Queen Mary
University of London

MSc/MSci Examination

26th May 2016 14:30 – 17:00

SPA7023P/SPA7023U Stellar Structure & Evolution Duration: 2 hours 30 minutes

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Instructions:

Answer ALL questions from Section A. Answer ONLY TWO questions from Section B. Section A carries 50 marks, each question in section B carries 25 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Only non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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You may assume the following:
In all questions: M is the mass, $m(r)$ the mass interior to radius r, R is the radius, L the luminosity and T_{eff} the effective temperature of a star. P, ρ and T denote the pressure, density and temperature respectively. κ is the opacity per unit mass, ϵ the rate of energy production per unit mass and μ denotes the mean molecular weight. c_p, c_v are the specific heats at constant pressure and volume, $\gamma = c_p/c_v$ and $\mathcal{R}$ is the gas constant where $\mathcal{R} = \mu(c_p - c_v)$.
$L = 4\pi R^2 F_{Rad}$ and F_{Rad} is given by

$$F_{Rad} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}.$$

$c, G, \sigma = ac/4$ are respectively the velocity of light, the constant of gravity and the Stefan-Boltzmann radiation constant, a is the Stefan radiation constant. X, Y, Z are the mass fractions respectively of hydrogen, helium and the heavier elements.
The central density ρ_c, central temperature T_c and central pressure P_c of a polytrope of index n are

$$\rho_c = a_n\frac{3M}{4\pi R^3}, \qquad T_c = b_n\frac{\mu GM}{\mathcal{R}R}, \qquad P_c = c_n\frac{GM^2}{R^4},$$

where a_n, b_n and c_n are constants. The apparent magnitude m_{app}, absolute magnitude M_{abs} and distance in parsecs d are related by $m_{\text{app}} = M_{\text{abs}} + 5\log d - 5$. The following rounded numerical values, all in S.I. Units may be assumed throughout the paper.

$$c = 3\times 10^8, G = 7\times 10^{-11}, \sigma = 6\times 10^{-8}, M_\odot = 2\times 10^{30}, R_\odot = 7\times 10^8, L_\odot = 4\times 10^{26}.$$

You may also assume that 1 year is 3×10^7seconds.

SECTION A Answer ALL questions in Section A

Question A1

A star S is known to have an absolute magnitude of 0.58. The distance to star S is 7.7 parsecs.

a) What is the parallax of the star S? What is its apparent magnitude?

b) When observed from the same distance, the Sun would have an apparent magnitude of 4.1. What is the absolute magnitude of the Sun?

c) The star S is a blue star. Is its effective temperature greater or smaller than that of the Sun?

[6 marks]

Question A2

a) Show that for a fully ionized gas consisting of atomic hydrogen and helium only, the mean molecular weight μ is given by

$$\mu = \frac{4}{8 - 5Y}.$$

b) For a particular homogeneous star, $Y = 0.25$ while for a second homogeneous star that is more evolved, Y has increased to 0.35. Both stars have the same polytropic index. The central temperature is the same in both stars and may be assumed to be as given in the rubric. If the second star has twice the mass of the first star, calculate the ratio of their radii.

[8 marks]

Question A3

The Lane-Emden equation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,$$

where constant n is the polytropic index. In polytropic stars, the radial profiles of density and pressure are governed by the solution of the Lane-Emden equation $\theta(\xi)$ as

$$\rho(r) = \rho_c\theta^n(\xi), \quad P(r) = P_c\theta^{n+1}(\xi),$$

where ρ_c and P_c are central values of density and pressure, and $r = \alpha\xi$ with some constant α, such that ξ is a dimensionless radial coordinate. Consider a polytropic star with $n = 1$.

a) Show by direct substitution that the solution to the Lane-Emden equation is

$$\theta(\xi) = \frac{\sin\xi}{\xi}.$$

b) Deduce the value of ξ at the surface of the star.

c) Find the density (in terms of the central density) and temperature (in terms of the central temperature) at a distance from the centre of 50% of the radius. You may assume that the equation of state is that of an ideal gas, and that the mean molecular weight is uniform throughout the star.

[8 marks]

Question A4

a) Derive the equation of hydrostatic equilibrium for a spherical self-gravitating body in the form

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r).$$

b) Consider a stellar model with uniform density ($\rho =$ const). Show by direct integration of the equation of hydrostatic equilibrium that the pressure P_c at the centre of a star of mass M and radius R is given by

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}.$$

[9 marks]

Question A5

a) Explain briefly why the gravitational binding energy Ω of a spherically-symmetric star is

$$\Omega = -G \int_0^M \frac{m\,dm}{r},$$

where $m = m(r)$ is the mass interior to radius r.

b) Consider a stellar model with uniform density (ρ = const). Show that the gravitational binding energy of a star of mass M and radius R is given by

$$\Omega = -\frac{3GM^2}{5R}.$$

c) Using the equation of hydrostatic equilibrium $dP/dr = -Gm(r)\rho(r)/r^2$, show that the pressure profile $P(r)$ inside the star of uniform density is

$$P(r) = \frac{3GM^2}{8\pi R^6}\left(R^2 - r^2\right).$$

Deduce that

$$\Omega = -3 \int_V P\,dV,$$

where V is the volume of the star.

[9 marks]

Question A6

Consider a group of stars. Each star in the group has the same chemical composition, is homogeneous, and is composed of an ideal gas. Energy generation is by the p-p chain, with $\epsilon = \epsilon_0 \rho T^4$. All the energy is carried by radiation and the opacity is given by $\kappa = \kappa_0 \rho T^{-3.5}$, where κ_0 is some constant.

a) Show that

$$M \propto R^{13}.$$

b) Also show that

$$L \propto M^{71/13}.$$

c) Obtain the slope of the line in an H-R diagram ($\log L$ versus $\log T_{\text{eff}}$) that these stars lie on.

[10 marks]

SECTION B Answer TWO questions from Section B

Question B1

A star of mass M contracts under the force of gravity, deriving its energy only from the change in its gravitational potential energy. This potential energy for a star of radius R and polytropic index n is given by

$$\Omega = -\frac{3GM^2}{(5-n)R}.$$

The virial theorem holds and so you can assume that half the gravitational energy released is radiated away, while the polytropic index is 3/2. In such a star the energy is transported outwards by convection and the star may be assumed to evolve with T_{eff} remaining constant.

a) Show that the time, t_1, taken by such a star to evolve from a large radius to some smaller radius R_1 is given by

$$t_1 = \frac{GM^2}{7L_1R_1},$$

where L_1 is the luminosity when the radius is R_1.

[9 marks]

b) When the star reaches a radius R_1 it re-adjusts internally such that energy transport becomes radiative rather than convective so that the polytropic index changes to 3. Both the luminosity and effective temperature remain constant during this adjustment. Show that t_2, the time taken by the star in adjusting internally is given by

$$t_2 = \frac{9GM^2}{28L_1R_1}.$$

[7 marks]

c) In this radiative state, the star may be assumed to evolve with a constant luminosity, L_1. Show that t_3, the time taken by the star to evolve from a radius R_1 to a radius R_2 on the main sequence, is

$$t_3 = \frac{3GM^2}{4L_1R_1}\left(\frac{R_1}{R_2} - 1\right).$$

[7 marks]

d) State, with a reason, which of the three stages of evolution (a-c) described above is most likely to be observed.

[2 marks]

Question B2

a) The interaction of photons with atoms is described in terms of a cross section σ_R, which is defined such that

$$n\lambda_\mathrm{ph}\sigma_\mathrm{R} = 1,$$

where n is the number of atoms per unit volume, and λ_ph is the mean free path of a photon. Using geometrical arguments, explain the origin of this definition.

[3 marks]

b) The opacity κ is defined as

$$\kappa = \frac{1}{\rho\lambda_\mathrm{ph}}.$$

Show that κ is the total cross section per unit mass.

[3 marks]

c) The optical depth, τ, in a stellar atmosphere is defined as

$$\tau = \int_r^\infty \kappa\rho\, dr.$$

Starting from the equation for radiative flux, F, given in the rubric, show that in the atmosphere of a star

$$T^4 = \frac{3F}{ac}\left(\tau + B\right),$$

where B is a constant of integration, which you may assume without proof to be $B = 2/3$. Using $F = \sigma T_\mathrm{eff}^4$ as a definition of the effective temperature T_eff, show that

$$T^4 = \frac{3}{4}T_\mathrm{eff}^4\left(\tau + \frac{2}{3}\right).$$

[8 marks]

d) Assume now that this $T - \tau$ relation is valid everywhere in the radiative atmosphere, including optically thin layers. Explain the possible weaknesses of this assumption. Assume further that the atmosphere is an ideal gas and in hydrostatic equilibrium and that the mass and thickness of the atmosphere are both negligible compared to the mass and radius of the star. Given that $P = 0$ at $\tau = 0$ and that the opacity is given by $\kappa = \kappa_0\rho T^5$ with some constant κ_0, show that

$$P^2 = P_0^2 \ln\left(1 + \frac{3}{2}\tau\right),$$

where P_0 is another constant which you do not need to specify.

[11 marks]

Question B3

The Schwarzschild condition for the onset of convection in an ideal gas is

$$\frac{d\,\ln T}{d\,\ln P} > \frac{\gamma - 1}{\gamma}.$$

a) Assume that the temperature and pressure profiles in radiative stellar atmosphere are given by the relations

$$T^4 = \frac{3}{4}T_{\text{eff}}^4\left(\tau + \frac{2}{3}\right),$$

$$P^2 = P_0^2 \ln\left(1 + \frac{3}{2}\tau\right),$$

where τ is an optical depth, and P_0 is some constant. Assume further that $\gamma = 5/3$. Show that the convection sets in at a level where

$$\tau = \frac{2}{3}\left[\exp\left(\frac{4}{5}\right) - 1\right].$$

[11 marks]

b) Below the instability level, the energy is transported by convection. Assuming that in the convective zone $P = KT^{5/2}$ (i.e., the temperature gradient is purely adiabatic), and using the equation of hydrostatic support and an ideal-gas equation of state, show that in the convective region

$$\frac{dT}{dr} = -\frac{2Gm(r)\mu}{5\mathcal{R}r^2}.$$

[11 marks]

c) Assuming that the mass of the convective zone is small compared to M, show that at a depth h measured from the top of the convective zone, the temperature is approximately given by

$$T = T_S + \frac{2GM\mu}{5\mathcal{R}R^2}h$$

when h is small compared to R, and T_S is the temperature at the top of the convective zone.

[3 marks]

Question B4

In white dwarfs, where pressure P is dominated by the pressure of the degenerate non-relativistic electrons, the pressure and density are related by a polytropic law

$$P = K\rho^{5/3},$$

where constant K depends on chemical composition.

a) Show that for white dwarfs of the same chemical composition, the mass M and radius R satisfy the relationship

$$R \propto M^{-1/3}.$$

[5 marks]

b) Assuming that in white dwarfs the energy transport is such that $L \propto M^{5.5}R^{-0.5}$, find the slope of the line in the H-R diagram on which white dwarfs will lie.

[8 marks]

c) A star has a non-relativistic degenerate helium core of mass M_c and radius R_c surrounded by a hydrogen burning shell. The mass–radius relationship for this core may be assumed to be like that of a white dwarf. The energy generated by fusion of mass m of hydrogen to helium is approximately $0.02mc^2$, where c is the speed of light. Show that for a star with luminosity L, an additional degenerate mass is deposited on the core at the rate

$$\frac{dM_c}{dt} = \frac{50L}{c^2}.$$

[3 marks]

d) Assuming that helium is deposited gently and uniformly on the core, show that the rate of release of gravitational energy in the core is

$$\frac{100GM_cL}{c^2R_c}.$$

You may assume that the gravitational potential energy of the core is

$$-\frac{6GM_c^2}{7R_c}.$$

[9 marks]

End of Paper