Queen Mary
University of London

M. Sci. Examination by course unit 2010

ASTM109 Stellar Structure and Evolution

Duration: 3 hours

Date and time: 11 June 2010, 1430h–1730h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): S.V.Vorontsov

You may assume the following:

In all questions: M is the mass, $m(r)$ the mass interior to radius r, R is the radius, L the luminosity and T_{eff} the effective temperature of a star. P, ρ and T denote the pressure, density and temperature respectively. κ is the opacity per unit mass, ϵ the rate of energy production per unit mass and μ denotes the mean molecular weight. c_p, c_v are the specific heats at constant pressure and volume, $\gamma = c_p/c_v$ and $\mathcal{R}$ is the gas constant where $\mathcal{R} = \mu(c_p - c_v)$.

$L = 4\pi R^2 F_{Rad}$ and F_{Rad} is given by

$$F_{Rad} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}.$$

$c, G, \sigma = ac/4$ are respectively the velocity of light, the constant of gravity and the Stefan-Boltzmann radiation constant. X, Y, Z are the mass fractions respectively of hydrogen, helium and the heavier elements.

The central density ρ_c, central temperature T_c and central pressure P_c of a polytrope of index n are

$$\rho_c = a_n \frac{3M}{4\pi R^3}, \qquad T_c = b_n \frac{\mu G M}{\mathcal{R} R}, \qquad P_c = c_n \frac{GM^2}{R^4}.$$

The apparent magnitude m_{app}, absolute magnitude M_{abs} and distance in parsecs d are related by $m_{\text{app}} = M_{\text{abs}} + 5\log d - 5$. The following rounded numerical values, all in S.I. Units may be assumed throughout the paper.

$c = 3\times10^8, G = 7\times10^{-11}, \sigma = 6\times10^{-8}, M_\odot = 2\times10^{30}, R_\odot = 7\times10^8, L_\odot = 4\times10^{26}.$

You may also assume that 1 year is 3×10^7seconds.

Section A: This section carries 60 marks. You should attempt ALL 6 questions. Marks awarded are shown next to the question.

Question 1 A star S has a measured parallax of 0.13 arcsec, and an apparent magnitude of 0.03. Spectroscopic measurements show that the effective temperature of S is twice bigger than that of the Sun.

(a) What is the visible colour of the star S: is it redder or bluer than the Sun?

(b) What is the distance to the star S (in parsecs)? What is its absolute magnitude?

(c) The Sun has an absolute magnitude of 4.62. What is the apparent magnitude of a star identical to the Sun, when it is placed at the same distance as S?

(d) What is the luminosity of the star S? What is its radius?

[9]

Question 2 The mean molecular weight μ is defined as

$$\mu = \frac{\rho}{n m_H},$$

where n is the number of particles per unit volume and m_H is the mass of the hydrogen atom. Show that, for a fully ionized gas,

$$\frac{1}{\mu} = \sum_i X_i \frac{Z_i + 1}{A_i},$$

where X_i is the mass fraction of the element i which has atomic number Z_i and atomic weight A_i.
Two stars have homogeneous chemical compositions, which is a mixture of hydrogen and helium only. The first star has $Y = 0.25$ and the second $Y = 0.35$ and both are assumed to be polytropes of the same polytropic index. The central temperature is assumed to be the same in both stars and given by the formula in the rubric. If the second star has twice the mass of the first star, calculate the ratio of their radii.

[9]

Question 3 The pressure and density inside a star obey the relation

$$P = K\rho^{\frac{n+1}{n}},$$

where K is a constant. The star is in hydrostatic equilibrium, i.e.

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho.$$

Show that when the density profile inside the star is specified as $\rho = \rho_c\theta^n(\xi)$, where ρ_c is the central density, $\xi = r/\alpha$ is dimensionless radius and

$$\alpha = \left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1-n}{n}}\right]^{\frac{1}{2}},$$

then $\theta(\xi)$ is the solution to the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,$$

with boundary conditions $\theta = 1$ and $d\theta/d\xi = 0$ at $\xi = 0$.
Show by direct integration that for $n = 0$, the solution is

$$\theta = 1 - \frac{\xi^2}{6}.$$

[10]

Question 4 Near the centre of an evolved star, there is an isothermal helium core of mass M_S and radius R_S. The equation of state is that of an ideal gas in the form

$$P = \frac{\mathcal{R}}{\mu_S}\rho T,$$

where μ_S is the mean molecular weight in the core. Show that there is a maximum value for the pressure, P_S, at the surface of the core when the radius of the core is

$$R_S = \frac{2\mu_S G M_S}{3\mathcal{R}T}.$$

You may assume that the gravitational binding energy of the core is

$$\Omega = -\frac{3GM_S^2}{2R_S}$$

and that

$$\Omega = 4\pi R_S^3 P_S - 3\int_0^{R_S} 4\pi r^2 P\, dr.$$

Show that this maximum pressure is

$$P_S = \frac{GM_S^2}{8\pi R_S^4}.$$

[10]

Question 5 You are given that the gravitational binding energy Ω of a star of polytropic index n is

$$\Omega = -\frac{3GM^2}{(5-n)R},$$

and it is related to the internal energy U by the virial theorem

$$2U + \Omega = 0.$$

Consider a star with a polytropic index $n = 3$ and no nuclear energy sources. The opacity is assumed to be of the Kramers type so that $L \propto M^{5.5}R^{-0.5}$. Show that such a star will evolve along a line in the HR diagram with a slope of 0.8. Show that t, the time taken by such a star to evolve from a very large radius to some smaller radius R_0, is given by

$$t \propto M^{-3.5}R_0^{-0.5}.$$

[10]

Question 6 Consider a group of homogeneous stars, which are assumed to be composed of ideal gases, all with the same chemical composition. All the energy is carried by radiation and the main opacity is due to electron scattering so that κ is a constant. The energy generation is by the CNO cycle, with $\epsilon = \epsilon_0\rho T^{17}$. Show that

$$R \propto M^{0.8}, \quad L \propto M^3.$$

Determine the slope of the line in the HR diagram on which these stars will lie.

[12]

Section B: Each question carries 40 marks. Marks for each part are shown thus [N]. You may attempt all questions but ONLY the best ONE question will be counted.

Question 7 (a) Define the optical depth, τ, in a stellar atmosphere.

Starting from the equation for radiative flux, F, given in the rubric, show that in the atmosphere of a star

$$T^4 = \frac{3F}{ac}\left(\tau + B\right),$$

where B is a constant of integration, which you may assume without proof to be $B = 2/3$. Using $F = \sigma T_{\rm eff}^4$ as a definition of the effective temperature $T_{\rm eff}$, show that

$$T^4 = \frac{3}{4}T_{\rm eff}^4\left(\tau + \frac{2}{3}\right).$$

[8]

(b) Assume now that this $T - \tau$ relation is valid everywhere in the radiative atmosphere, including optically thin layers. Explain the possible weaknesses of this assumption. Assume further that the atmosphere is an ideal gas and in hydrostatic equilibrium and that the mass and thickness of the atmosphere are both negligible compared to the mass and radius of the star. Given that $P = 0$ at $\tau = 0$ and that the opacity is given by $\kappa = \kappa_0 \rho T^5$ with some constant κ_0, show that

$$P^2 = P_0^2 \ln\left(1 + \frac{3}{2}\tau\right),$$

where P_0 is another constant which you do not need to specify. [8]

(c) Derive the Schwarzschild condition for the onset of convection in an ideal gas, namely

$$\frac{d\ln T}{d\ln P} > \frac{\gamma - 1}{\gamma}.$$

[8]

(d) Given that $\gamma = 5/3$, show that the convection sets in at a level where

$$\tau = \frac{2}{3}\left[\exp\left(\frac{4}{5}\right) - 1\right].$$

[8]

(e) Show that at a depth h measured from the top of the convective zone, the temperature is approximately given by

$$T = T_S + \frac{2GM\mu}{5\mathcal{R}R^2}h,$$

where T_S is the temperature at the top of the convective zone. You may assume that the mass of the convective zone is small compared to M and the depth h is small compared to R, and that in this zone $P = KT^{5/2}$ (i.e., the temperature gradient is purely adiabatic). [8]

Question 8 According to Pauli's exclusion principle, at most two electrons can occupy a given energy state, and each particular energy state occupies volume h^3 in the 6-dimensional space of coordinates and momenta, where h is the Planck's constant.

(a) In a degenerate gas all the electron states are filled up to a threshold momentum p_F and none above. Show that the electron number density is

$$n_e = \frac{8\pi}{3h^3} p_F^3.$$

Explain why this simple description is applicable at low temperatures, namely when

$$kT \ll \frac{h^2 n_e^{2/3}}{m_e},$$

where k is the Boltzmann's constant and m_e is the electron mass. [10]

(b) Show that when the pressure P is dominated by the electron pressure, and the electrons are moving with speeds small compared to the speed of light,

$$P = \frac{8\pi}{15h^3 m_e} p_F^5.$$

You may use the general expression

$$P = \frac{1}{3}\int_0^\infty vpn(p)dp,$$

where $n(p)$ is number density of particles which momentum p is in the interval $(p, p+dp)$, and v is their velocity. [7]

(c) Show that for a white dwarf, assumed to be made of non-relativistic degenerate material, the mass M and radius R satisfy the relationship $R \propto M^{-1/3}$. [5]

(d) Assuming that in white dwarfs the energy transport is such that $L \propto M^{5.5} R^{-0.5}$, find the slope of the line in the H-R diagram on which white dwarfs will lie. [8]

(e) A star has a non-relativistic degenerate helium core of mass M_c and radius R_c surrounded by a hydrogen burning shell. The mass–radius relationship for this core may be assumed to be like that of a white dwarf. The energy generated by fusion of mass m of hydrogen to helium is approximately $0.02mc^2$, where c is the speed of light. Show that for a star with luminosity L, an additional degenerate mass is deposited on the core at the rate

$$\frac{dM_c}{dt} = \frac{50L}{c^2}.$$

Assuming that helium is deposited gently and uniformly on the core, show that the rate of release of gravitational energy in the core is

$$\frac{100GM_cL}{c^2R_c}.$$

You may assume that the gravitational potential energy of the core is

$$-\frac{6GM_c^2}{7R_c}.$$

[10]

Question 9 (a) Consider the gravitational field created by a thin spherical shell of radius r', thickness dr' and density ρ. Show that at distance r from the centre of the shell, the gravitational potential is

$$d\psi = -G\frac{dm}{r}, \quad \text{when} \quad r \geq r', \quad \text{and}$$

$$d\psi = -G\frac{dm}{r'}, \quad \text{when} \quad r < r',$$

where dm is the mass inside the shell. [10]

(b) Show that the gravitational potential in the interior a spherically symmetric star, measured at distance r from its centre, is

$$\psi = -G\frac{m}{r} - G\int_m^M \frac{dm}{r},$$

where $m = m(r)$ is the mass coordinate (mass inside the sphere of radius r). [4]

(c) Using integration by parts, show that

$$\int_0^M \psi\, dm = -2G\int_0^M \frac{m\, dm}{r}.$$

[6]

(d) Explain, using physical arguments, why the gravitational binding energy of a spherically-symmetric star is

$$\Omega = -G\int_0^M \frac{m\, dm}{r}.$$

The star is in hydrostatic equilibrium. Using integration by parts, show that

$$\Omega = -12\pi\int_0^R Pr^2\, dr = -3\int_0^M \frac{P}{\rho} dm.$$

[5]

(e) The star is a polytrope of index n (i.e. $P = K\rho^{\frac{n+1}{n}}$). The equation of hydrostatic support can be written as

$$\frac{1}{\rho}\frac{dP}{dr} + \frac{d\psi}{dr} = 0.$$

Show that the first term on the left-hand side of this equation can be written as a complete derivative:

$$\frac{1}{\rho}\frac{dP}{dr} = (n+1)\frac{d}{dr}\left(\frac{P}{\rho}\right).$$

Integrating the equation of hydrostatic support, show that

$$(n+1)\frac{P}{\rho} + \psi = -\frac{GM}{R}.$$

[10]

(f) Show finally that the gravitational binding energy of a polytropic star is

$$\Omega = -\frac{3GM^2}{(5-n)R}.$$

[5]

End of Paper