Queen Mary
University of London

MSc Examination by Course Unit

Tuesday 30th April 2013 18:15 – 21:15

ASTM109 Stellar Structure and Evolution Duration: 3 hours

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Instructions:

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorised use of material stored in pre-programmable memory constitutes an examination office. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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You may assume the following:

In all questions: M is the mass, $m(r)$ the mass interior to radius r, R is the radius, L the luminosity and T_{eff} the effective temperature of a star. P, ρ and T denote the pressure, density and temperature respectively. κ is the opacity per unit mass, ϵ the rate of energy production per unit mass and μ denotes the mean molecular weight. c_p, c_v are the specific heats at constant pressure and volume, $\gamma = c_p/c_v$ and $\mathcal{R}$ is the gas constant where $\mathcal{R} = \mu(c_p - c_v)$.

$L = 4\pi R^2 F_{Rad}$ and F_{Rad} is given by

$$F_{Rad} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}.$$

$c, G, \sigma = ac/4$ are respectively the velocity of light, the constant of gravity and the Stefan-Boltzmann radiation constant. X, Y, Z are the mass fractions respectively of hydrogen, helium and the heavier elements.

The central density ρ_c, central temperature T_c and central pressure P_c of a polytrope of index n are

$$\rho_c = a_n\frac{3M}{4\pi R^3}, \qquad T_c = b_n\frac{\mu GM}{\mathcal{R}R}, \qquad P_c = c_n\frac{GM^2}{R^4}.$$

The apparent magnitude m_{app}, absolute magnitude M_{abs} and distance in parsecs d are related by $m_{\text{app}} = M_{\text{abs}} + 5\text{log}d - 5$. The following rounded numerical values, all in S.I. Units may be assumed throughout the paper.

$$c = 3\times10^8, G = 7\times10^{-11}, \sigma = 6\times10^{-8}, M_\odot = 2\times10^{30}, R_\odot = 7\times10^8, L_\odot = 4\times10^{26}.$$

You may also assume that 1 year is 3×10^7seconds.

SECTION A Each question carries 10 marks and you should attempt ALL questions.

Question A1

A CCD image of a region of sky containing stars S_1 and S_2 was obtained. Star S_1 is known to have an absolute magnitude of 12 and a parallax of 0.01 arcsec.

a) What is the apparent magnitude of the star S_1?

b) S_1 gave a photon count of 12500 and S_2 a photon count of 25000 in a same time interval. What is the apparent magnitude of S_2?

c) Both stars lie on the main sequence in the H-R diagram ($\log L$ vs. $\log T_{\text{eff}}$ plot), which may be assumed to be a line of slope 5. The effective temperature of S_2 is twice that of S_1. What is the absolute magnitude of S_2?

d) What is the distance of the star S_2?

[10 marks]

Question A2

a) Show that for a fully ionized gas consisting of atomic hydrogen, helium and carbon $^{12}_{6}C$ only, the mean molecular weight μ is given by

$$\mu = \frac{12}{9 + 15X - 2Z}.$$

b) After arriving to the main sequence, a star has uniform chemical composition with $X = 0.72$ and $Y = 0.26$. As the star evolves, hydrogen is converted to helium, so that X and Y change. Calculate the change in μ between the present state and when Y has increased to 0.33.

[10 marks]

Question A3

The Lane-Emden equation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,$$

where constant n is the polytropic index. In polytropic stars, the radial profiles of density and pressure are governed by the solution of the Lane-Emden equation $\theta(\xi)$ as

$$\rho(r) = \rho_c\theta^n(\xi), \quad P(r) = P_c\theta^{n+1}(\xi),$$

where ρ_c and P_c are central values of density and pressure, and $r = \alpha\xi$ with some constant α. Consider a polytropic star with $n = 0$.

a) Show by direct integration of the Lane-Emden equation that

$$\theta(\xi) = 1 - \frac{\xi^2}{6}.$$

b) Deduce the value of ξ at the surface of the star.

c) Find the temperature (in terms of the central temperature) at a distance from the centre of 20% of the radius. You may assume that the equation of state is that of an ideal gas, and that the mean molecular weight is uniform throughout the star.

d) Find the value of ξ at the point at which the temperature is 80% of the central temperature. What percentage of the star's radius does this value of ξ correspond to?

[10 marks]

Question A4

a) Derive the equation of hydrostatic equilibrium for a spherical self-gravitating body in the form

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r).$$

b) Consider a stellar model with uniform density ($\rho =$const). Show by direct integration of the equation of hydrostatic equilibrium that pressure P at the stellar centre is

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4},$$

where M is stellar mass and R is stellar radius.

[10 marks]

Question A5

The optical depth, τ, in a stellar atmosphere is defined as

$$\tau = \int_r^\infty \kappa\rho \, dr.$$

a) Starting from the equation for radiative flux, F, given in the rubric, show that in the atmosphere of a star

$$T^4 = \frac{3F}{ac}\left(\tau + B\right),$$

where B is a constant of integration.

b) Assume without proof that $B = 2/3$. Using $F = \sigma T_{\text{eff}}^4$ as a definition of the effective temperature T_{eff}, show that

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3}\right).$$

c) Discuss possible weaknesses of this derivation of the $T - \tau$ relation in stellar atmospheres.

[10 marks]

Question A6

Consider a group of homogeneous stars, which are assumed to be composed of ideal gases, all with the same chemical composition. All the energy is carried by radiation and the main opacity is due to electron scattering so that κ is a constant. The energy generation is by the CNO cycle, with $\epsilon = \epsilon_0 \rho T^{17}$.

a) Show that

$$R \propto M^{0.8}, \quad L \propto M^3.$$

b) Determine the slope of the line in the HR diagram on which these stars will lie.

[10 marks]

SECTION B Each question carries 40 marks. You may attempt all questions but ONLY the best ONE question will be counted.

Question B1

a) Explain, using physical arguments, why the gravitational binding energy Ω of a spherically-symmetric star is

$$\Omega = -G\int_0^M \frac{m\,dm}{r},$$

where $m = m(r)$ is the mass interior to radius r.

[7 marks]

b) The star is in hydrostatic equilibrium. Using integration by parts, show that

$$\Omega = -3\int_V P\,dv,$$

where V is the spherical volume occupied by the star.

[10 marks]

c) Consider a stellar model which is composed of classical particles. For a gas of classical particles, pressure P and internal energy density u are related as $u = 3P/2$, the relation which you can use without proof. Show that

$$\Omega = -2U,$$

where U is the thermal energy of the star. What is the total energy of the star?

Before arriving to the main sequence, the star is contracting slowly under its own gravity. Show that half of the gravitational energy which is released in the contraction is radiated away, and another half goes to heat up the gas.

[10 marks]

d) Consider a stellar model which is composed of ultra-relativistic particles. For a gas of ultra-relativistic particles, $u = 3P$ (a relation which you may also use without proof). Show that in this case,

$$\Omega = -U.$$

What is the total energy of the star?

Explain, using physical arguments, how this result puts an upper limit on the mass of a white dwarf (Chandrasekhar limit).

[13 marks]

Question B2

A star of mass M contracts under the forces of gravity, deriving its energy only from the change in its gravitational potential energy. This potential energy for a star of radius R and polytropic index n is given by

$$\Omega = -\frac{3GM^2}{(5-n)R}.$$

The virial theorem holds and so you can assume that half the gravitational energy released is radiated away, while the polytropic index is 3/2. In such a star the energy is transported outwards by convection and the star may be assumed to evolve with T_{eff} remaining constant.

a) Show that the time, t_1, taken by such a star to evolve from a large radius to some smaller radius R_1 is given by

$$t_1 = \frac{GM^2}{7L_1R_1},$$

where L_1 is the luminosity when the radius is R_1.

[15 marks]

b) When the star reaches a radius R_1 it re-adjusts internally such that energy transport becomes radiative rather than convective so that the polytropic index changes to 3. Both the luminosity and effective temperature remained constant during this adjustment. Show that t_2, the time taken by the star in adjusting internally is given by

$$t_2 = \frac{9GM^2}{28L_1R_1}.$$

[10 marks]

c) In this radiative state, the star may be assumed to evolve with a constant luminosity, L_1. Calculate the time taken by the star to evolve from a radius R_1 to a radius R_2 on the main sequence.

[12 marks]

d) State, with a reason, which of the three stages of evolution (a-c) described above is most likely to be observed.

[3 marks]

Question B3

According to Pauli's exclusion principle, at most two electrons can occupy a given energy state, and each particular energy state occupies volume h^3 in the 6-dimensional space of coordinates and momenta, where h is the Planck's constant.

a) In a degenerate gas all the electron states are filled up to a threshold momentum p_F and none above. Show that the electron number density is

$$n_e = \frac{8\pi}{3h^3} p_F^3.$$

[8 marks]

b) Explain why this simple description is applicable at low temperatures, namely when

$$kT \ll \frac{h^2 n_e^{2/3}}{m_e},$$

where k is the Boltzmann's constant and m_e is the electron mass.

[7 marks]

c) Show that when the pressure P is dominated by the electron pressure, and the electrons are moving with speeds small compared to the speed of light,

$$P = \frac{8\pi}{15h^3 m_e} p_F^5.$$

You may use the general expression

$$P = \frac{1}{3} \int_0^\infty vpn(p)dp,$$

where $n(p)$ is number density of particles with momentum p in the interval $(p, p + dp)$, and v is their velocity.

[8 marks]

d) Show that for a white dwarf, assumed to be made of non-relativistic degenerate material, the mass M and radius R satisfy the relationship $R \propto M^{-1/3}$.

[7 marks]

e) Assuming that in white dwarfs the energy transport is such that $L \propto M^{5.5} R^{-0.5}$, find the slope of the line in the H-R diagram on which white dwarfs will lie.

[10 marks]

End of Paper