Exercise 3

Physical constants

$M_\odot = 2\times10^{30}$ kg $R_\odot = 7\times10^{8}$ m $M_{Sun} = 4.63$ (absolute magnitude) $L_\odot = 3.83\times10^{26}$ J s^{-1}
1 AU $=1.5\times10^{11}$ m $G = 6.67\times10^{-11}$ N m^2 kg^{-2} $\sigma = 5.7\times10^{-8}$ kg s^{-3} K^{-4} (S-B constant)
$k_B = 1.38\times10^{-23}$ m^2 kg s^{-2} K^{-1} $m_H = 1.67\times10^{-27}$ kg

Assessed questions

Question 1

The Lane-Emden equation is given by

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

where $\rho = \rho_c\theta^n$, $r = \alpha\xi$ and n is the polytropic index.
(i) Show that for $n = 1$, the solution to the Lane-Emden equation is given by

$$\theta = \frac{\sin\xi}{\xi}.$$

Solution
Demonstrate by substitution.

$$\frac{d\theta}{d\xi} = \frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}, \quad \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi\sin\xi + \cos\xi + \cos\xi = -\xi\sin\xi.$$

Hence

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\frac{\sin\xi}{\xi},$$

and for $n = 1$ we also have

$$-\theta^n = -\frac{\sin\xi}{\xi}.$$

Therefore we have proved that the given θ is a solution. 20 marks

(ii) Calculate the value of ξ_1, where $\theta(\xi_1) = 0$. Calculate the value of $-(d\theta/d\xi)_{\xi=\xi_1}$.

Solution
By inspection we see that $\xi = \pi$ corresponds to $\theta(\xi) = 0$. Hence $\xi_1 = \pi$. We have

$$\frac{d\theta}{d\xi} = \frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2},$$

hence

$$-\left(\frac{d\theta}{d\xi}\right)_{\xi_1} = 1/\pi$$

15 marks

(ii) Assuming that the polytropic model described above has a Solar mass and radius, and has mass fractions of hydrogen and helium that are 70% and 30%, respectively, determine the values of central density ρ_c, pressure P_c and temperature T_c, giving your answers in S.I. units.

Solution
We have

$$\rho_c = -\frac{\xi_1}{3}\left(\frac{d\theta}{d\xi}\right)^{-1}_{\xi_1}\frac{3M}{4\pi R^3}.$$

Hence

$$\rho_c = \frac{\pi}{3}\times\pi\times\frac{3\times2\times10^{30}}{4\pi\times(7\times10^8)^3} = 4580\,\mathrm{kg\,m^{-3}}.$$

$$P_c = \frac{1}{4\pi(n+1)} \left(-\frac{d\theta}{d\xi}\right)_{\xi_1}^{-2} \frac{GM^2}{R^4}.$$

Hence

$$P_c = \frac{1}{4\pi \times (1+1)} \times \pi^2 \times \frac{6.67 \times 10^{-11} \times (2 \times 10^{30})^2}{(7 \times 10^8)^4} = 4.4 \times 10^{14} \,\text{pascals}$$

$$T_c = \left[(n+1)\xi_1 \left(-\frac{d\theta}{d\xi}\right)_{\xi_1} \right]^{-1} \frac{GM\mu m_H}{k_B R} \quad \text{and} \quad \mu = \frac{4}{8X+3Y} = 0.62.$$

Hence

$$T_c = \frac{1}{(1+1) \times \pi \times (1/\pi)} \times \frac{6.67 \times 10^{-11} \times 2 \times 10^{30} \times 0.62 \times 1.67 \times 10^{-27}}{1.38 \times 10^{-23} \times 7 \times 10^8} = 7 \times 10^6 \,\text{K}.$$

15 marks

Question 2

Energy generation in stars occurs because fusion reactions convert hydrogen to helium during their main sequence life times.

(i) Considering a head-on collision between two arbitrary nuclei with atomic numbers Z_1 and Z_2, and radii r_1 and r_2, obtain an expression for the temperature that a plasma must have in order that typical collisions between the nuclei 1 and 2 result in the nuclei physically colliding.

Solution

Coulomb energy is given by

$$E = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 (r_1 + r_2)}$$

which is obtained by estimating the work done in bringing two charges, with radii r_1 and r_2 from infinity until they just touch. The mean thermal energy for a gas whose constituent particles have a Maxwell-Boltzmann distribution of energies is given by $\langle U \rangle = (3k_B T)/2$. Equating the thermal and Coulomb energies and solving for the temperature gives

$$T = \frac{Z_1 Z_2 e^2}{6\pi\epsilon_0 k_B (r_1 + r_2)}.$$

20 marks

(ii) Using your expression, estimate the temperature required for two typical hydrogen nuclei to collide.

Solution

Using $r_1 = r_2 = 1 \times 10^{-15}$ m for hydrogen nuclei, we obtain

$$T = \frac{1 \times 1 \times (1.6 \times 10^{-19})^2}{6\pi \times 1.38 \times 10^{-23} \times 2 \times 10^{-15}} = 5.6 \times 10^9 \,\text{K}.$$

5 marks

(iii) Assuming that atomic nuclei can be treated as constant density spheres in which the density is independent of the atomic mass, estimate the temperature required for collisions to occur between ^{12}C and ^{1}H nuclei.

Solution

Here we take $Z_1 = 1$ and $Z_2 = 6$ for the charges, and $r_2 = 12^{1/3} \times r_1$ for the radius of the ^{12}C nucleus, and we take $r_1 = 10^{-15}$ m for the hydrogen nucleus. Inserting these into the temperature expression obtained above gives

$$T = 2 \times 10^{10} \,\text{K}.$$

10 marks

(iv) Based on your answers to parts (ii) and (iii), describe in words how you expect the PP- and CNO-chains to contribute to the energy output of main sequence stars as a function of their masses.

Solution

Very simply, we would expect the CNO cycle to become important or dominant at higher temperatures because of the larger Coulomb barrier that must be overcome. More massive stars have higher central temperatures, and therefore we would expect the CNO cycle to be important in higher mass stars.

5 marks

(v) The energy generation rate per unit mass from the PP-chain is given by $\epsilon_{\rm pp} = 2.6 \times 10^{-37} X^2 \rho T^{4.5}$ J s^{-1} kg^{-1}, and that from the CNO cycle is $\epsilon_{\rm CNO} = 7.9 \times 10^{-118} X Z \rho T^{16}$ J s^{-1} kg^{-1}. Above which temperature do we expect energy output from the CNO cycle to exceed that from the PP-chain per kg of stellar matter? You should assume the hydrogen mass fraction is 70% and the mass fraction of heavy elements is 2%.

Solution

Equating the two expressions

$$2.6 \times 10^{-37} X^2 \rho T^{4.5} = 7.9 \times 10^{-118} X Z \rho T^{16}$$

and solving for the temperature

$$T = \left(\frac{2.6 \times 10^{-37}}{7.9 \times 10^{-118}} \frac{X}{Z} \right)^{1./11.5}.$$

This evaluates to

$$T = 1.4 \times 10^7 \, K$$

. 10 marks

Non-assessed questions

(i) Make sure that you are able to derive the Lane-Emden equation from the equations of mass conservation, hydrostatic equilibrium and the equation of state. Make sure that you can derive an expression for the Lane-Emden equation using discrete steps in ξ and θ that can be used to solve the equation numerically, and ensure that you fully understand how numerical solutions are obtained.

(ii) Make sure that you can write down the nuclear reactions that make up the PP-chains and the CNO-cycle. Make sure you understand how the energy generation rates per unit mass can be used to evaluate the total luminosity of a stellar model, and in particular how it can be used for polytropic models.