SPA5218 Mathematical Techniques 3
Exercise Sheet 10

1. Write down the Lagrangian for the motion of a projectile of mass m in the x-y plane. The projectile is launched from $(0, 0)$ with initial velocity $\mathbf{v} = (v, 0)$ and subsequently falls due to gravity. Use the Euler–Lagrange equations to derive the equations of motion in the x and y directions. Solve these, and using the initial conditions, show that the path taken by the projectile is a parabola given by $y = -\frac{1}{2}gx^2/v^2$.

2. A ray of light follows a straight-line path in a first homogeneous medium, is refracted at an interface, and then follows a new straight-line path in the second medium. See Fig. 22.7 in the 7^{th} edition of Arfken. Use Fermat's principle of optics to derive Snell's law of refraction:

   $$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

   We have set up this problem in class. Assume that the light starts off from point (x_1, y_1), passes through the interface at point $(x_0, 0)$, and finally ends up at point (x_2, y_2). Vary the value of x_0 to find proves Snell's law.

3. A surface of revolution, whose equation in cylindrical polar coordinates is $\rho = \rho(z)$, is bounded by the circles $\rho = a$, $z = \pm c$ with $a > c$, that is, both circles are of radius a and are located at $\pm c$ along the z-axis.

   Show that the function that makes the surface integral

   $$I = \int \rho^{-1/2} dS$$

   stationary with respect to small variations is given by $\rho(z) = k + z^2/(4k)$, where $k = [a \pm (a^2 - c^2)^{1/2}]/2$.

   Hint: Show that the surface element dS in cylinderical polar coordinates is $dS = 2\pi\rho\sqrt{(dz)^2 + (d\rho)^2}$.

4. Derive the minimum value that the integral

   $$J = \int_0^1 [x^4(y'')^2 + 4x^2(y')^2]\mathrm{d}x$$

   can have, given that y is not singular at $x = 0$ and that $y(1) = y'(1) = 1$. Assume that the Euler–Lagrange equation gives the lower limit.

   Hints:
   Set $u(x) = y'(x)$. This will cast the integrand in a form that is suitable for the Euler–Lagrange equations. Then determine $u(x)$, integrate it using the ansatz $u = Ax^n$, and hence determine the curve $y(x)$. Then use the boundary conditions and the fact that y is non-singular to find $y(x)$. Then find the value of the integral.