# Exercise 5

### Physical constants

$M_\odot = 2\times10^{30}$ kg $R_\odot = 7\times10^{8}$ m $M_{\rm Sun} = 4.63$ (absolute magnitude) $L_\odot = 3.83\times10^{26}$ J s$^{-1}$
1 AU $=1.5\times10^{11}$ m $G = 6.67\times10^{-11}$ N m$^2$ kg$^{-2}$ $\sigma = 5.7\times10^{-8}$ kg s$^{-3}$ K$^{-4}$ (S-B constant)
$k_{\rm B} = 1.38\times10^{-23}$ m$^2$ kg s$^{-2}$ K$^{-1}$ $m_{\rm H} = 1.67\times10^{-27}$ kg

## Assessed questions

### Question 1

(i) Consider a fully degenerate electron gas. The number of electrons in volume $V$ with momenta between $p$ and $p+dp$ is given by

$$\begin{aligned} N_p dp &= \frac{8\pi p^2 V}{h^3} dp \quad \text{for } p \le p_0 \\ N_p dp &= 0 \qquad\qquad \text{for } p > p_0, \end{aligned} \tag{1}$$

where $p_0$ is the Fermi momentum and $h$ is Planck's constant. Show that the number of electrons per unit volume, $n_e$, is given by

$$n_e = \frac{8\pi p_0^3}{3h^3}. \tag{2}$$

**Solution**

Divide both sides by $V$ and integrate

$$n_e = \int_0^{p_0} \frac{N_p dp}{V} = \int_0^{p_0} \frac{8\pi p^2 V}{h^3} dp = n_e = \frac{8\pi p_0^3}{3h^3}.$$

15 marks

(ii) Obtain an expression for $p_0$ in terms of the electron number density and fundamental constants.

**Solution**

Simply solve for $n_e$

$$n_e = \frac{8\pi p_0^3}{3h^3} \implies p_0 = \frac{h}{2}\left(\frac{3n_e}{\pi}\right)^{1/3}$$

15 marks

(iii) The electron degeneracy pressure is given by

$$P = \frac{8\pi}{3h^3 m_e} \int_0^{p_0} \frac{p^4\, dp}{\left(1+\frac{p^2}{m_e^2c^2}\right)^{1/2}}. \tag{3}$$

Showing all of your working, demonstrate that in the non-relativistic limit, $P$ can be expressed as

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{h^2}{m_e}\right) n_e^{5/3}. \tag{4}$$

**Solution**

We have $p^2 \ll m_e^2c^2$ in the non-relativistic limit. Hence

$$P = \frac{8\pi}{3h^3 m_e} \int_0^{p_0} \frac{p^4\, dp}{\left(1+\frac{p^2}{m_e^2c^2}\right)^{1/2}} \longrightarrow \frac{8\pi}{3h^3 m_e}\int_0^{p_0} p^4\, dp = \frac{8\pi}{15h^3 m_e} p_0^5.$$

Substituting above expression for $p_0$ gives

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{h^2}{m_e}\right) n_e^{5/3}.$$

20 marks

Show that in the ultra-relativistic limit, $P$ can be expressed as

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn_e^{4/3}. \tag{5}$$

**Solution**

In the ultra-relativistic limit $p^2 \gg m_e^2 c^4$. Hence

$$P = \frac{8\pi}{3h^3 m_e}\int_0^{p_0} \frac{p^4\, dp}{\left(1+\frac{p^2}{m_e^2c^2}\right)^{1/2}} \longrightarrow \frac{8\pi}{3h^3 m_e}\int_0^{p_0} (m_e c)\, p^3\, dp = \frac{2\pi c}{3h^3} p_0^4.$$

Substituting above expression for $p_0$ gives

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn_e^{4/3}.$$

20 marks

(iv) Equation (4) can be written in the form

$$P = K_1 \rho^{5/3} \tag{6}$$

where

$$K_1 = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{1+X}{2m_{\rm H}}\right)^{5/3},$$

demonstrating that a white dwarf star supported by non-relativistic degeneracy pressure has the structure of an $n = 3/2$ polytrope. Using the above expression for $K_1$, combined with equation (20) from the lecture notes of week 4 (see the online version), and the relevant information contained in the final row of table 2 from those lecture notes, obtain an expression that relates the radius to the mass of a white dwarf. Assuming $X = 0$, find the radius of a 1 $M_\odot$ white dwarf star. How large is it compared to the Sun and the Earth?

**Solution**

Equation (20) from week 4 lecture notes gives:

$$K = (4\pi)^{\frac{1}{n}} \frac{G}{n+1}\xi_1^{-\frac{n+1}{n}}\left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}^{\frac{1-n}{n}} M^{\frac{n-1}{n}} R^{\frac{3-n}{n}}.$$

From table 2 we also have $\xi_1 = 3.654$ and $(-d\theta/d\xi)_{\xi_1} = 0.2$. We equate $K$ with $K_1$ defined above:

$$(4\pi)^{\frac{1}{n}} \frac{G}{n+1}\xi_1^{-\frac{n+1}{n}}\left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}^{\frac{1-n}{n}} M^{\frac{n-1}{n}} R^{\frac{3-n}{n}} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{1+X}{2m_{\rm H}}\right)^{5/3}$$

and then solve for $R$ in terms of $M$ (which gives $R \propto M^{1/3}$). Inserting $n = 3/2$, values for all the constants, and the quantities obtained from the polytropic model, we should obtain a value: $R \approx 8623937$ metres $= 0.0123\ R_\odot = 1.35\ R_\oplus$.

30 marks

**Non-assessed questions**

(i) Make sure that you fully understand, and are able to reproduce, all steps involved in the derivation of the electron degeneracy pressure covered in the week 9 lecture notes.

(ii) Make sure that you understand the consequences for increasing the mass of the white dwarf for the number density of electrons, and for the Fermi momentum, and how this leads to a change from a non-relativistic system to an ultra-relativistic one. Make sure you understand the consequences for the masses of white dwarfs of the equation of state for an ultra-relativistic fully-degenerate electron gas, and how one obtains an estimate of the Chandrasekhar mass.