Exercise 4

Physical constants

$M_\odot = 2\times10^{30}$ kg $R_\odot = 7\times10^{8}$ m $M_{Sun} = 4.63$ (absolute magnitude) $L_\odot = 3.83\times10^{26}$ J s^{-1}
1 AU $=1.5\times10^{11}$ m $G = 6.67\times10^{-11}$ N m^2 kg^{-2} $\sigma = 5.7\times10^{-8}$ kg s^{-3} K^{-4} (S-B constant)
$k_B = 1.38\times10^{-23}$ m^2 kg s^{-2} K^{-1} $m_H = 1.67\times10^{-27}$ kg

Assessed questions

Question 1

Consider a group of stars. Each star in the group has the same chemical composition, is homogeneous, and is composed of an ideal gas. Energy generation occurs through the P-P chain, with $\epsilon = \epsilon_{pp}\rho T^4$ and ϵ_{pp} being a constant. All of the energy is transported by radiation (and hence the temperature gradient is given by the expression derived in lectures for radiative zones) and the opacity is given by $\kappa = \kappa_0\rho T^{-3.5}$, where κ_0 is a constant.

(i) Show that

$$M \propto R^{13}.$$

Solution

We have

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi a c r^2 T^3}, \tag{1}$$

$$\epsilon = \epsilon_{pp}\rho T^4, \tag{2}$$

$$\kappa = \kappa_0\rho T^{-3.5}, \tag{3}$$

$$T \approx \frac{GM\mu m_H}{k_B R}, \tag{4}$$

and

$$\rho \approx \frac{M}{R^3}. \tag{5}$$

From eqn (1) we have

$$L = -\frac{4\pi r^2 ac}{3\kappa\rho}\frac{dT^4}{dr} \implies L \approx \frac{ac}{\kappa_0}\left(\frac{G\mu m_H}{k_B}\right)^{1/2} R^{-0.5}M^{5.5}$$

hence

$$L \propto R^{-0.5}M^{5.5}.$$

Energy generation gives

$$L = \int_M \epsilon dm = \int_V \epsilon\rho dV = 4\pi\int_0^R \epsilon\rho r^2 dr \approx \epsilon\rho R^3. \tag{6}$$

Hence

$$L \approx \epsilon_{pp}\rho^2 T^4 R^3 \implies L \propto M^6 R^{-7}.$$

Combining expressions for L gives

$$M \propto R^{13}.$$

20 marks

(ii) Also show that

$$L \propto M^{71/13}.$$

Solution

Combining

$$L \propto R^{-0.5}M^{5.5}$$

and

$$M \propto R^{13}$$

gives

$$L \propto M^{-1/26} M^{11/2} \implies L \propto M^{142/26} \implies L \propto M^{71/13}.$$

20 marks

(iii) Obtain the slope of the line in an H-R diagram ($\log_{10} L$ versus $\log_{10} T_{\rm eff}$) that these stars lie on.

Solution

We have

$$L \propto T_{\rm eff}^4 R^2$$

. Combining

$$M \propto R^{13}, \quad L \propto M^{71/13}$$

gives

$$T_{\rm eff} \propto M^{69/52}.$$

Hence we have

$$L \propto T^{284/69}.$$

Hence slope in H-R diagram will be $142/31 \approx 4.12$.

20 marks

Question 2

The Schwarzschild condition for convective instability is

$$\frac{d \ln T}{d \ln P} > \frac{\gamma - 1}{\gamma}.$$

Consider a stellar atmosphere that is in radiative equilibrium, and which has a temperature profile

$$T^4 = \frac{3}{4} T_{\rm eff}^4 \left(\tau + \frac{2}{3} \right),$$

and pressure profile

$$P^2 = P_0^2 \ln \left(1 + \frac{3}{2} \tau \right),$$

where τ is the optical depth, and P_0 is a constant. Given that $\gamma = 5/3$, show that convection sets in where

$$\tau = \frac{2}{3} \left[\exp \left(\frac{4}{5} \right) - 1 \right].$$

Solution

Here we treat the optical depth, τ, as the independent variable, and differentiate T and P with respect to τ to construct an expression for $d \ln T / dlnP$, where we note that

$$\frac{1}{T} \frac{dT}{d\tau} \left(\frac{1}{P} \frac{dP}{d\tau} \right)^{-1} = \frac{d \ln T}{d\tau} \left(\frac{d \ln P}{d\tau} \right)^{-1} = \frac{d \ln T}{d \ln P}.$$

After differentiating and some algebra we obtain

$$\frac{d \ln P}{d\tau} = \frac{3}{4(1 + 3\tau/2)} \frac{1}{\ln (1 + 3\tau/2)},$$

and

$$\frac{d \ln T}{d\tau} = \frac{1}{4(\tau + 2/3)}.$$

We then obtain

$$\frac{d \ln T}{d \ln P} = \frac{1}{2} \ln (1 + 3\tau/2) = \frac{\gamma - 1}{\gamma}.$$

This gives

$$\ln\left(1 + 3\tau/2\right) = \frac{2(\gamma - 1)}{\gamma} = \frac{4}{5}$$

since $\gamma = 5/3$. Hence we finally obtain

$$\tau = \frac{2}{3}\left[\exp\left(\frac{4}{5}\right) - 1\right].$$

40 marks

Non-assessed questions

(i) Make sure that you are able to derive the Schwarzschild criterion for convective instability.

(ii) Make sure that you understand which parameters control the radiative temperature gradient, and hence what determines whether or not a star has a convection zone or not.

(iii) For radiative models of stars, make sure you understand how scaling relations between mass, radius, temperature, luminosity can be obtained for specified opacity and energy generation rates.