SPA5218 Mathematical Techniques 3
Exercise Sheet 9

1. Show that the integral

$$J = \int_{x_1}^{x_2} f(y, y_x, x)\mathrm{d}x,$$

with $f = y(x)$ has no extreme values.

Now choose

$$f(y, y_x, x) = f_1(x, y) + f_2(x, y)y_x.$$

Show that the Euler equation leads to

$$\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} = 0.$$

2. By extremizing the integral

$$I = \int_{t_1}^{t_2} L(x, \dot{x}, t)\,\mathrm{d}t$$

under variations of the paths $x(t)$ subject to the condition that $\delta x(t_1) = \delta x(t_2) = 0$, derive the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\,.$$

Show that this equation can be re-written in the form

$$\frac{\partial L}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right)\,.$$

3. The action for a particle of mass m moving in one dimension is

$$S = \int L(x, \dot{x}, t)\,\mathrm{d}t = \int \left(\frac{1}{2}m\dot{x}^2 - kx^2\right)\,\mathrm{d}t\,.$$

Show that the Euler-Lagrange equation gives the Newtonian equation of motion. Find the solution which obeys the boundary condition $x(t = 0) = A$, $\dot{x}(t = 0) = 0$.

Here, L is the Lagrangian and S is the action. This is what Feynman talks about in the lecture on the Principle of Least Action (which I hope you have read). Hamilton's principle states that for the physical system the action is extremized.