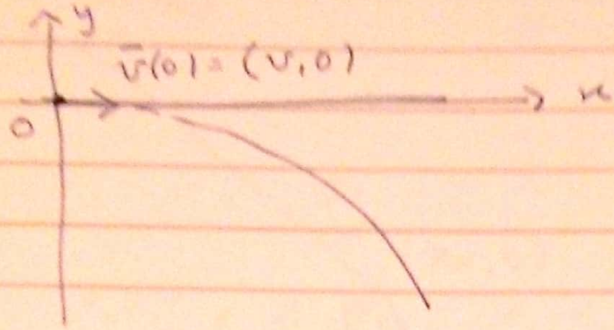
MT3 HW10

1



$\bar{v}(0)=(v,0)$

$x(0)=0=y(0)$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)=0 \quad \& \quad \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)=0$$

$\therefore 0 - \frac{d}{dt}(m\dot{x}) = 0$

$\therefore m\dot{x} = c$

$\therefore x = \frac{c}{m}t + c'$

$\therefore -mg - m\ddot{y} = 0$

$\therefore \cancel{mg}\ m\ddot{y} = -mg$

$\therefore \ddot{y} = -g$

$\therefore y = -\frac{1}{2}gt^2 + dt + d'$

Since $x(0)=0$

$\therefore c'=0$

$\dot{x}(0)=v$

$\therefore \frac{c}{m} = v$

ie $c = vm$

$\therefore x = vt$

$y(0)=0$

$\therefore d'=0$

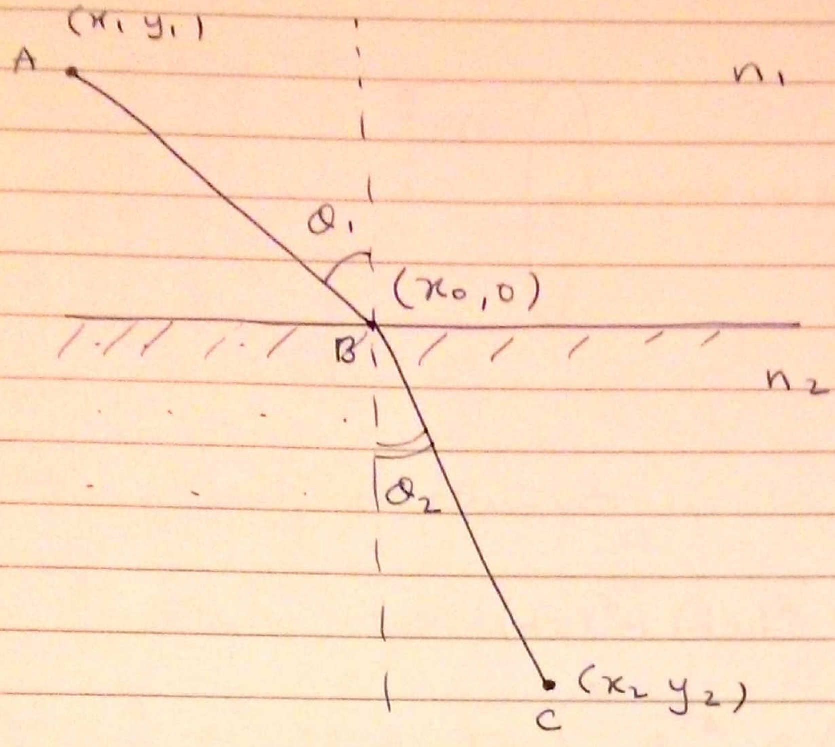
$\dot{y}(0)=0$

$\therefore d=0$

$\therefore y = -\frac{1}{2}gt^2$

$$\therefore y(x) = -\frac{1}{2}g\frac{x^2}{v^2}$$

2.



Optical path is $n \times s$

Total optical path

$$\Delta S = n_1 R(A \to B) + n_2 R(B \to C)$$
$$= n_1 \sqrt{(x_1 - x_0)^2 + y_1^2} + n_2 \sqrt{(x_0 - x_2)^2 + y_2^2}$$

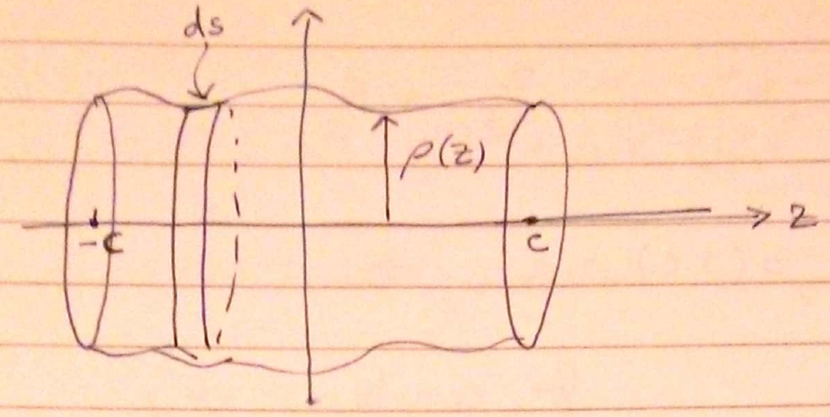
Minimize (actually extremize) ΔS w.r.t x_0:

$$0 = \frac{\partial \Delta S}{\partial x_0} = \frac{-n_1 (x_1 - x_0)}{\sqrt{(x_1 - x_0)^2 + y_1^2}} + \frac{n_2 (x_0 - x_2)}{\sqrt{(x_0 - x_2)^2 + y_2^2}}$$

$$= -n_1 \cos\left(\frac{\pi}{2} - \theta_1\right) + n_2 \cos\left(\frac{\pi}{2} - \theta_2\right)$$

$$\therefore n_1 \sin \theta_1 = n_2 \sin \theta_2$$

3.



The surface element is

$$dS = 2\pi\rho\sqrt{(dz)^2+(d\rho)^2} = 2\pi\rho\left(1+\rho'^2\right)^{1/2}dz$$

$$\therefore I = \int \rho^{-1/2}\,dS = 2\pi\int_{-c}^{c}\underbrace{\rho^{-1/2}\rho\left(1+\rho'^2\right)^{1/2}}_{f(\rho,\rho',z)=\rho^{1/2}(1+\rho'^2)^{1/2}}dz$$

There is no z-dependence
∴ use the alternative form of the Euler-Lagrange eqn

$$f - \rho'\frac{\partial f}{\partial\rho'} = k \quad \text{(constant)}$$

$$\therefore \rho^{1/2}\sqrt{1+\rho'^2} - \rho' \times \frac{\rho^{1/2}\rho'}{\sqrt{1+\rho'^2}} = k$$

$$\therefore \rho^{1/2}\left(1+\rho'^2-\rho'^2\right) = k\sqrt{1+\rho'^2}$$

$$\therefore \rho = k^2\left(1+\rho'^2\right)$$

$$\therefore \rho' = \frac{d\rho}{dz} = \left(\frac{\rho-k^2}{k^2}\right)^{1/2}$$

$$\therefore \int\frac{d\rho}{\sqrt{\rho-k^2}} = \int\frac{dz}{k}$$

$$\therefore\ 2\sqrt{\rho - k^2} = \frac{z}{k} + c$$

B.C.s $\rho(\pm c) = a \implies c = 0$

$$\& \quad c - k^2 = \frac{c^2}{4k^2}$$

$$\therefore\ 4k^4 - 4ak^2 + c^2 = 0$$

$$\therefore\ k^2 = \frac{1}{2}\left[a \pm (a^2 - c^2)^{1/2}\right]$$

Re-define $k^2 \to k$ and ~~we~~

$$\therefore\ 2\sqrt{\rho - k} = \frac{z}{\sqrt{k}} \quad , k =$$

$$\therefore\ \rho - k = \frac{z^2}{4k}$$

or $\rho(z) = \dfrac{z^2}{4k} + k$

4.

$$J = \int_0^1 [x^4 (y'')^2 + 4x^2 (y')^2]\, dx$$

$y(x)$ is non-singular

$y(1) = y'(1) = 1$

Set $u(x) = y'(x)$

$$\therefore J = \int_0^1 \underbrace{[x^4 (u')^2 + 4x^2 u^2]}_{f(u, u', x)}\, dx$$

E-L: $$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) = 0$$

$$8x^2 u - \underbrace{\frac{d}{dx}\left(2x^4 u'\right)}_{= 8x^3 u' + 2x^4 u''} = 0$$

$$\therefore \quad x^2 u'' + 4x u' - 4u = 0$$

Ansatz $u(x) = A x^n$

$\therefore \quad n(n-1) + 4n - 4 = 0$

$\therefore \quad n = -4$ or 1.

$$\therefore \quad u(x) = A x^{-4} + Bx = \frac{dy}{dx}$$

$$\therefore \quad y(x) = -\frac{A}{3} x^{-3} + \frac{B}{2} x^2 + C$$

Since y is not singular $\therefore$ $A = 0$.

B.C.s $y(1) = 1 = y'(1)$

$\therefore \frac{B}{2} + C = 1$ & $B = 1$

$\therefore C = 1/2$

So, $y(x) = \frac{1}{2}(1 + x^2)$

$\therefore y' = x$ & $y'' = 1$

$\therefore$ The extremized value of J is

$$J_{\text{extremized}} = \int_0^1 [x^4 (1)^2 + 4x^2 (x)^2] dx$$

$$= \int_0^1 5x^4 \, dx$$

$$= 1.$$