M.Sc. EXAMINATION BY COURSE UNIT

ASTM116/MAS429 Astrophysical Plasmas

24 May 2007 18:15 – 21:15 (3 hours)

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each section.

Calculators are NOT permitted in this examination.

Numerical answers where required may be determined approximately, to within factors ~ 5, or left in terms of trigonometric or other transcendental functions.

You may quote the following results unless the question specifically asks you to derive it. All notation is standard. Vectors are denoted by boldface type, e.g., **A**, *while scalars, including the magnitude of a vector, are in italics, e.g.,* $|\mathbf{E}| = E$.

(i) The Lorentz force on a particle of charge q moving in electric and magnetic fields **E** *and* **B** *respectively is given by*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(ii) Maxwell's Equations

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{E} &= \frac{\rho_q}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

where $\mu_0 \varepsilon_0 = 1/c^2$.

(iii) The electric and magnetic fields **E** *and* **B** *as measured in a laboratory frame are related to the fields* $\mathbf{E}'$ *and* $\mathbf{B}'$ *measured in a frame moving relative to the laboraty frame at a velocity* **u** *by the transformation laws*

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \frac{\mathbf{E}_{\perp} + \mathbf{u} \times \mathbf{B}}{\sqrt{1 - (u^2/c^2)}} \end{aligned}$$

$$\mathbf{B}'_{\perp} = \frac{\mathbf{B}_{\perp} - \frac{\mathbf{u}\times\mathbf{E}}{c^2}}{\sqrt{1-(u^2/c^2)}}$$

(iv) The MHD equations for a plasma with electrical conductivity σ*:*

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) &= 0 \\
\rho\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} &= -\nabla p + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} \\
\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\left(p\rho^{-\gamma}\right) &= 0 \\
\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times(\mathbf{V}\times\mathbf{B}) + \frac{1}{\mu_0\sigma}\nabla^2\mathbf{B} \\
\mathbf{E} + \mathbf{V}\times\mathbf{B} &= \mathbf{j}/\sigma
\end{aligned}$$

(v) Divergence of a vector in spherical coordinates:

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

(vi) The following vector identities and relations

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \\
\nabla\cdot(\mathbf{a}\times\mathbf{b}) &= \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}) \\
\nabla\times(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}(\nabla\cdot\mathbf{b}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - \mathbf{b}(\nabla\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{b} \\
(\nabla\times\mathbf{B})\times\mathbf{B} &= (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla\left(\frac{B^2}{2}\right) \\
\nabla\times(\nabla\times\mathbf{B}) &= \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} \\
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \\
(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}
\end{aligned}$$

(vii) The following numerical values of physical constants and parameter values:

Name	symbol	value
Electronic Charge	e	1.6×10^{-19} C
Electron volt	eV	1.6×10^{-19} Joules
Electron mass	m_e	9.1×10^{-31} kg
Proton mass	m_p	1.67×10^{-27} kg
Permeability of free space	μ_0	$4\pi\times10^{-7}$ Henry/m
Permittivity of free space	ε_0	8.85×10^{-12} Farad/m
Speed of light in vacuo	c	3×10^{8} m/s
Earth Radius	R_e	6371 km
Astronomical Unit	AU	1.5×10^{11} m
Solar Radius	$R_\odot$	6.96×10^{8}m

SECTION A

Each question carries 10 marks. You should attempt ALL questions.

A1. The Magnetic Reynolds number R_m is used to distinguish different types of plasma behaviour. Define R_m, explaining all the terms used. Describe the behaviour of the two limiting cases of R_m. Explain how the "plasma cell" model of astrophysical interactions depends on R_m.

A2. A hot beam-like flow of plasma is observed in space. Illustrate, with the aid of a sketch, how this may be explained by a reconnection process. Indicate any other observations that may support that explanation.

A3. Consider a particle of mass m and charge q moving non-relativistically in a static, uniform magnetic field $\mathbf{B} \equiv B_o\hat{z}$ with a zero electric field. Show that

$$\begin{aligned} v_x &= v_\perp \sin(\Omega_c\, t) \\ v_y &= v_\perp \cos(\Omega_c\, t) \end{aligned}$$

satisfies the equations of motion, when the initial velocity at $t = 0$ is $(0, v_\perp, 0)$. Give a mathematical expression for Ω_c, and explain its significance. From this solution find the radius of gyration of the motion.

A4. Explain what is meant in MHD by "flux freezing." Give two astrophysical examples, including a brief explanation or description of how flux freezing manifests itself there and what the consequences are.

A5. Express the magnetic moment of a particle in terms of its pitch angle and energy. Why is the magnetic moment important, and under what circumstances? Discuss briefly one astrophysical example of particle motion where the magnetic moment can be used to constrain particle motion. In your answer relate any variations of the particle pitch angle distribution to the magnetic field configuration.

SECTION B

Each question carries 25 marks. You may attempt all questions but only marks for the best TWO questions will be counted.

B1. The interstellar medium (ISM) has a neutral hydrogen component which can penetrate close to the Sun, where it is ionized by UV radiation. Consider the motion of a newly ionized particle, of mass m and charge q, in the interplanetary electric and magnetic fields. This is the so-called "pick-up" mechanism.

Use a coordinate system in which the magnetic field $\mathbf{B}$ is in the $\hat{\mathbf{x}}$ direction, and the solar wind velocity $\mathbf{V}_{sw}$ is in the $x-y$ plane and makes an angle α to $\mathbf{B}$. You can assume that the velocity of the neutral ISM relative to the Sun is negligible, and that $\mathbf{B}$ and $\mathbf{V}_{sw}$ are constant.

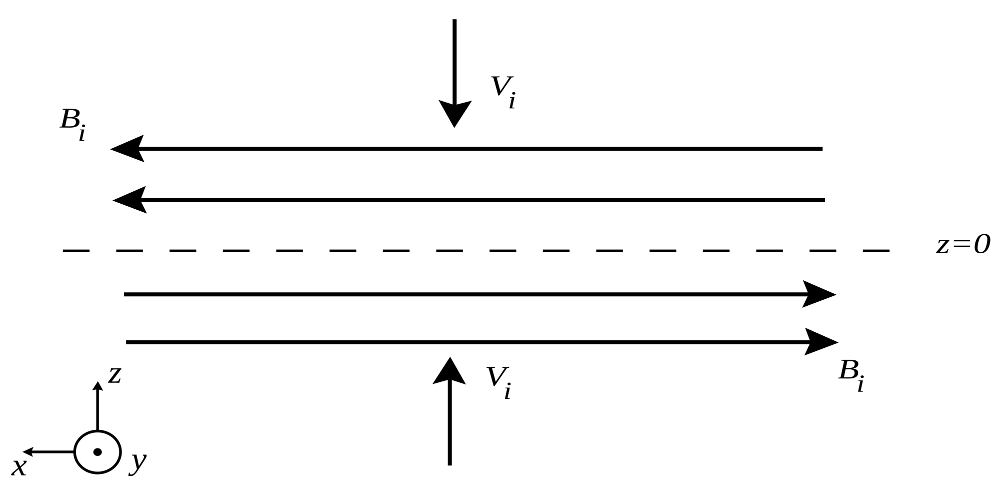
(a) [*12 marks*] Starting from the Lorentz force, solve the particle's equations of motion to find its velocity as a function of time. You may assume that ideal MHD applies for the solar wind, and that the particle is ionized at the origin at $t = 0$. You will need to find the electric field based on the given magnetic field and solar wind velocity.

(b) [*4 marks*] Describe qualitatively the subsequent motion of the newly ionized particle.

(c) [*4 marks*] What is the maximum and average kinetic energy of the ionized particle? What might be the observational signatures of these particles?

(d) [*5 marks*] From your knowledge of the variation of the interplanetary magnetic field (both large scale configuration and short term variations) comment on what you would expect for (i) the evolution of the pick-up ion motion with distance, and (ii) the differences in motion expected for particles ionized at different distances and latitudes relative to the Sun.

B2. A one-dimensional magnetic annihilation configuration is shown below:



 [This question continues overleaf . . .]

All quantities are functions of z only. Plasma flows in symmetrically from either side of the $z=0$ plane at a constant speed V_i (subscript refers to input region). The oppositely directed fields, lying parallel/antiparallel to the x axis in the inflow region, annihilate at $z=0$. Far from the $z=0$ plane the field strength is B_i. The magnetic field components $B_y = B_z = 0$ everywhere.

(a) [*10 marks*] Assuming steady state, determine the governing equation for B_x in the system described above, and show that it is satisfied by the expression

$$B_x = \pm B_i \left[1 - \exp(\mp\mu_0 \sigma V_i z)\right]$$

where the upper/lower signs refer to the $z \geq 0$ and $z \leq 0$ regions, respectively.

(b) [*2 marks*] Comment, briefly, why such a configuration is unrealistic.

(c) [*7 marks*] Sketch the Sweet-Parker model of reconnection, showing flows and representative field lines.

(d) [*6 marks*] For the Sweet-Parker model, by balancing the Poynting flux $(\mathbf{E}\times\mathbf{B})/\mu_0$ in the inflow region with the energy flux of the outflow plasma, show that the outflow speed $V_0 \approx \sqrt{2} V_{Ai}$, where V_{Ai} is the Alfvén speed in the inflow region. Assume the plasma is incompressible with mass density ρ and that ideal MHD is valid in the inflow region.

B3. A simple model of the solar wind can be derived assuming that the solar corona, described as an isothermal gas obeying the ideal gas pressure law $p = 2nk_BT$, undergoes a steady-state, spherically symmetric and purely radial expansion. The mass density and number density are related by $\rho = nm$, where m is the mean mass of a particle. Neglect the magnetic field.

(a) [*12 marks*] Show that the outflow speed V, as a function of radius r, is governed by:

$$\left(V^2 - \frac{2k_BT}{m}\right)\frac{1}{V}\frac{dV}{dr} = \frac{4k_BT}{mr} - \frac{GM_\odot}{r^2}$$

Ensure that you state clearly any additional assumptions you make.

(b) [*6 marks*] Sketch the general forms of the two possible solutions of the equation for $V(r)$, marking the critical radius r_c and the point at which the flow becomes supersonic. Discuss which solution best approximates the real solar wind.

(c) [*4 marks*] The rotation of the sun causes the interplanetary magnetic field to be wound up into the Parker spiral configuration, since in a frame corotating with the Sun the solar wind has an additional azimuthal speed $V_\phi = -\Omega r$. (Ω is the angular velocity of solar rotation.) In a spherically symmetric model, the frozen-in flux condition implies

$$\frac{B_\phi}{B_r} = \frac{V_\phi}{V_r} = \frac{-\Omega r}{V_r}$$

 [This question continues overleaf ...]

By using Maxwell's equations to determine the variation of the radial component B_r of the field with radial distance, use the above relation to determine the radial dependence of the azimuthal component.

(d) [*3 marks*] Given that spacecraft in near-Earth orbits observe $|B_r| \sim |B_\phi| \sim 5$nT on average, calculate the average strength of the 2 magnetic field components expected to be observed by the Galileo spacecraft at Jupiter. (Assume that Jupiter is at 5AU from the Sun, and that V_r is constant between the Earth and Jupiter.)

B4. For some high frequency phenomena a suitable model of a plasma is a cold electron fluid and a uniform background of ions at rest to ensure quasi-neutrality. The governing equations for the electron density n_e, and velocity $\mathbf{v}_e$ are

$$\begin{aligned} \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) &= 0 \\ \frac{\partial \mathbf{v_e}}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e &= -\frac{e}{m_e}(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) \end{aligned}$$

In addition to Maxwell's equations, the charge density is given by $\rho_q = e\, n_i - e\, n_e$, where n_i is the ion (proton) density, and e the electronic charge. The ion density n_i is uniform and constant.

The waves of this system can be found by linearizing these equations, so that each quantity is written as $Q = Q_0 + Q_1$, where Q_0 is the equilibrium value and Q_1 is a small perturbation.

(a) [*6 marks*] Produce linearized equations for $\nabla \cdot \mathbf{E}$, n_e, and $\mathbf{v}_e$ in the electrostatic, zero magnetic field approximation, i.e., $\mathbf{B} = 0$. You may assume that in equilibrium the density is uniform $n_e = n_i = n_0$, and that the plasma is at rest.

(b) [*10 marks*] Show that n_{e1} obeys the equation for SHM. Give its frequency and phase velocity. What property of the plasma may be determined using this mode?

(c) [*3 marks*] By use of the plane wave ansatz $Q = \tilde{Q}\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, show that the corresponding electric field disturbance is longitudinal in this case.

(d) [*6 marks*] For plane wave solutions, by considering the divergence of the equation for electron velocity, show that the properties of an electrostatic wave (i.e., with $\mathbf{B}_1 = 0$) propagating parallel to the magnetic field are unchanged by the addition of a uniform, constant background magnetic field $\mathbf{B}_0$.

[End of examination paper]