SPA5218 MATHEMATICAL TECHNIQUES 3

EXERCISE SHEET 8 : SOLUTIONS

Q1.

(a) $$\frac{\partial P(x,t)}{\partial x} + x = 0 \qquad (1)$$

$$\frac{\partial P(x,t)}{\partial t} = t \qquad (2)$$

First integrate (1) with respect to x holding t fixed:

$$\int dP(x,t) = -\int x \, dx$$

$$\Rightarrow \quad P(x,t) = -\frac{x^2}{2} + f(t)$$

where $f(t)$ is an arbitrary function of t, for now. We then substitute this answer in (2) to find an equation for $f(t)$:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial f(t)}{\partial t} = t$$

$$\Rightarrow \quad \int df(t) = \int t \, dt$$

$$\Rightarrow \quad f(t) = \frac{t^2}{2} + c$$

where c is a constant. Therefore, the solution is

$$P(x,t) = -\frac{x^2}{2} + \frac{t^2}{2} + c$$

(b) $$\frac{\partial U(x,y)}{\partial x} = U(x,y) \qquad (3)$$

$$\frac{\partial U}{\partial y}(x,y) = y\,U(x,y) \qquad (4)$$

We integrate (3) with respect to x holding y fixed:

$$\int \frac{dU(x,y)}{U(x,y)} = \int dx$$

$$\Rightarrow \ln[U(x,y)] = x + f(y)$$

$$\Rightarrow U(x,y) = e^{x+f(y)}$$

where $f(y)$ is an arbitrary function of y. Substitute in (4) to find:

$$\frac{\partial U(x,y)}{\partial y} = \frac{df(y)}{dy}\, e^{x+f(y)}$$

$$\Rightarrow \frac{df}{dy}(y)\, U(x,y) = y\,U(x,y)$$

$$\Rightarrow \frac{df(y)}{dy} = y$$

$$\Rightarrow \int df(y) = \int y\, dy$$

$$\Rightarrow f(y) = \frac{y^2}{2} + c$$

$$\Rightarrow U(x,y) = A\, e^{\left(x + \frac{y^2}{2}\right)} \quad \text{where } A = e^{c}$$

(c) $\dfrac{\partial R(x,t)}{\partial x} = t \qquad (5)$

$$\frac{\partial R(x,t)}{\partial t} - \frac{R(x,t)}{t} = 0 \qquad (6)$$

$$R(0,t) = 3t$$

Integrating (5) holding t fixed:

$$\int dR(x,t) = \int t\, d \quad \Rightarrow \quad R(x,t) = tx + f(t)$$

Substitute into (6):

$$\frac{\partial R(x,t)}{\partial t} = x + \frac{df(t)}{dt}, \qquad \frac{R(x,t)}{t} = x + \frac{f(t)}{t}$$

$$\Rightarrow \quad x + \frac{df(t)}{dt} - \left(x + \frac{f(t)}{t}\right) = 0$$

$$\Rightarrow \quad \frac{df(t)}{dt} = \frac{f(t)}{t} \qquad \Rightarrow \int \frac{df(t)}{f(t)} = \int \frac{dt}{t}$$

$$\Rightarrow \quad \ln(f(t)) = \ln t + c$$

$\Rightarrow \quad f(t) = At \quad$ where $A = e^c$ is a constant

The solution is $\quad R(x,t) = tx + At$

The boundary condition $R(0,t) = 3t$ gives $A = 3$ and so the solution is

$$R(x,t) = (x+3)t$$

Q2. $$\frac{\partial V(x,y)}{\partial x} = \frac{-Qx}{x^2+y^2} \qquad (7)$$

$$\frac{\partial V(x,y)}{\partial y} = \frac{-Qy}{x^2+y^2} \qquad (8)$$

Integrating (7) with respect to x while keeping y fixed we have

$$\int dV(x,y) = -Q\int \frac{x}{x^2+y^2}\,dx$$

Change variables

$$u = (x^2+y^2) \; ; \quad du = 2x\,dx \, , \quad \frac{du}{2} = x\,dx$$

$$\Rightarrow \int dV(x,y) = -\frac{Q}{2}\int \frac{du}{u}$$

$$\Rightarrow V(x,y) = -\frac{Q}{2}\ln u + f(y)$$

$$\Rightarrow V(x,y) = -\frac{Q}{2}\ln(x^2+y^2) + f(y)$$

Now substitute this into (8) and get

$$\frac{\partial V(x,y)}{\partial y} = -\frac{Qy}{x^2+y^2} + \frac{df(y)}{dy} \Rightarrow \frac{df(y)}{dy} = 0 \Rightarrow f(y) = c$$
(const.)

Therefore $V(x,y) = -\frac{Q}{2}\ln(x^2+y^2) + c$

[CHECK: In polars, $V = -\frac{Q}{2}\ln r^2 + c$

$$\Rightarrow F_r = -\frac{\partial V}{\partial r} = -\frac{Q}{r} \Rightarrow \underset{\sim}{F} = -\frac{Q}{r^2}\,\hat{\underset{\sim}{r}}$$]

Q3. $$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho \frac{\partial^2 u}{\partial t^2}$$

We seek solutions $u(x,y,t)$ that are periodic in time and have

$$u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0$$

Write $u(x,y,t) = X(x)\,Y(y)\,S(t)$

and substitute in differential equation:

$$T\left(X''YS + XY''S\right) = \rho XYS''$$

Dividing through by XYS gives

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{\rho}{T}\frac{S''}{S} = -\frac{\rho}{T}\omega^2$$ (in order to get periodic solution for S)

$$\Rightarrow S(t) = A\cos(\omega t) + B\sin(\omega t)$$

[CHECK: $S'(t) = -A\omega \sin\omega t + B\omega\cos\omega t$

$S''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$

$= -\omega^2 S(t)$]

Likewise, to get a periodic solution for X and Y we must have

$$\frac{X''}{X} = \lambda \quad \text{and} \quad \frac{Y''}{Y} = \mu$$ (where λ and μ are constants)

where $\lambda + \mu = -\omega^2 \dfrac{\rho}{T}$ and

$$X = A\sin px, \quad Y = B\sin qy$$

where $p^2 = -\lambda$ and $q^2 = -\mu$

These satisfy $u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0$

(Each solution has to have two zeroes at two different values of its argument.)

Further, since $u(a,y,t) = u(x,b,t) = 0$

we must have $p = n\pi/a$ and $q = m\pi/b$

where n and m are integers.

But $\lambda + \mu = -\dfrac{\omega^2 \rho}{T}$ and so

$$\lambda = -p^2 \, , \quad \mu = -q^2$$

$$\Rightarrow \quad -p^2 - q^2 = -\omega^2 \frac{\rho}{T}$$

$$\Rightarrow \quad \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right) = \omega^2 \frac{\rho}{T}$$