ODE with $g(x) = 0$,

$$\frac{dy}{dx} + f(x) y = 0$$

is called a homogeneous differential equation, as each term contains the same power of $y(x)$.

Consider the example of a R-L circuit:



Applying Kirchoff's current/voltage laws gives:

$$L \frac{dI(t)}{dt} + RI(t) = V(t)$$

This is a linear, 1st order ODE for $I(t)$.

Our function

$$\alpha(t) = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

The general solution is

$$I(t) = e^{-Rt/L} \left\{ \int e^{Rt/L} \frac{V(t)}{L} dt + c \right\}$$

Special case: What if we choose $V(t) = V_0$, a constant voltage?

$$I(t) = e^{-Rt/L} \left\{ \int e^{Rt/L} \frac{V_0}{L} . dt + c \right\}$$

$$= Ce^{-Rt/L} + \frac{V_0}{R}$$

Here the integration constant C can be given in terms of initial value of the current I at $t = 0$:

$$I(t=0) = I_0 = C + \frac{V_0}{R}$$

$$\Rightarrow C = \left(I_0 - \frac{V_0}{R}\right)$$

and so $$I(t) = \left(-\frac{V_0}{R} + I_0\right)e^{-Rt/L} + \frac{V_0}{R}$$

This example has shown us an important general feature. A first order ODE has <u>one</u> unknown constant (constant of "integration") in the expression for the general solution. This constant depends on so-called "initial" values or "boundary values" of our solution at a particular value of t. So C was determined by $I(t=0)$. In fact if we know I at <u>any</u> given time t_0, this is enough to fix C, because C is a constant and <u>does not change</u> <u>with time</u>.

Consider the example of the equation of motion for a body of mass m, falling under the influence of a constant gravitational field but with resistive "drag" present is:

$$m \frac{dv}{dt} = mg - \beta v$$

Here g = acceleration due to gravity = constant and β is the drag coefficient. Here we can think of v as the vertical component of the object's velocity as it falls through, e.g. the Earth's atmosphere.

$$\frac{dv(t)}{dt} = g - \frac{\beta}{m} v$$

or $$\frac{dv}{dt} + \frac{\beta}{m} v = g$$

Hence $$\alpha(t) = e^{\int \beta/m \, dt} = e^{(\beta/m) t}$$

The general solution is

$$v(t) = e^{-\frac{\beta}{m}t} \left(\int e^{\frac{\beta}{m}t} g \, dt + c \right)$$

$$= e^{-\frac{\beta}{m}t} \left(\frac{m}{\beta} g e^{\frac{\beta}{m}t} + c \right)$$ (since g is a constant)

i.e. $$v(t) = \frac{mg}{\beta} + c e^{-\frac{\beta}{m}t}$$

If we take $t = 0$, $v(0) = v_0$ = initial velocity

$$= \frac{mg}{\beta} + c$$

Hence $$c = v_0 - \frac{mg}{\beta}$$

$$\Rightarrow \quad v(t) = \frac{mg}{\beta} + \left(v_0 - \frac{mg}{\beta}\right) e^{-\frac{\beta}{m} t}$$

Notice that if $v_0 = \frac{mg}{\beta}$ then $v(t) =$ constant.

Physically the forces on the falling object are zero in this case – the acceleration due to gravity is balanced by the friction of the air.

2.3 SOLVING 2ND ORDER ODEs

The most general, linear, 2nd order ODE can be written as

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = F(x)$$

where P, Q and F are functions of x only.

When $F = 0$, the above is a homogeneous 2nd order ODE.

We will discuss the inhomogeneous case $(F \neq 0)$ later. For now, let's concentrate on the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

We can consider some simplifying special cases

(i) $Q(x) = 0$

Then $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} = 0$$

If we call

$$u(x) = \frac{dy}{dx}$$

then

$$\frac{du}{dx} + P(x)u(x) = 0$$

This is a homogeneous 1st order ODE which we can solve, e.g. by methods discussed earlier.

Even if we take $F(x) \neq 0$ we can still solve

$$\frac{du}{dx} + P(x)u(x) = F(x)$$

Once an exact solution is known, we finally have to solve the differential equation:

$$\frac{dy(x)}{dx} = u(x), \qquad y(x) = \int u(x)\,dx + c_2$$

to get $y(x)$.

Since the solution for $u(x)$ will involve some unknown constant of integration, call it c_1 (see last section), we can see that $y(x)$ depends on 2 arbitrary constants, c_1 and c_2. In the previous section we found that solutions of a 1st order linear ODE depend only on <u>one</u> arbitrary constant. Both c_1 and c_2 can be fixed by <u>boundary conditions</u>.

e.g. The equation of motion for a body of mass m moving under the force of friction depending linearly on its velocity is

$$m \frac{d^2 y}{dt^2} - \beta \frac{dy}{dt} = 0$$ (here for simplicity we only consider motion in 1-D)

So let $$u(t) = \frac{dy}{dt}$$

i.e. $$m \frac{du}{dt} - \beta u = 0$$

Solving $$\int \frac{du}{u} = \int \frac{\beta}{m} dt = \frac{\beta}{m} t + c_1$$

$$\Rightarrow \ln u = \frac{\beta}{m} t + c_1 \quad \Rightarrow u(t) = e^{\frac{\beta}{m} t + c_1}$$

Finally we have to solve

$$u(t) = \frac{dy}{dt} = e^{\frac{\beta}{m} t + c_1}$$ to obtain $y(t)$.

$$\int dy = \int e^{\frac{\beta}{m} t + c_1} dt \Rightarrow y(t) = \left(\frac{m}{\beta} e^{\frac{\beta}{m} t + c_1} + c_2 \right)$$

Note that $$y(t=0) = \frac{m}{\beta} e^{c_1} + c_2$$

and $$\left.\frac{dy}{dt}\right|_{t=0} = u(t=0) = e^{c_1}$$

So by specifying the values of $y(t=0)$ and $\left(\frac{dy}{dt}\right)_{t=0}$, c_1 and c_2 are determined.

(ii) $P(x) = Q(x) = 0, \; F(x) \neq 0$

In this case the equation becomes

$$\frac{d^2y}{dx^2} = F(x)$$

Integrating directly gives:

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \int F(x)\,dx$$

$$\Rightarrow y(x) = \int \frac{dy}{dx}\,dx = \int \left(\int F(x)\,dx \right) dx.$$

As long as we can evaluate $\int F(x)\,dx$ and $\int \left(\int F(x)\,dx \right) dx$, $y(x)$ is obtained.

<u>More general cases</u>: Series Solutions

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0.$$

Let's try to find a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$ where a_n are coefficients yet to be determined.

This method could have been used to solve the general 1st order ODE of the last section if all other methods fail. But the series method depends on being able to solve for a_n which is not always exactly possible.

The idea is to substitute our series expansion into the ODE and obtain a (hopefully) solvable set of algebraic equations. For general functions $P(x)$, $Q(x)$ this is difficult to carry out. But let's look at a familiar example to see how this method works.

Consider the case $P(x) = 0$, $Q(x) = \omega^2 > 0$ where ω is a constant.

$$\frac{d^2y}{dx^2} = -\omega^2 y.$$

(Note: If $x = t$ this is related to the familiar equation of motion of a simple harmonic oscillator.)

We know the general solution is simply

$$y(x) = A\sin(\omega x) + B\cos(\omega x)$$

where A, B are the familiar integration constants (2 constants because y satisfies a 2nd order ODE).

But let's see how the series method can reproduce the above solution.

If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, $\quad \frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1}$

and $$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

The differential equation becomes

$$\underbrace{\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}}_{d^2y/dx^2} = \underbrace{-\omega^2 \sum_{n=0}^{\infty} a_n x^n}_{-\omega^2 y(x)}$$

Now, on the LHS there is no contribution from $n=0$, $n=1$ terms. Therefore

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\omega^2 \sum_{n=0}^{\infty} a_n x^n$$

Let $n' = n-2$. Then

$$\sum_{n'=0}^{\infty} (n'+2)(n'+1) a_{n'+2} x^{n'} = -\omega^2 \sum_{n=0}^{\infty} a_n x^n$$

i.e. $$\sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} x^n + \omega^2 a_n x^n \right) = 0$$

i.e. $$\sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} + \omega^2 a_n \right) x^n = 0$$

In order for an infinite power series to vanish for any value of x, all of its coefficients must vanish. This gives

$$(n+2)(n+1)\, a_{n+2} + \omega^2 a_n = 0 \qquad (n = 0, 1, 2, \ldots)$$

This equation defines a so-called recursion-relation for the a_n. Putting in values ~~of~~ for $n = 0, 1, 2, \ldots$ we find

$n=0 \qquad 2\cdot 1\cdot a_2 = -\omega^2 a_0 \quad \Rightarrow \quad a_2 = \dfrac{-\omega^2}{2\cdot 1} a_0$

$n=1 \qquad 3\cdot 2\cdot a_3 = -\omega^2 a_1 \quad \Rightarrow \quad a_3 = \dfrac{-\omega^2}{3\cdot 2} a_1$

$n=2 \qquad 4\cdot 3\cdot a_4 = -\omega^2 a_2$

$$= \frac{(-\omega^2)^2}{2\cdot 1} a_0 \quad \Rightarrow \quad a_4 = \frac{(-\omega^2)^2}{4\cdot 3\cdot 2\cdot 1} a_0$$

$n=3 \qquad 5\cdot 4\cdot a_5 = -\omega^2 a_3$

$$= \frac{(-\omega^2)^2}{3\cdot 2} a_1 \quad \Rightarrow \quad a_5 = \frac{(-\omega^2)^2}{5\cdot 4\cdot 3\cdot 2} a_1$$

$n=4 \qquad 6\cdot 5\cdot a_6 = -\omega^2 a_4$

$$= \frac{(-\omega^2)^3}{4\cdot 3\cdot 2\cdot 1} a_0 \quad \Rightarrow \quad a_6 = \frac{(-\omega^2)^3}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} a_0$$

$n=5 \qquad 7\cdot 6\cdot a_7 = -\omega^2 a_5$

$$= \frac{(-\omega^2)^3}{5\cdot 4\cdot 3\cdot 2} a_1 \quad \Rightarrow \quad a_7 = \frac{(-\omega^2)^3}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} a_1$$

etc.

So <u>all</u> the a_n for n odd are related to a_1 while all the a_n for n even are related to a_0.

The solutions for the recursion relations for a_n are

$$a_n = \frac{(-1)^{\frac{n-1}{2}} \omega^{n-1}}{n!} a_1 \qquad n = 1, 3, 5, \ldots$$

$$a_n = \frac{(-1)^{\frac{n}{2}} \omega^{n}}{n!} a_0 \qquad n = 0, 2, 4, \ldots$$

So our solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \underset{\substack{n=0,2,4,\ldots \\ (\text{sum over} \\ n \text{ even})}}{\sum^{\infty}} a_n x^n + \underset{\substack{n=1,3,5,\ldots \\ (\text{sum over} \\ n \text{ odd})}}{\sum^{\infty}} a_n x^n$$

$$y(x) = \sum_{n \text{ even}} \frac{(-1)^{n/2} \omega^n}{n!} a_0 x^n + \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}}{n!} \omega^{n-1} a_1 x^n$$

$$= a_0 \sum_{n \text{ even}} \frac{(-1)^{n/2} \omega^n}{n!} x^n + a_1 \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}} \omega^{n-1}}{n!} x^n$$

Now recall the series expansion for $\sin \omega x$ and $\cos \omega x$:

$$\sin(\omega x) = \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}}{n!} (\omega x)^n$$

$$\cos(\omega x) = \sum_{n \text{ even}} \frac{(-1)^{\frac{n}{2}}}{n!} (\omega x)^n$$

Therefore we have found our solution :

$$y(x) = \frac{a_1}{\omega}\sin(\omega x) + a_0 \cos(\omega x)$$

where a_1, a_0 are arbitrary constants and play the role of constants of integration.

Note also that we can define a slightly more general series expansion than the one we have used so far:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{k+n}$$

where k could be any real number (not necessarily integer valued). In our simple harmonic motion example we took $k=0$ and still obtained most general solution. For other 2nd order ODEs we may have to take $k \neq 0$.

2.4 HERMITE'S DIFFERENTIAL EQUATION

Hermite's equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\alpha y(x) = 0 .$$ for some real number α.

This is clearly a 2nd order, linear (and homogeneous) ODE.

A trick to simplify the problem of solving this equation is to introduce a new function $u(x)$ related to $y(x)$ by :

$$y(x) = e^{\frac{x^2}{2}} u(x)$$

Then $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(e^{x^2/2} u(x)\right) = x e^{x^2/2} u(x) + e^{x^2/2} \dfrac{du}{dx}$

So $\dfrac{d^2y}{dx^2} = x^2 e^{x^2/2} u(x) + e^{x^2/2} u(x) + x e^{x^2/2} \dfrac{du}{dx} + x e^{x^2/2} \dfrac{du}{dx} + e^{x^2/2} \dfrac{d^2u}{dx^2}$

$$= (x^2+1) e^{x^2/2} u(x) + 2x e^{x^2/2} \frac{du}{dx} + e^{x^2/2} \frac{d^2u}{dx^2}$$

Hence Hermite's DE becomes :

$$(x^2+1) e^{x^2/2} u(x) + \cancel{2x e^{x^2/2} \frac{du}{dx}} + e^{x^2/2} \frac{d^2u}{dx^2}$$

$$- 2x \left(x e^{x^2/2} u + \cancel{e^{x^2/2} \frac{du}{dx}}\right) + 2\alpha e^{x^2/2} u = 0$$

$$\Rightarrow e^{x^2/2} \left(\frac{d^2u}{dx^2} + (2\alpha + 1 - x^2) u\right) = 0$$

i.e. $$\frac{d^2u}{dx^2} + (2\alpha + 1 - x^2) u = 0$$

So if we can solve this equation for $u(x)$ we have our solution because $y(x) = e^{x^2/2} u(x)$.

This last equation has important applications in Physics; in particular it is related to

The Schrödinger equation for the Quantum Mechanical Simple Harmonic Oscillator (see QMA, QMB).

Schrödinger equation : $\hat{H}\psi(x) = E\psi(x)$.

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x).$$

For a particle moving in a simple harmonic potential, $V(x) = \frac{1}{2}m\omega^2 x^2$.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left(\frac{1}{2}m\omega^2x^2 - E\right)\psi(x) = 0$$

or $$\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right)\psi = 0$$

To make this look like our equation for $u(x)$, let $x = \sqrt{\frac{\hbar}{m\omega}}\, z$ (i.e. a rescaled coordinate)

$$\Rightarrow \quad \frac{d^2\psi}{dz^2} + (\lambda - z^2)\psi = 0, \qquad \lambda = \frac{2E}{\hbar\omega}$$

<u>CHECK</u> $\quad x^2 = \frac{\hbar}{m\omega}z^2 \quad$ and $\quad \frac{d}{dx} = \sqrt{\frac{m\omega}{\hbar}}\frac{d}{dz}$

i.e. $$\frac{d^2}{dx^2} = \frac{m\omega}{\hbar}\frac{d^2}{dz^2}$$

i.e. $\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x\right)\psi = 0$ becomes

$$\frac{m\omega}{\hbar}\frac{d^2\psi}{dz^2} + \left(\lambda\frac{m\omega}{\hbar} - \frac{m^2\omega^2}{\hbar^2}\cdot\frac{\hbar}{m\omega}z^2\right)\psi = 0$$

or $\frac{m\omega}{\hbar}\left\{\frac{d^2\psi}{dz^2} + \left(\lambda - z^2\right)\psi\right\} = 0$

$$\Rightarrow \quad \frac{d^2\psi}{dz^2} + (\lambda - z^2)\psi = 0$$

So, $\psi(x) = u(x)$ if we identify $\lambda = 2\alpha + 1$

Therefore finding a solution to Hermite's DE, $y(x)$, allows us to obtain a wave function of the quantum mechanical harmonic oscillator,

$$\psi(x) = u(x) = e^{-x^2/2}\, y(x) .$$

Let's try to find a series solution :

$$y(x) = \sum_{n=0}^{\infty} a_n x^{k+n}$$

Substituting this into the Hermite DE we find

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (k+n)\, a_n\, x^{k+n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (k+n)(k+n-1)\, a_n\, x^{k+n-2}$$

So the Hermite equation becomes

$$\left(\sum_{n=0}^{\infty} (k+n)(k+n-1)\, a_n x^{k+n-2}\right) - 2\left(\sum_{n=0}^{\infty} (k+n)\, a_n x^{k+n}\right) + 2\alpha\left(\sum_{n=0}^{\infty} a_n x^{k+n}\right) = 0$$

All the coefficients must vanish to satisfy the equation.

Shifting the index on the first term, we obtain (with $m = n-2$)

$$\sum_{m=-2}^{\infty} (m+2+k)(m+1+k)\,a_{m+2}\,x^{m+k}$$

For the coefficients to vanish always (for all m) consider the terms for $m=-2$ and $m=-1$

$$m=-2, \qquad k(k-1)\,a_0\,x^{k-2} = 0 \qquad (a)$$

$$m=-1, \qquad k(k+1)\,a_1\,x^{k-1} = 0 \qquad (b)$$

The coefficient of x^{m+k} gives (from DE)

$$(m+2+k)(m+1+k)\,a_{m+2} = 2(m+k)\,a_m - 2\alpha\,a_m$$

$$\Rightarrow a_{m+2} = 2a_m \frac{(m+k-\alpha)}{(m+2+k)(m+1+k)}$$

In order to solve the equation each power of x must have coefficient zero.

So (a) gives the <u>indicial equation</u>

$$k(k-1)\,a_0\,x^{k-2} = 0$$

$$\Rightarrow k(k-1) = 0$$

Therefore $k=0$ or $k=1$. (Note that if $k=1$ we must have $a_1 = 0$ from (b)).

However in each case we would not have an

$m = -2$ term (the coefficient of x^{k-2}).

So we set $k = 0$ and are careful about the lower limits on our sums.

$$\Rightarrow \quad y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

giving $$a_{n+2} = \frac{2a_n(n-\alpha)}{(n+1)(n+2)} \qquad n = 0, 2, 4, \ldots \quad (a_1 = a_3 = \cdots = 0)$$

For the case $k = 1$ we have

$$a_{n+2} = \frac{2a_n(n+1-\alpha)}{(n+2)(n+3)} \qquad n = 0, 2, 4, \ldots \quad (a_1 = a_3 = \cdots = 0)$$

So, for $k = 0$ we have the series solution

$$y_{even}(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 \left[1 + \frac{2(-\alpha)}{2!} x^2 + \frac{2^2(-\alpha)(2-\alpha)x^4}{4!} + \cdots \right]$$

and for $k = 1$ we have the series solution

$$y_{odd}(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$

$$= a_0 \left[x + \frac{2(1-\alpha)x^3}{3!} + \frac{2^2(1-\alpha)(3-\alpha)x^5}{5!} + \dots \right]$$

Note that "even" and "odd" subscripts above indicate that the solution is either even or odd function (i.e. $f(-x) = f(x)$ or $f(-x) = -f(x)$)

<u>Special case of α = integer</u>

For general real values of the parameter α, both of the solutions above, $y_{even}(x)$ and $y_{odd}(x)$, are infinite power series in x. For the special case when $\alpha = n$, where n is some positive integer, a dramatic simplification happens – the series terminates after a finite number of terms!

For example, take $\alpha = n = 0$, then for $k = 0$

$$y_{even}(x) = a_0 = \text{const.}$$

$n = 1$, $\quad y_{odd}(x) = a_0 x$

$n = 2$, $\quad y_{even}(x) = a_0 - 2a_0 x^2 = a_0(1 - 2x^2)$

$n = 3$, $\quad y_{odd}(x) = a_0 x - \frac{2}{3} a_0 x^3 = a_0\left(x - \frac{2}{3}x^3\right)$

etc.

In fact, it turns out that we can write our solution as

$$y_{(n=0)}(x) = a_0 = a_0 H_0(x)$$

$$y_{(n=1)}(x) = a_0 x = \frac{1}{2} a_0 H_1(x)$$

$$y_{(n=2)}(x) = a_0 (1 - 2x^2) = -\frac{1}{2} a_0 H_2(x)$$

$$y_{(n=3)}(x) = a_0 \left(x - \frac{2}{3} x^3\right) = -\frac{1}{12} a_0 H_3(x)$$

etc.

The functions $H_0(x), H_1(x), H_2(x), \ldots$ are the so-called <u>Hermite polynomials</u>:

$$H_0 = 1$$

$$H_1 = 2x$$

$$H_2 = 4x^2 - 2$$

$$H_3 = 8x^3 - 12x$$

etc.

So we see that in the special case $\alpha = n$, Hermite's equation is solved by simple polynomial functions.