Queen Mary
University of London

MSc/MSci Examination

Day 28th April 2015 18:30 – 21:00

SPA7023P/SPA7023U/ASTM109 Stellar Structure and Evolution Duration: 2.5 hours

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Instructions:

Answer ALL question from Section A. Answer ONLY TWO questions from Section B. Section A carries 50 marks; each question in Section B carries 25 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Calculators ARE permitted in this examination. The unauthorised use of material stored in pre-programmable memory constitutes an examination office. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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EXAM PAPERS MUST NOT BE REMOVED FROM THE EXAM ROOM.

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You may assume the following:
In all questions: M is the mass, $m(r)$ the mass interior to radius r, R is the radius, L the luminosity and T_{eff} the effective temperature of a star. P, ρ and T denote the pressure, density and temperature respectively. κ is the opacity per unit mass, ϵ the rate of energy production per unit mass and μ denotes the mean molecular weight. c_p, c_v are the specific heats at constant pressure and volume, $\gamma = c_p/c_v$ and $\mathcal{R}$ is the gas constant where $\mathcal{R} = \mu(c_p - c_v)$.
$L = 4\pi R^2 F_{Rad}$ and F_{Rad} is given by

$$F_{Rad} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}.$$

$c, G, \sigma = ac/4$ are respectively the velocity of light, the constant of gravity and the Stefan-Boltzmann radiation constant. X, Y, Z are the mass fractions respectively of hydrogen, helium and the heavier elements.
The central density ρ_c, central temperature T_c and central pressure P_c of a polytrope of index n are

$$\rho_c = a_n \frac{3M}{4\pi R^3}, \qquad T_c = b_n \frac{\mu G M}{\mathcal{R} R}, \qquad P_c = c_n \frac{GM^2}{R^4}.$$

The apparent magnitude m_{app}, absolute magnitude M_{abs} and distance in parsecs d are related by $m_{\text{app}} = M_{\text{abs}} + 5\log d - 5$. The following rounded numerical values, all in S.I. Units may be assumed throughout the paper.

$$c = 3 \times 10^8, G = 7 \times 10^{-11}, \sigma = 6 \times 10^{-8}, M_\odot = 2 \times 10^{30}, R_\odot = 7 \times 10^8, L_\odot = 4 \times 10^{26}.$$

You may also assume that 1 year is 3×10^7seconds.

SECTION A Answer ALL questions in Section A

Question A1

A CCD image of a region of sky containing stars S_1 and S_2 was obtained. Star S_1 is known to have an absolute magnitude of 12 and a parallax of 0.01 arcsec.

a) What is the apparent magnitude of the star S_1?

b) S_1 gave a photon count of 12500 and S_2 a photon count of 25000 in the same time interval. What is the apparent magnitude of S_2?

c) Both stars lie on the main sequence in the H-R diagram, which may be assumed to be a line of slope 5. The effective temperature of S_2 is twice that of S_1. What is the absolute magnitude of S_2?

d) What is the distance of the star S_2?

[9 marks]

Question A2

a) Show that for a fully ionized gas consisting of atomic hydrogen, helium and carbon $^{12}_{6}C$ only, the mean molecular weight μ is given by

$$\mu = \frac{12}{9 + 15X - 2Z}.$$

b) After arriving to the main sequence, a star has uniform chemical composition with $X = 0.72$ and $Y = 0.26$. As the star evolves, hydrogen is converted to helium, so that X and Y change. Calculate the change in μ between the present state and when Y has increased to 0.33.

[8 marks]

Question A3

The temperature profile $T(r)$ in a polytropic star composed of an ideal gas is

$$T(r) = T_c\theta(\xi),$$

where T_c is central temperature, $\xi = r/\alpha$ with some constant α, and $\theta(\xi)$ is the solution to the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

with boundary conditions $\theta(0) = 1$ and $d\theta/d\xi = 0$ at $\xi = 0$. The constant n is the polytropic index. Consider a polytropic star with $n = 0$.

a) Show by direct integration of the Lane-Emden equation that

$$\theta(\xi) = 1 - \frac{\xi^2}{6}.$$

b) Deduce the value of ξ at the surface of the star.

c) Find the temperature (in terms of the central temperature) at a distance from the centre of 50% of the radius.

[8 marks]

Question A4

The Schwarzschild condition for convective instability is

$$\frac{d\,\ln T}{d\,\ln P} > \frac{\gamma - 1}{\gamma}.$$

Consider a stellar atmosphere which is in radiative equilibrium, and has a temperature profile

$$T^4 = \frac{3}{4}T_{\text{eff}}^4\left(\tau + \frac{2}{3}\right),$$

and pressure profile

$$P^2 = P_0^2\ln\left(1 + \frac{3}{2}\tau\right),$$

where τ is optical depth, and P_0 is some constant.
Given that $\gamma = 5/3$, show that the convection sets in at a level where

$$\tau = \frac{2}{3}\left[\exp\left(\frac{4}{5}\right) - 1\right].$$

[7 marks]

Question A5

The optical depth, τ, in a stellar atmosphere is defined as

$$\tau = \int_r^\infty \kappa\rho \, dr.$$

a) Starting from the equation for radiative flux, F, given in the rubric, show that in the atmosphere of a star

$$T^4 = \frac{3F}{ac}\left(\tau + B\right),$$

where B is a constant of integration.

b) Assume without proof that $B = 2/3$. Using $F = \sigma T_{\text{eff}}^4$ as a definition of the effective temperature T_{eff}, show that

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3}\right).$$

c) Discuss possible weaknesses of this derivation of the $T - \tau$ relation in stellar atmospheres.

[9 marks]

Question A6

Consider a group of homogeneous stars. Each star in the group has the same chemical composition, is homogeneous, and is composed of an ideal gas. Energy generation is by the CNO cycle, with $\epsilon = \epsilon_0 \rho T^{17}$. All the energy is carried by radiation and the main opacity is due to electron scattering so that κ is a constant.

a) Show that

$$R \propto M^{0.8}.$$

b) Also show that

$$L \propto M^3.$$

c) Obtain the slope of the line in an H-R diagram ($\log L$ versus $\log T_{\text{eff}}$) that these stars lie on.

[9 marks]

SECTION B Answer TWO questions from Section B

Question B1

a) By considering the forces acting on a volume element, show that for a spherically symmetric star to be in hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}.$$

[7 marks]

Consider a hypothetical star of mass M and radius R, where density profile is represented by a linear function of radial coordinate, $\rho = \rho_c(1 - r/R)$.

b) Show that the mass coordinate, i.e. the mass interior to radius r, is

$$m(r) = \frac{4}{3}\pi r^3 \rho_c \left(1 - \frac{3r}{4R}\right).$$

[4 marks]

c) Deduce that the mean density of the star is given by $\rho_c/4$.

[2 marks]

d) Given that pressure P is zero when $r = R$, show that

$$P_c = \frac{5GM^2}{4\pi R^4}.$$

[7 marks]

e) The star is composed of an ideal gas with equation of state $P = \frac{\mathcal{R}}{\mu}\rho T$. Show that the central temperature is

$$T_c = \frac{5\mu}{12\mathcal{R}}\frac{GM}{R}.$$

[2 marks]

f) This particular star has the same mass, but twice the radius of the Sun and a composition such that $\frac{\mathcal{R}}{\mu}$ has a value of 10^4. Calculate the value of the central temperature. What does this tell you about the likely source of energy within the star?

[3 marks]

Question B2

a) Explain, using physical arguments, why the gravitational binding energy Ω of a spherically-symmetric star is

$$\Omega = -G \int_0^M \frac{m\,dm}{r},$$

where $m = m(r)$ is the mass interior to radius r.

[3 marks]

b) Consider a stellar model with uniform density, $\rho =$const. Show that the mass coordinate is $m(r) = \frac{4}{3}\pi r^3 \rho$, and the gravitational binding energy of the star is

$$\Omega = -\frac{3GM^2}{5R},$$

where M is stellar mass, and R is stellar radius.

[6 marks]

c) Using the equation of hydrostatic support $dP/dr = -Gm\rho/r^2$, show that the pressure profile $P(r)$ inside the star is

$$P(r) = \frac{3GM^2}{8\pi R^6}\left(R^2 - r^2\right).$$

[6 marks]

d) This star is composed of an ideal gas of non-relativistic particles, so that the internal energy density is $u = 3P/2$. Show that the total internal energy of the star is

$$U = \frac{3GM^2}{10R},$$

which confirms a more general result $U = -\Omega/2$, valid for a star with an arbitrary density profile.

[6 marks]

e) Now consider a scenario when the star is composed of an ideal gas of ultra-relativistic particles, with internal energy density is $u = 3P$. Show that in this case,

$$U = -\Omega.$$

Explain, using physical arguments, why this star is unstable to uncontrolled gravitational collapse.

[4 marks]

Question B3

a) A star is in hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r),$$

where $m(r)$ is the mass interior to radius r. The gravitational binding energy is defined as

$$\Omega = -G\int_0^M \frac{m\,dm}{r}.$$

Using integration by parts, show that

$$\Omega = -3\int_V P\,dv,$$

where V is the spherical volume occupied by the star.

[7 marks]

b) The star is composed of classical particles. For a gas of classical particles, pressure P and internal energy density u are related as $u = 3P/2$, the relation which you can use without proof. Show that

$$\Omega = -2U,$$

where U is the thermal energy of the star. What is the total energy of the star? Before arriving to the main sequence, the star is contracting slowly under its own gravity. Show that half of the gravitational energy which is released in the contraction is radiated away, and another half goes to heat up the gas.

[5 marks]

c) The star of mass M contracts under forces of gravity, deriving its energy only from the change in its gravitational binding energy. The energy is transported outwards by convection, so that the star can be described by a polytropic model with polytropic index of $3/2$. The gravitational binding energy for a star of radius R and polytropic index n is given by

$$\Omega = -\frac{3GM^2}{(5-n)R}.$$

The virial theorem holds and so you can assume that half the gravitational energy released is radiated away. Show that the luminosity of the star at time t is

$$L = -\frac{3GM^2}{7R^2(t)}\frac{dR(t)}{dt}.$$

[4 marks]

d) The star may be assumed to evolve with effective temperature T_{eff} remaining constant. Show that the time, t_1, taken by such a star to evolve from a large radius to some smaller radius R_1, is given by

$$t_1 = \frac{GM^2}{7L_1R_1},$$

where L_1 is the luminosity when the radius is R_1.

[9 marks]

Question B4

According to Pauli's exclusion principle, at most two electrons can occupy a given energy state, and each particular energy state occupies volume h^3 in the 6-dimensional space of coordinates and momenta, where h is the Planck constant.

a) In a degenerate gas all the electron states are filled up to a threshold momentum p_F and none above. Show that the number density of electrons for which momentum p is in the interval $(p, p+dp)$ is

$$n_e(p)\,dp = \frac{8\pi p^2}{h^3}\,dp$$

when $p \leq p_F$. Show that the total (i.e., integrated over all the possible momenta) electron number density is

$$n_e = \frac{8\pi}{3h^3}p_F^3.$$

[5 marks]

b) Show that when the pressure P is dominated by the electron pressure, and the electrons are moving with speeds comparable to the speed of light, c,

$$P = \frac{2\pi c}{3h^3}p_F^4.$$

You may use the general expression

$$P = \frac{1}{3}\int_0^{\infty} vpn(p)dp,$$

where $n(p)\,dp$ is number density of particles which momentum p is in the interval $(p, p+dp)$, and v is their velocity.

[4 marks]

c) Show that in the completely ionized gas of a given chemical composition, the electron number density, n_e, is proportional to mass density, ρ.

[2 marks]

d) Show that in the stellar core where pressure P is dominated by the pressure of the degenerate relativistic electrons, the pressure and density are related by a polytropic law

$$P = K\rho^{\frac{4}{3}},$$

where K is a constant which you do not need to specify.

[6 marks]

e) Show that the degenerate electrons become relativistic when

$$n_e \gg \left(\frac{m_e c}{h}\right)^3,$$

where m_e is electron mass, and c is speed of light.

[8 marks]

End of Paper