(a).

$$\Omega = -\frac{3GM^2}{(5-n)R}$$

$$2U + \Omega = 0.$$

$$L \propto M^{5.5} R^{-0.5}$$

$$L = 4\pi R^2 \sigma T_{eff}^4$$

We require a relation between L and T_{eff}.

Note that M = Constant during evolution.

$$\therefore \quad L \propto R^{-0.5} \propto R^2 T_{eff}^4$$

$$\Rightarrow \quad R^{-0.5} \propto R^2 T_{eff}^4$$

$$\therefore \quad T_{eff}^4 \propto R^{-2.5}$$

$$R^{2.5} \propto T_{eff}^{-4}$$

$$R^2 \propto T_{eff}^{-16/5}$$

$$\Rightarrow \quad L \propto T_{eff}^{-16/5} \, T_{eff}^4$$

$$\therefore \quad L \propto T_{eff}^{+4/5}.$$

$$\log L \propto +\frac{4}{5} \log T_{eff} \Rightarrow \boxed{\text{Slope} = \frac{4}{5}}$$

②

(b). For $n=3$, $\Omega = -\dfrac{3GM^2}{2R}$

$U = -\dfrac{\Omega}{2}$ from Virial theorem

and $E_{TOT} = U + \Omega$

$\therefore E_{TOT} = \dfrac{\Omega}{2} = -\dfrac{3GM^2}{4R}$.

Now $L = -\dfrac{dE_{TOT}}{dt} = -\left(\dfrac{d}{dt}\left[-\dfrac{3GM^2}{4R}\right]\right)$

$= -\dfrac{3GM^2}{4R^2}\dfrac{dR}{dt}$

Now we have $L \propto M^{5.5} R^{-0.5}$

$\Rightarrow L = C_1 M^{5.5} R^{-0.5}$ $C_1 =$ Constant.

$\therefore C_1 M^{5.5} R^{-0.5} = -\dfrac{3GM^2}{4R^2}\dfrac{dR}{dt}$.

$\Rightarrow \displaystyle\int_0^t dt = -\frac{3GM^2}{4C_1M^{5.5}}\int_{\infty}^{R_0} R^{-1.5}\,dR$

$\therefore t = \dfrac{3GM^{-3.5}}{2C_1}\left[R^{-0.5}\right]_{\infty}^{R_0}$

$\therefore \boxed{t = C_2 M^{-3.5} R_0^{-0.5}}$

$$\epsilon = \epsilon_0 \rho T^4$$

$$K = K_0 \rho T^{-3.5}$$

$$P = \frac{R}{\mu} \rho T$$

$$L = \int_m \epsilon\, dm = \int \epsilon_0 \rho T^4 dm \quad \propto \rho T^4 m$$

$$L \propto \rho T^4 m.$$

$$\rho \propto \frac{m}{R^3}$$

$$\frac{dP}{dr} = -\frac{G m(r)}{r^2} \rho \quad \Rightarrow \quad \frac{P}{R} \sim \frac{Gm}{R^2} \frac{m}{R^3}$$

$$\therefore P \propto \frac{m^2}{R^4}. \qquad \text{But } P \propto \rho T$$

$$\Rightarrow T \propto P/\rho.$$

$$\Rightarrow L \propto \frac{m}{R^3} \left(\frac{m^2}{R^4} \Big/ \frac{m}{R^3} \right)^4 m$$

$$\therefore L \propto \frac{m}{R^3} \frac{m^4}{R^4} m \quad \propto \frac{m^6}{R^7}$$

$$\underline{L \propto m^6 R^{-7}}$$

$$L = 4\pi R^2 \frac{4acT^3}{3\kappa\rho}\frac{dT}{dR} \propto \frac{R^2T^3}{\kappa\rho}\frac{T}{R}$$

$$\therefore L \propto \frac{R^2 T^3}{\rho^2 T^{-3.5}}\frac{T}{R}$$

$$\therefore L \propto \frac{R^2 \left(M/R\right)^{8.5 \to 7.5}}{\left(\frac{M}{R^3}\right)^2 R} \propto M^{5.5 \to 5.5} R^{-0.5}$$

$$\therefore \underline{L \propto M^{5.5} R^{-0.5}}$$

(a). $$L \propto M^6 R^{-7} \propto M^{5.5} R^{-0.5}$$

$$\Rightarrow M^{0.5} \propto R^{6.5}$$

$$\therefore \underline{M \propto R^{13}}$$

(b). $$L \propto M^6 R^{-7} \qquad R \propto M^{1/13}$$

$$\therefore L \propto M^6 M^{-7/13} \propto M^{78/13} M^{-7/13}$$

$$\therefore \underline{L \propto M^{71/13}}$$

(c). $L = 4\pi R^2 T_{eff}^4$

$\therefore L \propto R^2 T_{eff}^4$

<u>NOTE:</u> $T_{eff} \neq T$!!!

We need to get a relation between R and L so that the R^2 term can be eliminated.

We have $L \propto M^{71/13}$ and $M \propto R^{13}$

$\therefore L \propto R^{71} \Rightarrow R \propto L^{1/71}$

$\therefore L \propto L^{2/71} T_{eff}^4$

$L^{71/71 - 2/71} \propto T_{eff}^4$

$L^{69/71} \propto T_{eff}^4$

$\therefore L \propto T_{eff}^{4 \times 71/69} = T_{eff}^{4.12}$.

$\log L \propto 4.12 \log T_{eff} \Rightarrow \boxed{\text{Slope} = 4.12}$