SPA5218 Mathematical Techniques 3
Exercise Sheet 8

1. Solve the following sets of differential equations:

   (a)
   $$\frac{\partial P(x,t)}{\partial x} + x = 0\,, \quad \frac{\partial P(x,t)}{\partial t} = t\,.$$

   (b)
   $$\frac{\partial U(x,y)}{\partial x} = U(x,y)\,, \quad \frac{\partial U(x,y)}{\partial y} = yU(x,y)\,.$$

   (c)
   $$\frac{\partial R(x,t)}{\partial x} = t\,, \quad \frac{\partial R(x,t)}{\partial t} - \frac{R(x,t)}{t} = 0\,, \quad \text{where } R(0,t) = 3t\,.$$

2. In two dimensions, a point charge $+Q$ is placed at the origin. The electric potential $V(x,y)$ is found to satisfy :
   $$\frac{\partial V(x,y)}{\partial x} = \frac{-Qx}{x^2+y^2}\,, \quad \frac{\partial V(x,y)}{\partial y} = \frac{-Qy}{x^2+y^2}\,.$$
   Find the electric potential as a function of the coordinates x and y.

3. The wave equation describing the transverse vibrations of a stretched membrane under tension T and having a uniform surface density ρ is
   $$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho\frac{\partial^2 u}{\partial t^2}\,.$$
   Find a separable solution of the form $u(x,y,t) = X(x)Y(y)S(t)$ appropriate to a membrane stretched on a frame of length a and width b, showing that the natural angular frequencies of such a membrane are given by
   $$\omega^2 = \frac{\pi^2 T}{\rho}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\,,$$
   where n and m are any positive integers. You may assume that each of X''/X, Y''/Y and S''/S is a different constant where $'$ denotes partial differentiation with respect to the appropriate variable (x, y or t).