B1 2016

We have $\Omega = -\dfrac{3GM^2}{(5-n)R}$ and $n = \dfrac{3}{2}$

Hence $\Omega = -\dfrac{3GM^2}{(7/2)R} = -\dfrac{6GM^2}{7R}.$

The Virial theorem says $2U + \Omega = 0$

$$\therefore \quad U = -\frac{\Omega}{2}.$$

Total Energy $E = U + \Omega = -\dfrac{\Omega}{2} + \Omega = \dfrac{\Omega}{2}.$

$$\therefore \quad E = \frac{\Omega}{2} = -\underline{\frac{3GM^2}{7R}}.$$

(a). Luminosity $L = -\dfrac{dE}{dt}.$

$$= + \frac{3GM^2}{7R^2}\frac{dR}{dt}$$

$$\therefore \quad L = 4\pi R^2 \sigma T_{eff}^4 = \frac{3GM^2}{7R^2}\frac{dR}{dt}$$

$$\Rightarrow \int_0^{t_1} dt = \frac{3GM^2}{7} \cdot \frac{1}{4\pi\sigma T_{eff}^4} \int_\infty^{R_1} \frac{dR}{R^4}$$

$$\therefore \ t_1 = \frac{3GM^2}{7} \frac{1}{4\pi\sigma T_{eff}^4} \left[-\frac{1}{3R^3} \right]_{\infty}^{R_1}$$

$$= \frac{GM^2}{7R_1} \frac{1}{4\pi\sigma T_{eff}^4 R_1^2}$$

$$\therefore \ t_1 = \frac{GM^2}{7L_1 R_1}$$

(b). Here we note the following:

$$\left.\begin{aligned} \text{Luminosity} &= \text{Constant.} \\ T_{eff} &= \text{Constant} \end{aligned}\right\} \Rightarrow R_1 = \text{Constant.}$$

We have $E = +\frac{\Omega}{2} = -\frac{1}{2} \frac{3GM^2}{(5-n)R}$.

and $L = -\frac{dE}{dt}$. Hence, because $M =$ Constant and $R =$ Constant, the luminosity is being sourced by the changing internal structure which arises because n is changing.

$$\therefore \quad L_1 = -\frac{dE}{dt} = -\frac{3GM^2}{2(5-n)^2 R_1} \frac{d(5-n)}{dt}.$$

i.e we differentiate the $(5-n)$ term w.r.t. time to obtain the luminosity

$$\therefore \int_0^{t_2} dt = -\frac{3GM^2}{2R_1} \int_{n=3/2}^{n=3} \frac{d(5-n)}{(5-n)^2}$$

$$= \frac{3GM^2}{2R_1} \left[\frac{1}{5-n} \right]_{n=3/2}^{n=3}$$

$$= \frac{3GM^2}{2R_1} \left[\frac{1}{2} - \frac{2}{7} \right]$$

$$\therefore t_2 = \frac{9GM^2}{28R_1}$$

(c) Now we have $L_1 =$ Constant, $M =$ Constant.

$$L_1 = -\frac{dE}{dt} = \frac{1}{2}\frac{d\Omega}{dt} = \frac{1}{2}\frac{3GM^2}{2R^2}\frac{dR}{dt}.$$

Note that $n = 3$.

$$\int dt = \frac{3GM^2}{4L_1}\int_{R_1}^{R_2}\frac{dR}{R^2}$$

$$= \frac{3GM^2}{4L_1}\left[\frac{1}{R_2} - \frac{1}{R_1}\right]$$

$$\therefore t_3 = \frac{3GM^2}{4L_1}\left[\frac{R_1}{R_2} - 1\right]$$

(d). If we assume $R_1 >> R_2$, then we see that the phase corresponding to t_3 will be the longest, and therefore the most easily observable.

B4 (c) The rate of converting hydrogen to helium also corresponds to the rate at which helium is added to the core.

$$\therefore \quad L = 0.02 \frac{dM_c}{dt} c^2$$

$$\Rightarrow \quad \frac{dM_c}{dt} = \frac{50L}{c^2}.$$

(d). $$\Omega = -\frac{6}{7}\frac{GM_c^2}{\cancel{4} R_c}$$

As helium is added to the core, both the mass and radius change according to

$$R_c \propto M_c^{-1/3}.$$

We write this as $R_c \cancel{\propto} = C M_c^{-1/3}$.

$$\therefore \quad \Omega = -\frac{6}{7}\frac{GM_c^2}{C M_c^{-1/3}} = \frac{6}{7}\frac{GM_c^{7/3}}{C}$$

$$\therefore \quad \frac{d\Omega}{dt} = \frac{7}{3} \cdot \frac{6}{7} \frac{G M_c^{4/3}}{C} \frac{dM_c}{dt}$$

$$\therefore \frac{d\Omega}{dt} = \frac{2\,G M_c^{4/3}}{C} \times \frac{50L}{C^2}$$

$$\frac{M_c^{1/3}}{C} = \frac{1}{R_c}$$

$$\therefore \frac{d\Omega}{dt} = \frac{100\,G M_c L}{C^2 R_c}$$