SPA5218 MATHEMATICAL TECHNIQUES 3

EXERCISE SHEET 9 : SOLUTIONS

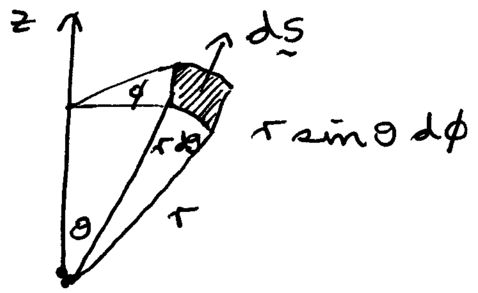
Q1.

Gauss's theorem implies

$$\iiint_V \nabla.(\nabla G)\, dV = \iint_S \nabla G . d\underset{\sim}{S} = \iiint_V \delta(x)\delta(y)\delta(z)\, dV = 1$$

For a spherical surface

$$d\underset{\sim}{S} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\underset{\sim}{r}}$$



and

$$\nabla G = \frac{dG}{dr}\hat{\underset{\sim}{r}} \quad \text{since } G = G(r) \text{ by spherical symmetry.}$$

Therefore

$$\iint_S \nabla G . d\underset{\sim}{S} = \iint_S \frac{dG}{dr} . r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{dG}{dr} r^2 \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi r^2 \frac{dG}{dr} = 1$$

Integrating gives $\quad G(r) = \int \frac{dr}{4\pi r^2} = -\frac{1}{4\pi r}$

We have

$$\nabla . \underset{\sim}{E} = -\nabla.\nabla V = -\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial z^2} = \frac{\rho}{\varepsilon_0}$$

For charge Q at origin,

$$\rho(x, y, z) = Q\,\delta(x)\,\delta(y)\,\delta(z)$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{Q}{\varepsilon_0}\,\delta(x)\,\delta(y)\,\delta(z)$$

Hence $V(r) = -\dfrac{Q}{\varepsilon_0} G(r) = \dfrac{Q}{4\pi\varepsilon_0 r} = \dfrac{Q}{4\pi\varepsilon_0\sqrt{x^2+y^2+z^2}}$

For a charge Q at $\underset{\sim}{r}' = (x', y', z')$, the potential at $\underset{\sim}{r} = (x, y, z)$ is

$$V(\underset{\sim}{r}) = \frac{Q}{4\pi\varepsilon_0 |\underset{\sim}{r} - \underset{\sim}{r}'|} = \frac{Q}{4\pi\varepsilon_0\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

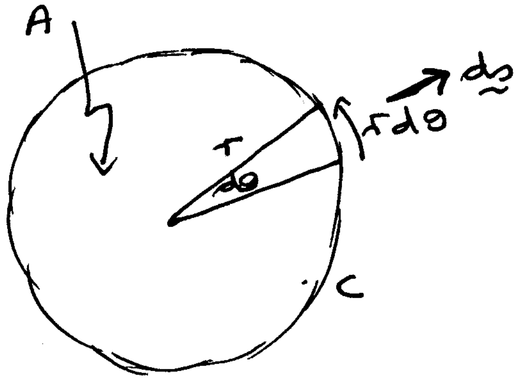
Q2. Integrate the differential equation for the Green's function over a circle A of radius r and centre at the origin:

$$\iint_A \nabla^2 G(x, y)\, dA = \iint_A \delta(x)\,\delta(y)\, dA$$

$$\Rightarrow \quad \iint_A \nabla\cdot\left(\nabla G(x, y)\right) dA = 1$$

We can use Gauss's theorem with $\underset{\sim}{V} = \nabla G(x, y)$ where the boundary C is the circumference of the circle A.

$$\int_C \nabla G(x,y) \cdot d\underset{\sim}{s} = 1$$



The measure $d\underset{\sim}{s}$ is a vector directed orthogonally out from A and C with a magnitude equal to the arclength of an element of C.

We can write $d\underset{\sim}{s} = \hat{r}\, r\, d\theta$.

Spherical symmetry requires $G(x,y) \to G(r)$ and so

$$\int_0^{2\pi} \frac{\partial G(r)}{\partial r}\, r\, d\theta = 1$$

$$\Rightarrow \frac{\partial G}{\partial r} = \frac{1}{2\pi r} \quad \Rightarrow \quad G(r) = \frac{1}{2\pi} \ln r$$

For the two charges ($-Q$ at $x = x_1$ and Q at $x = -x_1$) the charge density function can be written

$$\rho(x,y) = -Q\,\delta(x-x_1)\,\delta(y) + Q\,\delta(x+x_1)\,\delta(y)$$

Hence, the potential is

$$V(x,y) = \int dx'\, dy'\, G(x-x', y-y') \left(-\frac{\rho(x',y')}{\varepsilon_0} \right)$$

Since $\partial/\partial x \equiv \partial/\partial(x-x')$ and $\partial/\partial y \equiv \partial/\partial(y-y')$

this is easily shown to be a solution of

$$\nabla^2 V = -\rho/\varepsilon_0 :$$

$$\nabla^2 V(x,y) = \nabla^2 \int dx'\,dy'\; G(x-x', y-y')\left(-\frac{\rho(x',y')}{\varepsilon_0}\right)$$

$$= \int dx'\,dy' \left(\nabla^2 G(x-x', y-y')\right)\left(-\frac{\rho(x',y')}{\varepsilon_0}\right)$$

$$= \int dx'\,dy'\; \delta(x-x')\,\delta(y-y')\left(-\frac{\rho(x',y')}{\varepsilon_0}\right)$$

$$= -\frac{\rho(x,y)}{\varepsilon_0}$$

Using the explicit expressions for $G(x,y)$ and $\rho(x,y)$ we have

$$V(x,y) = -\frac{1}{2\pi\varepsilon_0}\int dx'\,dy'\; \ln\sqrt{(x-x')^2+(y-y')^2}$$
$$\times\left(-Q\delta(x-x_1)\delta y + Q\,\delta(x+x_1)\,\delta(y)\right)$$

$$= +\frac{1}{2\pi\varepsilon_0}\left(Q\ln\sqrt{(x-x_1)^2+y^2} - Q\ln\sqrt{(x+x_1)^2+y^2}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\ln\left[(x-x_1)^2+y^2\right] - \ln\left[(x+x_1)^2+y^2\right]\right)$$

For $y=0$ and $x \gg x_1$, this gives

$$V(x) = \frac{Q}{4\pi\varepsilon_0}\left[2\ln x\left(1-\frac{x_1}{x}\right) - 2\ln x\left(1+\frac{x_1}{x}\right)\right]$$

$$= \frac{Q}{2\pi\varepsilon_0}\left[\ln\left(1-\frac{x_1}{x}\right) - \ln\left(1+\frac{x_1}{x}\right)\right] \simeq -\frac{Qx_1}{\pi\varepsilon_0 x}$$

Q3. $$I = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$$

$$\delta I = \int_{t_1}^{t_2} \delta L(x, \dot{x}, t)\, dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \frac{d}{dt} (\delta x) \right) dt$$

Integrating the second term by parts gives

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x\, dt + \left[\frac{\partial L}{\partial \dot{x}} \delta x \right]_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x\, dt$$

where the boundary conditions $\delta x(t_1) = \delta x(t_2) = 0$ are applied.

At an extremum of I we require $\delta I = 0$ for any δx and so this requires

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$$

$$\Rightarrow \quad \frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}$$

Now show that the Euler-Lagrange equation can be written in the form

$$\frac{\partial L}{\partial t} = \frac{d}{dt}\left(L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right)$$

First apply the chain rule to dL/dt :

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x}$$

$$\Rightarrow \quad \frac{\partial L}{\partial t} = \frac{d}{dt}L - \frac{\partial L}{\partial x}\dot{x} - \frac{\partial L}{\partial \dot{x}}\ddot{x}$$

Then we use the Euler-Lagrange equation on the second term on the RHS to replace $\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$, hence

$$\frac{\partial L}{\partial t} = \frac{d}{dt}L - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} - \frac{\partial L}{\partial \dot{x}}\frac{d}{dt}(\dot{x})$$

$$= \frac{dL}{dt} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\dot{x}\right)$$

$$\Rightarrow \quad \frac{\partial L}{\partial t} = \frac{d}{dt}\left(L - \frac{\partial L}{\partial \dot{x}}\dot{x}\right)$$

Q4. $$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

$$L = \frac{1}{2}m\dot{x}^2 - kx^2 \quad \Rightarrow \quad \frac{\partial L}{\partial x} = -2kx$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

and $\quad m\ddot{x} = -2kx$

This is equivalent to Newton's equation $F = ma$ in the case $F = -2kx$, i.e. Hooke's Law.

By writing $\omega = \sqrt{2k/m}$ we obtain

$$\ddot{x} = -\omega^2 x(t)$$

which has the general solution

$$x(t) = A' \cos\omega t + B' \sin\omega t$$

CHECK: $\quad \dot{x}(t) = -\omega A' \sin\omega t + \omega B' \cos\omega t$

$$\ddot{x}(t) = -\omega^2 A' \cos\omega t - \omega^2 B' \sin\omega t = -\omega^2 x(t)$$

Applying the boundary conditions $\dot{x}(t=0) = 0$ gives

$$\dot{x}(0) = -\omega A' \sin 0 + \omega B' \cos 0 = \omega B' = 0$$

$$\Rightarrow \quad B' = 0$$

$$\Rightarrow \quad x(t) = A' \cos\omega t$$

Now apply $x(t=0) = A \quad \Rightarrow x(0) = A' \cos 0 = A' = A$

Hence the solution is

$$x(t) = A \cos\omega t.$$
